Type - IV :

If $f(x) = f_1(x) \pm f_2(x)$

then

$$P.I = \frac{f(x)}{\phi(E)}$$

$$= \frac{f_1(x) + f_2(x)}{\phi(E)}$$

$$= \frac{f_1(x)}{\phi(E)} + \frac{f_2(x)}{\phi(E)}$$

$$P.I = P.I_1 + P.I_2$$

Q: Solve the differential equation

(i) $(E^2 - 6E + 8)y_x = 2^x + 6x$

The differential equation is:

$$(E^2 - 6E + 8)y_x = 2^x + 6x$$

(1) For complementary function:

Consider,

$$(E^2 - 6E + 8)y_x = 0$$

The characteristic equation is:

$$m^2 - 6m + 8 = 0$$

$$m^2 - 4m - 2m + 8 = 0$$

$$m(m-4) - 2(m-4) = 0$$

$$(m-2)(m-4) = 0$$

$$m - 2 = 0 \quad ; \quad m - 4 = 0$$

$$m = 2 \quad , \quad m = 4$$

$$y_c = c_1 2^x + c_2 4^x$$

(2) For particular solution:

$$P.I = \frac{2^x + 6x}{E^2 - 6E + 8}$$

$$= \frac{2^x}{E^2 - 6E + 8} + \frac{6x}{E^2 - 6E + 8}$$

$$P.I = PI_1 + PI_2$$

Consider,

$$P.I_1 = \frac{2^x}{E^2 - 6E + 8}$$

$$= \frac{2^x}{(E-2)(E-4)} \quad (\text{failure case})$$

$$= \frac{2^x}{(2-4)(E-2)}$$

$$= \frac{-1}{2} \cdot \frac{2^x}{(E-2)} \qquad \frac{a^x}{(x-a)^r} = \frac{x(x-1)\cdots(x-(r-1))}{r!} a^{x-r}$$

$$P.I_1 = -\frac{1}{2} x\, 2^{x-1}$$

$$P.I_2 = \frac{6x}{E^2 - 6E + 8}$$

$$= \frac{6x}{(1+\Delta)^2 - 6(1+\Delta) + 8}$$

$$= \frac{6x}{1 + \Delta^2 + 2\Delta - 6 - 6\Delta + 8}$$

$$= \frac{6x}{\Delta^2 - 4\Delta + 3}$$

$$= \frac{6x}{3\left(\frac{\Delta^2}{3} - \frac{4\Delta}{3} + 1\right)}$$

$$= 2x\left[1 + \left(\frac{\Delta^2}{3} - \frac{1\Delta}{3}\right)\right]^{-1}$$

$$= 2x\left[1 + (-1)\left(\frac{\Delta^2}{3} - \frac{4\Delta}{3}\right) + \frac{(-1)(-1-1)}{2!}\left(\frac{\Delta^2}{3} - \frac{4\Delta}{3}\right)^2\right]$$

$$= 2x\left(1 - \frac{\Delta^2}{3} + \frac{4\Delta}{3} + \frac{16\Delta^2}{9}\right)$$

$$= 2x - \frac{2}{3}\Delta^2 x + \frac{8}{3}\Delta x + \frac{32}{9}\Delta^2 x$$

$$= 2x - \frac{2}{3}(0) + \frac{8}{3}(1) + \frac{32}{9}(0)$$

$$P.I_2 = 2x + \frac{8}{3}$$

$$P.I = -\frac{1}{2} x\, 2^{x-1} + 2x + \frac{8}{3}$$

$$\therefore P.I = -x\, 2^x + 2x + \frac{8}{2}$$

(3) Complete Solution:

$$y_x = y_c + y_p$$

$$y_x = c_1 2^x + c_2 4^x - x 2^x + 2x + \frac{8}{3}$$

(2) $(E^2 - 7E + 10) y_x = 2^x + 3^x + x$

The differential equation is:

$$(E^2 - 7E + 10) y_x = 2^x + 3^x + x$$

(1) For complementary function:

