Taylor Series Method: (36)

Consider initial value problem:

$$\frac{dy}{dx} = f(x,y) \quad ; \quad y(x_0) = y_0$$

Taylor series formula:

$$y_1 = y(x_1) = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots$$

$$y_2 = y(x_2) = y(x_1) + \frac{h}{1!} y'(x_1) + \frac{h^2}{2!} y''(x_1) + \frac{h^3}{3!} y'''(x_1) + \cdots$$

In general,

$$y_{n+1} = y(x_{n+1}) = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots$$

where, $y_n = y(x_n)$

$x_n = x_0 + nh$

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Q: Solve $y' = x + y$; $y(0) = 1$
using Taylor series method
Find $y(0.1)$, $y(0.2)$

Sol: The differential equation is:

$$y' = x + y \quad ; \quad y(0) = 1$$

$$x_0 = 0 \,,\; y_0 = 1 \,,\; h = 0.1$$

$x_n = x_0 + nh$
$x_1 = 0 + h$
$h = 0.1$

Taylor series Formula:

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \dots \quad (1)$$

Now,

$$y' = x + y \quad \text{(given)}$$

$$y'' = 1 + y'$$

$$y''' = y''$$

Putting $n = 0$ in (1)

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots \quad (2)$$

$$y_0 = y(x_0) = y(0) = 1$$

$$y_0' = x_0 + y_0 = 0 + 1 = 1$$

$$y_0'' = 1 + y_0' = 1 + 1 = 2$$

$$y_0''' = y_0'' = 2$$

Then (2) becomes:

$$y_1 = 1 + (0.1)(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2)$$

$$y_1 = 1.1103$$

$$\boxed{y_1 = y(x_1) = y(0.1) = 1.1103}$$

$x_n = x_0 + nh$
$x_1 = x_0 + h$
$x_1 = 0.1$

Putting $n = 1$ in (1)

$$y_2 = y_1 + h y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \dots \quad (3)$$

$$y_1 = y(x_1) = y(0.1) = 1.1103$$

$$y_1' = x_1 + y_1 = 0.1 + 1.1103 = 1.2103$$

$y_1'' = 1 + y_1' = 1 + 1.2103 = 2.2103$

$y_1''' = y_1'' = 2.2103$

③ becomes:

$$y_2 = 1.1103 + (0.1)(1.2103) + \frac{(0.1)^2}{2!}(2.2103) + \frac{(0.1)^3}{3!}(2.2103)$$

$y_2 = 1.2427$

$\boxed{y_2 = y(x_2) = y(0.2) = 1.2427}$

$x_n = x_0 + nh$
$x_2 = x_0 + 2h$
$x_2 = 2(0.1) = 0.2$

Q: Use Taylor series method to find $y(0.1)$, $y(0.2)$ of the initial value problem $\frac{dy}{dx} = 3e^x + 3y$, $y(0) = 0$ correct to 4 decimal places?

Sol:

The differential equation is:

$$y' = \frac{dy}{dx} = 3e^x + 3y \quad ; \quad y(0) = 0$$

$x_0 = 0$, $y_0 = 0$, $h = 0.1$

Taylor Series Formula:

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \cdots \quad ①$$

Now,

$y' = 3e^x + 3y$ (given)

$y'' = 3e^x + 3y'$

$y''' = 3e^x + 3y''$

Putting $n = 0$ in ①

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \quad ②$$

$y_0 = y(x_0) = y(0) = 0$

$y_0' = 3e^{x_0} + 3y_0 = 3e^0 + 3(0) = 3$

$y_0'' = 3e^{x_0} + 3y_0' = 3e^0 + 3(3) = 12$

$y_0''' = 3e^{x_0} + 3y_0'' = 3e^0 + 3(12) = 39$

Then ② becomes:

$$y_1 = 0 + (0.1)\,3 + \frac{(0.1)^2}{2!}(12) + \frac{(0.1)^3}{3!}(39)$$

$y_1 = 0.3665$

$$\boxed{y_1 = y(x_1) = y(0.1) = 0.3665}$$

$x_n = x_0 + nh$
$x_1 = 0 + 0.1$

Putting $n=1$ in (1)

