Runge - Kutta Methods: (R-K methods)

Consider initial value problem

$$\frac{dy}{dx} = f(x_0, y_0) \quad ; \quad y(x_0) = y_0$$

Runge-Kutta method of first order:

$$y_{n+1} = y_n + hf(x_n, y_n) \quad ; \quad n = 0, 1, 2, \ldots$$

This is also known as Euler method.

R-K method of order one is defined as.

$$k_1 = hf(x_n, y_n)$$

$$y_{n+1} = y_n + k_1 \quad ; \quad n = 0, 1, 2, \ldots$$

Runge-Kutta method of second order:

Runge - Kutta method of second order is:

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k = \frac{1}{2}(k_1 + k_2)$$

and

$$y_{n+1} = y_n + k \quad ; \quad n = 0, 1, 2, \ldots$$

This method is also known as Modified Euler's Method.

Runge-Kutta Method of third order:

Runge-Kutta method of third order is:

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf(x_n + h, y_n + 2k_2 - k_1)$$

$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

or

$$y_{n+1} = y_n + k \quad ; \quad n = 0, 1, 2, \ldots$$

Runge - Kutta method of forth order:

Runge - Kutta method of forth order is

$k_1 = hf(x_n, y_n)$

$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$

$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$

$k_4 = hf(x_n + h, y_n + k_3)$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

or

$y_{n+1} = y_n + k \quad ; \quad n = 0, 1, 2, \ldots$

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Q: Use R-K method of order four

Solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$, $y(0)=1$, $h=0.2$

at $x=0.2$ and $x=0.4$

Sol:

The differential equation is:

$$\frac{dy}{dx} = f(x,y) = \frac{y^2-x^2}{y^2+x^2}$$

$y(0)=1$

$x_0=0$, $y_0=1$, $h=0.2$

Formula of R-K method of order four is:

$$\left.\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf\left(x_n+\frac{h}{2}, y_n+\frac{k_1}{2}\right) \\ k_3 &= hf\left(x_n+\frac{h}{2}, y_n+\frac{k_2}{2}\right) \\ k_4 &= hf(x_n+h, y_n+k_3) \\ k &= \frac{1}{6}(k_1+2k_2+2k_3+k_4) \\ y_{n+1} &= y_n+k \end{aligned}\right\} \quad (A)$$

$k_1 = hf(x_n, y_n)$

$$k_1 = h\left(\frac{y_n^2-x_n^2}{y_n^2+x_n^2}\right)$$

Put $n=0$ in (A)

$$k_1 = h\left(\frac{y_0^2-x_0^2}{y_0^2+x_0^2}\right)$$

$$= (0.2)\left(\frac{1-0}{1+0}\right)$$

$k_1 = 0.2$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$$

$$= hf\left(0+\frac{0.2}{2}, 1+\frac{0.2}{2}\right)$$

$k_2 = hf(0.1, 1.1)$

$= (0.2)\left[\dfrac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2}\right]$

$k_2 = 0.197$

$k_3 = hf\left(x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_2}{2}\right)$

$= hf\left(0 + \dfrac{0.2}{2}, 1 + \dfrac{0.197}{2}\right)$

$= hf(0.1, 1.0985)$

$= (0.2)\left[\dfrac{(1.0985)^2 - (0.1)^2}{(1.0985)^2 + (0.1)^2}\right]$

$k_3 = 0.197$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= hf(0 + 0.2, 1 + 0.197)$

$= hf(0.2, 1.197)$

$= (0.2)\left[\dfrac{(1.197)^2 - (0.2)^2}{(1.197)^2 + (0.2)^2}\right]$

$k_4 = 0.1891$

$k = \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \dfrac{1}{6}\left[0.2 + 2(0.197) + 2(0.197) + 0.1891\right]$

$k = 0.1962$

$y_{n+1} = y_n + k$

$y_1 = y_0 + k$

$y_1 = 1 + 0.1962$

$y_1 = 1.1962$

$\boxed{y_1 = y(x_1) = y(0.2) = 1.1962}$

$x_n = x_0 + nh$

$x_1 = 0 + (1)(0.2)$

$x_1 = 0.2$

Put $n=1$ in (A)

$$k_1 = hf(x_1, y_1)$$
$$= hf(0.2, 1.1962)$$
$$= (0.2)\left[\frac{(1.1962)^2 - (0.2)^2}{(1.1962)^2 + (0.2)^2}\right]$$

$k_1 = 0.1891$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= hf\left(0.2 + \frac{0.2}{2}, 1.1962 + \frac{0.1891}{2}\right)$$

$$= hf(0.3, 1.2908)$$
$$= (0.2)\left[\frac{(1.2908)^2 - (0.3)^2}{(1.2908)^2 + (0.3)^2}\right]$$

$k_2 = 0.1795$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= hf\left(0.2 + \frac{0.2}{2}, 1.1962 + \frac{0.1795}{2}\right)$$

$$= hf(0.3, 1.2860)$$
$$= (0.2)\left[\frac{(1.2860)^2 - (0.3)^2}{(1.2860)^2 + (0.3)^2}\right]$$

$k_3 = 0.1794$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$
$$= hf(0.2 + 0.2, 1.1962 + 0.1794)$$
$$= hf(0.4, 1.3756)$$
$$= (0.2)\left[\frac{(1.3756)^2 - (0.4)^2}{(1.3756)^2 + (0.4)^2}\right]$$

$k_4 = 0.1688$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}\left[0.1891 + 2(0.1795) + 2(0.1794) + 0.1688\right]$$

$k = 0.1793$

$$y_{n+1} = y_n + k$$

$$y_2 = y_1 + k$$

$$y_2 = 1.1962 + 0.1793$$

$$y_2 = 1.3755$$

$$x_n = x_0 + nh$$

$$x_2 = x_0 + 2h$$

$$= 0 + 2(0.2)$$

$$x_2 = 0.4$$

$$\boxed{y_2 = y(x_2) = y(0.4) = 1.3755}$$