Consider,

$$(E^2 - 7E + 10) y_x = 0$$

The characteristic equation is:

$$m^2 - 7m + 10 = 0$$

$$m^2 - 5m - 2m + 10 = 0$$

$$m(m-5) - 2(m-5) = 0$$

$$(m-2)(m-5) = 0$$

$$m - 2 = 0 \quad , \quad m - 5 = 0$$

$$m = 2 \quad , \quad m = 5$$

$$y_c = c_1 2^x + c_2 5^x$$

(2) For particular solution:

$$P.I = \frac{2^x + 3^x + x}{E^2 - 7E + 10}$$

$$P.I = \frac{2^x}{E^2 - 7E + 10} + \frac{3^x}{E^2 - 7E + 10} + \frac{x}{E^2 - 7E + 10}$$

$$P.I = P.I_1 + P.I_2 + P.I_3$$

Consider,

$$P.I_1 = \frac{2^x}{E^2 - 7E + 10}$$

$$= \frac{2^x}{(E-2)(E-5)} \quad \text{(failure case)}$$

$$= \frac{2^x}{(E-2)(2-5)}$$

$$= -\frac{1}{3}\ \frac{2^x}{(E-2)}$$

$$P.I_1 = -\frac{1}{3}\ x\,2^{x-1}$$

$$P.I_2 = \frac{3^x}{E^2 - 7E + 10} \rightarrow \text{Type-1}$$

Replace E by 3

$$P.I_2 = \frac{3^x}{9 - 7(3) + 10}$$

$$P.I_2 = \frac{3^x}{-2}$$

$$P.I_3 = \frac{x}{E^2 - 7E + 10}$$

$$= \frac{x}{(1+\Delta)^2 - 7(1+\Delta) + 10} \quad \because E = 1+\Delta$$

$$= \frac{x}{1 + \Delta^2 + 2\Delta - 7 - 7\Delta + 10}$$

$$= \frac{x}{\Delta^2 - 5\Delta + 4}$$

$$= \frac{x}{4\left[1 + \left(\frac{\Delta^2}{4} - \frac{5\Delta}{4}\right)\right]}$$

$$= \frac{1}{4}\,x\left[1 + \left(\frac{\Delta^2}{4} - \frac{5\Delta}{4}\right)\right]^{-1}$$

$$= \frac{1}{4}\,x\left[1 + (-1)\left(\frac{\Delta^2}{4} - \frac{5\Delta}{4}\right) + \frac{(-1)(-1-1)}{2!}\left(\frac{\Delta^2}{4} - \frac{5\Delta}{4}\right)^2\right]$$

$$= \frac{1}{4}\,x\left(1 - \frac{\Delta^2}{4} + \frac{5\Delta}{4} + \frac{25}{16}\Delta^2\right)$$

$$= \frac{1}{4}\left(x - \frac{\Delta^2 x}{4} + \frac{5\Delta x}{4} + \frac{25}{16}\Delta^2 x\right)$$

$$= \frac{1}{4}\left(x - \frac{0}{4} + \frac{5}{4}(1) + \frac{25}{16}(0)\right)$$

$$P.I_3 = \frac{1}{4}\left(x + \frac{5}{4}\right)$$

Type - V

1

If $f(x) = \cos ax$ or $\sin ax$

then

$$P.I = \frac{f(x)}{\phi(E)}$$

$$P.I = \frac{(\cos ax \text{ or } \sin ax)}{\phi(E)}$$

$$\boxed{\begin{array}{ll} e^{ix} = \cos x + i\sin x \;; & e^{-ix} = \cos x - i\sin x \\ Re\,(e^{ix}) = \cos x \;; & Img\,(e^{ix}) = \sin x \end{array}}$$