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots \quad (3)$$

$$y_1 = y(x_1) = y(0.1) = 0.3665$$
$$y_1' = 3e^{x_1} + 3y_1 = 3e^{0.1} + 3(0.3665) = 4.4150$$
$$y_1'' = 3e^{x_1} + 3y_1' = 3e^{0.1} + 3(4.4150) = 16.5605$$
$$y_1''' = 3e^{x_1} + 3y_1'' = 3e^{0.1} + 3(16.5605) = 52.9970$$

Then (3) becomes:

$$y_2 = 0.3665 + (0.1)(4.4150) + \frac{(0.1)^2}{2!}(16.5605) + \frac{(0.1)^3}{3!}(52.9970)$$

$y_2 = 0.8996$

$$\boxed{y_2 = y(x_2) = y(0.2) = 0.8996}$$

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Milne's Predictor - Corrector Method :-

Consider the differential equation:

$$\frac{dy}{dx} = f(x,y) \quad ; \quad y(x_0) = y_0$$

Euler's formula:

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$n = 1, 2, 3, \dots$

{ Euler method = Predictor method

Modified Euler's method:

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k = \frac{1}{2}(k_1 + k_2)$$

$$y_{n+1} = y_n + k \quad ; \quad n = 0, 1, 2, \dots$$

{ Corrector method = Modified Euler's method.

Milne's Predictor Formula:

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

$\Rightarrow$ we'll always find y_4 from this formula

$$y_4 = y_0 + \frac{4h}{3}\left(2y'_1 - y'_2 + 2y'_3\right)$$

Milne's Corrector Formula:

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

$$y_4 = y_2 + \frac{h}{3}\left[y'_2 + 4y'_3 + y'_4\right]$$

This method is known as multi-step method.

Q: Solve $y' = \frac{1}{2}(1+x^2)y^2$; $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$. Compute $y(0.4)$ using Milne's Predictor-Corrector method.

The given differential equation is:

$$y' = \frac{1}{2}(1+x^2)y^2$$

$$f(x,y) = \frac{1}{2}(1+x^2)y^2 \quad , \quad h = 0.1$$

	x_0	x_1	x_2	x_3
x	0	0.1	0.2	0.3
y	1	1.06	1.12	1.21
	y_0	y_1	y_2	y_3

Milne's Predictor Method:

$$y_4 = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right] \quad (1)$$

Now,

$$y_1' = \frac{1}{2}(1+x_1^2)y_1^2 = \frac{1}{2}\left[1+(0.1)^2\right](1.06)^2 = 0.5674$$

$$y_2' = \frac{1}{2}(1+x_2^2)y_2^2 = \frac{1}{2}\left[1+(0.2)^2\right](1.12)^2 = 0.6523$$

$$y_3' = \frac{1}{2}(1+x_3^2)y_3^2 = \frac{1}{2}\left[1+(0.3)^2\right](1.21)^2 = 0.7979$$

Putting above values in (1)

$$y_4 = 1 + \frac{4(0.1)}{3}\left[2(0.5674) - 0.6523 + 2(0.7979)\right]$$

$$\boxed{y_4 = 1.2771}$$

Milne's Corrector Method:

$$y_4 = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right] \quad (2)$$

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$x_n = x_0 + nh$

$x_4 = x_0 + 4h$

$= 0 + 4(0.1)$

$x_4 = 0.4$

$y_4' = \frac{1}{2}(1 + x_4^2) y_4^2 = \frac{1}{2}[1 + (0.4)^2](1.2771)^2 = 0.9459$

Putting values in ② we have:

$y_4 = 1.12 + \frac{(0.1)}{3}[0.6523 + 4(0.7979) + 0.9459]$

$\boxed{y_4 = 1.2797}$

Q. Solve differential equation $5xy' + y^2 = 2$; $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$. Compute $y(4.4)$ using Milne's Predictor Corrector Method.