- if $f(x) = \cos ax$, then

$$P.I = \frac{\cos ax}{\phi(E)} = Re\left(\frac{e^{iax}}{\phi(E)}\right)$$

- if $f(x) = \sin ax$, then

$$P.I = \frac{\sin ax}{\phi(E)} = Img\left(\frac{e^{iax}}{\phi(E)}\right)$$

Remark:

- $\Delta^{-1} a^x = \frac{a^x}{a-1}$, $a \neq 1$
- $\Delta^{-1} [x]^n = \frac{[x]^{n+1}}{n+1}$
- $\Delta^{-1} \sin ax = \frac{\cos a\left(x - \frac{1}{2}\right)}{2\sin\frac{a}{2}}$
- $\Delta^{-1} \cos ax = \frac{\sin a\left(x - \frac{1}{2}\right)}{2\sin\frac{a}{2}}$

Q: Solve difference equations:

(i) $y_{x+1} - y_x = \sin 2x$

The differential equation is:

$$y_{x+1} - y_x = \sin 2x$$

or

$$Ey_x - y_x = \sin 2x$$

$$(E-1)y_x = \sin 2x$$

(1) For complementary function:

Consider,

$$(E-1)y_x = 0$$

The characteristic equation is:

$$m-1=0$$

$$m=1$$

$$y_c = c_1(1)^x$$

$$y_c = c_1$$

(2) For particular solution:

$$P.I = \frac{\sin 2x}{E-1}$$

$$= \frac{\sin 2x}{\cancel{1}+\Delta-\cancel{1}} \qquad \because E = 1+\Delta$$

$$P.I = \frac{\sin 2x}{\Delta}$$

$$= \Delta^{-1}(\sin 2x)$$

$$= \frac{\cos 2\left(x-\frac{1}{2}\right)}{2\sin\left(\frac{2}{2}\right)} \qquad \because \Delta^{-1}\sin ax = \frac{\cos a\left(x-\frac{1}{2}\right)}{2\sin\frac{a}{2}}$$

$$y_p = P.I = \frac{\cos(2x-1)}{2\sin(1)}$$

3

(3) Complete Solution:

$$y_x = y_c + y_p$$
$$= c_1 + \frac{\cos(2x-1)}{2\sin(1)}$$

(ii) $\Delta y_x + \Delta^2 y_x = \cos x$

The difference equation is:

$\because E = 1+\Delta$

$$(E-1)y_x + (E-1)^2 y_x = \cos x$$
$$E y_x - \cancel{y_x} + E^2 y_x + \cancel{y_x} - 2E y_x = \cos x$$
$$E^2 y_x - E y_x = \cos x$$

$$(E^2 - E) y_x = \cos x$$

(1) For complementary function:

Consider,

$$(E^2 - E) y_x = 0$$

The characteristic equation is:

$$m^2 - m = 0$$
$$m(m-1) = 0$$
$$m = 0, \quad m - 1 = 0$$
$$m = 1$$

$$y_c = c_1(0)^x + c_2(1)^x$$
$$y_c = c_2$$

(2) For particular solution:

$$P.I = \frac{\cos x}{E^2 - E}$$

$\because Re(e^{ix}) = \cos x$

$$= Re\left(\frac{e^{ix}}{E^2 - E}\right)$$

$$= Re\left(\frac{(e^i)^x}{E^2 - E}\right)$$

$\because \frac{a^x}{\phi(E)} = \frac{a^x}{\phi(a)}$

$$= Re\left(\frac{(e^i)^x}{(e^i)^2 - (e^i)^1}\right)$$

$$= Re\left(\frac{e^{ix}}{e^{2i} - e^{i}}\right)$$

$$= Re\left(\frac{\cos x + i\sin x}{\cos 2 + i\sin 2 - \cos 1 - i\sin 1}\right)$$

$$= Re\left(\frac{\cos x + i\sin x}{(\cos 2 - \cos 1) + i(\sin 2 - \sin 1)}\right)$$

$$= Re\left(\frac{\cos x + i\sin x}{(\cos 2 - \cos 1) + i(\sin 2 - \sin 1)}\right) \times \frac{(\cos 2 - \cos 1) - i(\sin 2 - \sin 1)}{(\cos 2 - \cos 1) - i(\sin 2 - \sin 1)}$$

$$= Re\left(\frac{\cos x(\cos 2 - \cos 1) - i\cos x(\sin 2 - \sin 1) + i\sin x(\cos 2 - \cos 1) + \sin x(\sin 2 - \sin 1)}{(\cos 2 - \cos 1)^2 - i^2(\sin 2 - \sin 1)^2}\right)$$

$$Re\left(\frac{\cos x\cos 2 - \cos x\cos 1 - i\cos x\sin 2 + i\cos x\sin 1 + i\sin x\cos 2 - i\sin x\cos 1 + \sin x\sin 2 - \sin x\sin 1}{(\cos 2 - \cos 1)^2 + (\sin 2 - \sin 1)^2}\right)$$

$$= Re\left(\frac{\cos(x-2) - \cos(x-1) + i\sin(x-2) + i\sin(1-x)}{\cos^2 2 + \cos^2 1 - 2\cos 2\cos 1 + \sin^2 2 + \sin^2 1 - 2\sin 2\sin 1}\right)$$

$$= \left(\frac{\cos(x-2) - \cos(x-1)}{1 + 1 - 2(\cos 2\cos 1 + \sin 2\sin 1)}\right)$$

$$= \frac{\cos(x-2) - \cos(x-1)}{2 - 2\cos(2-1)}$$

$$y_p = P.I = \frac{\cos(x-2) - \cos(x-1)}{2(1 - \cos 1)}$$

(3) Complete Solution:

$$y_x = y_c + y_p$$

$$y_x = c_2 + \frac{\cos(x-2) - \cos(x-1)}{2(1 - \cos 1)}$$

(iii) $(E^2-1)y_x = \cos\frac{x}{2}$

The difference equation is:

$$(E^2-1)y_x = \cos\frac{x}{2}$$

(1) For complementary function:

Consider,

$$(E^2-1)y_x = 0$$

The characteristic equation is:

$$m^2-1=0$$

$$m^2=1$$

$$m=\pm 1$$

$$y_c = c_1(1)^x + c_2(-1)^x$$

(2) For particular solution:

$$P.I = \frac{\cos\frac{x}{2}}{E^2-1}$$

$$= Re\left(\frac{e^{\frac{ix}{2}}}{E^2-1}\right) \qquad \because Re(e^{ix}) = \cos x$$

$$= Re\left(\frac{(e^{i/2})^x}{E^2-1}\right)$$

$$= Re\left(\frac{(e^{i/2})^x}{(e^{i/2})^2-1}\right) \qquad \frac{a^x}{\phi(E)} = \frac{a^x}{\phi(a)}$$

$$= Re\left(\frac{\cos\frac{x}{2}+i\sin\frac{x}{2}}{e^i-1}\right)$$

$$= Re\left(\frac{\cos\frac{x}{2}+i\sin\frac{x}{2}}{\cos 1+i\sin 1-1}\right)$$

$$= Re\left(\frac{\cos\frac{x}{2} + i\sin\frac{x}{2}}{(\cos 1 - 1) + i\sin 1}\right)$$

$$= Re\left(\frac{\cos\frac{x}{2} + i\sin\frac{x}{2}}{(\cos 1 - 1) + i\sin 1} \times \frac{(\cos 1 - 1) - i\sin 1}{(\cos 1 - 1) - i\sin 1}\right)$$

$$Re\left(\frac{\cos\frac{x}{2}(\cos 1 - 1) - i\cos\frac{x}{2}\sin 1 + i\sin\frac{x}{2}(\cos 1 - 1) - i^2\sin\frac{x}{2}\sin 1}{(\cos 1 - 1)^2 - i^2\sin^2 1}\right)$$