Sol: The given differential equation is:

$5xy' + y^2 = 2$

$y' = \frac{2 - y^2}{5x}$

$f(x,y) = \frac{2 - y^2}{5x}$, $h = 0.1$

$x_n = x_0 + nh$

$x_1 = x_0 + h$

$4.1 = 4 + h$

$h = 0.1$

$x_2 = x_0 + 2h$

$4.2 = 4.0 + 2h$

$h = 0.1$

	x_0	x_1	x_2	x_3
x	4	4.1	4.2	4.3
y	1	1.0049	1.0097	1.0143
	y_0	y_1	y_2	y_3

Milne's Predictor Method:

$y_4 = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$ ——①

Now,

$y_1' = \frac{2 - y_1^2}{5x_1} = \frac{2 - (1.0049)^2}{5(4.1)} = 0.0483$

$y_2' = \frac{2 - y_2^2}{5x_2} = \frac{2 - (1.0097)^2}{5(4.2)} = 0.0467$

$y_3' = \frac{2 - y_3^2}{5x_3} = \frac{2 - (1.0143)^2}{5(4.3)} = 0.0452$

Putting above values in 1

$$y_4 = 1 + \frac{4(0.1)}{3}\left[2(0.0483) - 0.0467 + 2(0.0452)\right]$$

$$\boxed{y_4 = 1.0187}$$

Milne's Corrector Method:

$$y_4 = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right] \quad (2)$$

$$y_4' = \frac{2 - y_4^2}{5x_4} = \frac{2 - (1.0187)^2}{5(4.4)} = 0.0437$$

$x_n = x_0 + nh$
$x_4 = x_0 + 4h$
$x_4 = 4 + 4(0.1)$
$x_4 = 4.4$

Putting values in 2 we have:

$$y_4 = 1.0097 + \frac{(0.1)}{3}\left[0.0467 + 4(0.0452) + 0.0437\right]$$

$$\boxed{y_4 = 1.0187}$$

Q: Solve $y' = xy + y^2$; $y(0) = 1$, $h = 0.1$
Find $y(0.4)$ by Milnes' Predictor Corrector Method:

$$y' = xy + y^2$$
$$f(x,y) = xy + y^2$$
$$y(0) = 1$$
$$x_0 = 0, \quad y_0 = 1, \quad h = 0.1$$

Euler's formula:

$$y_{n+1} = y_n + h f(x_n, y_n) \quad (1)$$

Put $n = 0$ in 1

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= y_0 + h(x_0 y_0 + y_0^2)$$
$$= 1 + (0.1)\left[0(1) + (1)^2\right]$$
$$y_1 = 1.1$$
$$y_1 = y(x_1) = y(0.1) = 1.1$$

Put $n=1$ in ①

$$y_2 = y_1 + hf(x_1, y_1)$$
$$= y_1 + h(x_1 y_1 + y_1^2)$$
$$= 1.1 + (0.1)\left[(0.1)(1.1) + (1.1)^2\right]$$
$$y_2 = 1.232$$
$$y_2 = y(x_2) = y(0.2) = 1.232$$

Put $n=2$ in ①

$$y_3 = y_2 + hf(x_2, y_2)$$
$$= y_2 + h(x_2 y_2 + y_2^2)$$
$$= 1.232 + (0.1)\left[(0.2)(1.232) + (1.232)^2\right]$$
$$y_3 = 1.4084$$
$$y_3 = y(x_3) = y(0.3) = 1.4084$$

	x_0	x_1	x_2	x_3
x	0	0.1	0.2	0.3
y	1	1.1	1.232	1.4084
	y_0	y_1	y_2	y_3

Milne's Predictor Method:

$$y_4 = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right] \quad ①$$

Now,

$$y_1' = x_1 y_1 + y_1^2 = (0.1)(1.1) + (1.1)^2 = 1.32$$

$$y_2' = x_2 y_2 + y_2^2 = (0.2)(1.232) + (1.232)^2 = 1.7642$$

$$y_3' = x_3 y_3 + y_3^2 = (0.3)(1.4084) + (1.4084)^2 = 2.4061$$

Putting above values in ①

$$y_4 = 1 + \frac{4(0.1)}{3}\left[2(1.32) - 1.7642 + 2(2.4061)\right]$$

$$\boxed{y_4 = 1.7584}$$

Milne's Corrector Method:

$$y_4 = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right] \quad \text{— (2)}$$

$$x_n = x_0 + nh$$

$$x_4 = x_0 + 4h$$

$$= 0 + 4(0.1)$$

$$x_4 = 0.4$$

$$y_4' = x_4 y_4 + y_4^2 = (0.4)(1.7584) + (1.7584)^2 = 3.7953$$

Putting values in (2) we have:

$$y_4 = 1.232 + \frac{(0.1)}{3}\left[1.7642 + 4(2.4061) + 3.7953\right]$$

$$\boxed{y_4 = 1.7381}$$

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