$$Re\left(\frac{\cos\frac{x}{2}\cos 1 - \cos\frac{x}{2} - i\cos\frac{x}{2}\sin 1 + i\sin\frac{x}{2}\cos 1 - i\sin\frac{x}{2} + \sin\frac{x}{2}\sin 1}{\cos^2 1 + 1 - 2\cos 1 + \sin^2 1}\right)$$

$$Re\left(\frac{\cos\left(\frac{x}{2} - 1\right) - \cos\frac{x}{2} + i\sin\left(\frac{x}{2} - 1\right) - i\sin\frac{x}{2}}{1 + 1 - 2\cos 1}\right)$$

$$y_p = \frac{\cos\left(\frac{x}{2} - 1\right) - \cos\frac{x}{2}}{2(1 - \cos 1)}$$

(3) Complete Function:

$$y_x = y_c + y_p$$

$$y_x = c_1(1)^x + c_2(-1)^x + \frac{\cos\left(\frac{x}{2} - 1\right) - \cos\frac{x}{2}}{2(1 - \cos 1)}$$

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Numerical Solution of Ordinary Differential Equations. (24)

Consider the first order differential equation

$$\frac{dy}{dx} = f(x,y)$$

with initial condition $y(x_0) = y_0$

This is called initial value problem There are following numerical methods to obtain the numerical solution of the initial value problem.

(1) Euler's Method.
(2) Modified Euler's Method.
(3) Improved Euler's Method.
(4) Taylor Series Method.
(5) Runge-Kutta Methods (R-K methods)

Euler's Method:

Consider an initial value problem

$$\frac{dy}{dx} = f(x,y) \quad ; \quad y(x_0) = y_0$$

Euler's formula:

$$y_{n+1} = y_n + h f(x_n, y_n) \quad ; \quad n = 0, 1, 2, 3, \ldots$$

This is called Euler algorithm

$$x_n = x_0 + nh$$

$$y_n = y(x_n)$$

Q. Use Euler's Method to find $y(0.4)$ given that $\frac{dy}{dx} = \frac{y-x}{y+x}$; $y(0)=1$; $h=0.1$

Sol: The given ordinary differential equation is :

$$\frac{dy}{dx} = \frac{y-x}{y+x} = f(x,y)$$

$y(0) = 1$

i.e

$x_0 = 0$, $y_0 = 1$, $h = 0.1$

Euler's formula:

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$y_{n+1} = y_n + h\left(\frac{y_n - x_n}{y_n + x_n}\right) \quad \text{--- (1)}$$

Putting $n = 0, 1, 2, 3$ in (1)

n=0

$$y_1 = y_0 + h\left(\frac{y_0 - x_0}{y_0 + x_0}\right)$$

$$y_1 = 1 + (0.1)\left(\frac{1-0}{1+0}\right)$$

$x_n = x_0 + nh$
$x_1 = 0 + 0.1$

$y_1 = 1.1$

Thus, $y_1 = y(x_1) = y(0.1) = 1.1$

$\because y_0 = y(x_0)$

n=1

$$y_2 = y_1 + h\left(\frac{y_1 - x_1}{y_1 + x_1}\right)$$

$$= 1.1 + (0.1)\left(\frac{1.1 - 0.1}{1.1 + 0.1}\right)$$

$y_2 = 1.18$

Thus,

$y_2 = y(x_2) = y(0.2) = 1.18$

n=2

$$y_3 = y_2 + h\left(\frac{y_2 - x_2}{y_2 + x_2}\right)$$

$$y_3 = 1.18 + (0.1)\left(\frac{1.18 - 0.2}{1.18 + 0.2}\right)$$

$$y_3 = 1.25$$

Thus,

$$y_3 = y(x_3) = y(0.3) = 1.25$$

Put $n = 3$

$$y_4 = y_3 + h\left(\frac{y_3 - x_3}{y_3 + x_3}\right)$$

$$= 1.25 + (0.1)\left(\frac{1.25 - 0.3}{1.25 + 0.3}\right)$$

$$y_4 = 1.311$$

Thus,

$$y_4 = y(x_4) = y(0.4) = 1.311$$

Q: Solve $\frac{dy}{dx} = 1 - y$; $y(0) = 0$; $h = 0.1$

find $y(0.1)$

The ordinary differential equation is,

$$\frac{dy}{dx} = f(x, y) = 1 - y$$

$$y(0) = 0$$

i.e

$$y_0 = 0 \quad , \quad x_0 = 0 \quad ; \quad h = 0.1$$

Euler's formula:

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$y_{n+1} = y_n + h(1 - y_n) \quad \text{—} (1)$$

Put $n = 0$ in (1)

$$y_1 = y_0 + h(1 - y_0)$$

$$= 0 + (0.1)(1 - 0)$$

$$y_1 = 0.1$$

Thus,

$$y_1 = y(x_1) = y(0.1) = 0.1$$

Modified Euler's Method:

Consider ordinary differential equation:

$$\frac{dy}{dx} = f(x, y) \quad , \quad y(x_0) = y_0$$

Modified Euler's formula/algorithm:

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2} \, , \, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

where,

$$x_n = x_0 + nh \quad \text{and} \quad y_n = y(x_n)$$

Q: Compute $y(0.1)$ when $\frac{dy}{dx} = \frac{y-x}{y+x}$; $y(0) = 1$

$h = 0.05$ using Modified Euler's Method?

Solution:

$$y(0) = 1 \quad ; \quad h = 0.05 \; ; \quad \frac{dy}{dx} = f(x, y) = \frac{y-x}{y+x}$$

i.e

$$y_0 = 1 \, , \quad x_0 = 0 \; ; \quad h = 0.05$$

Modified Euler's formula:

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2} \, , \, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2} \, , \, y_n + \frac{h}{2}\left(\frac{y_n - x_n}{y_n + x_n}\right)\right) \quad \text{--- (1)}$$

Put $n = 0$ in (1)

$$y_1 = y_0 + hf\left(x_0 + \frac{h}{2} \, , \, y_0 + \frac{h}{2}\left(\frac{y_0 - x_0}{y_0 + x_0}\right)\right)$$

$$= 1 + (0.05) f\left(0 + \frac{0.05}{2} \, , \, 1 + \frac{0.05}{2}\left(\frac{1-0}{1+0}\right)\right)$$

$$= 1 + (0.05) f\left(0.025 \, , \, 1.025\right)$$

$$= 1 + (0.05)\left(\frac{1.025 - 0.025}{1.025 + 0.025}\right)$$

$y_1 = 1.0476$

Thus,

$x_n = x_0 + nh$
$x_1 = x_0 + (1)h$
$x_1 = 0 + 1(0.05) = 0.05$

$$y_1 = y(x_1) = y(0.05) = 1.0476$$

Put $n=1$ in ①

$$y_2 = y_1 + hf\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}\left(\frac{y_1 - x_1}{y_1 + x_1}\right)\right)$$

$$= 1.0476 + (0.05)f\left(0.05 + \frac{0.05}{2}, 1.0476 + \frac{0.05}{2}\left(\frac{1.0476 - 0.05}{1.0476 + 0.05}\right)\right)$$

$$= 1.0476 + (0.05)f(0.075, 1.0703)$$

$$= 1.0476 + (0.05)\left(\frac{1.0703 - 0.075}{1.0703 + 0.075}\right)$$

$y_2 = 1.091$

Thus,

$x_n = x_0 + nh$
$x_2 = x_0 + (2)h$
$= 0 + 2(0.05)$
$x_2 = 0.1$

$$y_2 = y(x_2) = y(0.1) = 1.091$$

Hence,

$$y(0.1) = 1.091.$$

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