Formations of Difference Equations.

A difference equation is formed by eliminating arbitrary constants between the relation of variables.

Q: Form the difference equation corresponding to the family of curves $y = ax + bx^2$.

Sol: The given curve is:

$$y = ax + bx^2$$

$$y_x = ax + bx^2$$

$$y_{x+1} = a(x+1) + b(x+1)^2$$

$$y_{x+2} = a(x+2) + b(x+2)^2$$

which is system of linear equation in a and b.

To eliminate the constant a and b we have:

$$\begin{vmatrix} y_x & x & x^2 \\ y_{x+1} & x+1 & (x+1)^2 \\ y_{x+2} & x+2 & (x+2)^2 \end{vmatrix} = 0$$

$$y_x\left[(x+1)(x+2)^2 - (x+2)(x+1)^2\right] - x\left[y_{x+1}(x+2)^2 - (x+1)^2 y_{x+2}\right] + x^2\left[(x+2)y_{x+1} - (x+1)y_{x+2}\right] = 0$$

$$(x^2+3x+2)y_x - x(x^2+4+4x)y_{x+1} + x(x^2+1+2x)y_{x+2} + (x^3+2x^2)y_{x+1} - (x^3+x^2)y_{x+2} = 0$$

$$(x^2+x)y_{x+2} - 2x(x+2)y_{x+1} + (x^2+3x+2)y_x$$

(i)

Q: Form the difference equation by eliminating a and b $y_x = ax + b2^x$

Sol: Given that:

$$y_x = ax + b2^x$$

$$y_{x+1} = a(x+1) + b2^{x+1}$$

$$y_{x+2} = a(x+2) + b2^{x+2}$$

which is the system of linear equation in a and b.

To eliminate the constant a and b we have:

$$\begin{vmatrix} y_x & x & 2^x \\ y_{x+1} & x+1 & 2^{x+1} \\ y_{x+2} & x+2 & 2^{x+2} \end{vmatrix} = 0$$

$$\begin{vmatrix} y_x & x & 2^x \\ y_{x+1} & x+1 & 2^x 2 \\ y_{x+2} & x+2 & 2^x \cdot 2^2 \end{vmatrix} = 0$$

$$2^x \begin{vmatrix} y_x & x & 1 \\ y_{x+1} & x+1 & 2 \\ y_{x+2} & x+2 & 4 \end{vmatrix} = 0$$

Since, $2^x \neq 0$

Therefore,

$$\begin{vmatrix} y_x & x & 1 \\ y_{x+1} & x+1 & 2 \\ y_{x+2} & x+2 & 4 \end{vmatrix} = 0$$

$$y_x\left[4(x+1) - 2(x+2)\right] - x\left[4y_{x+1} - 2y_{x+2}\right] + 1\left[(x+2)y_{x+1} - (x+1)y_{x+2}\right] = 0$$

$$2x\,y_x - 4x\,y_{x+1} + 2x\,y_{x+2} + (x+2)y_{x+1} - (x+1)y_{x+2} = 0$$

$$(x-1)y_{x+2} + (2-3x)y_{x+1} + 2x\,y_x = 0$$

(ii) $y_x = (A+Bx)2^x$

Given that:

$$y_x = (A+Bx)2^x$$
$$y_{x+1} = [A + B(x+1)]\,2^{x+1}$$
$$y_{x+2} = [A + B(x+2)]\,2^{x+2}$$

which is the system of linear equation in A and B.

To eliminate the constant A and B we have:

$$\begin{vmatrix} y_x & 2^x & x.2^x \\ y_{x+1} & 2^{x+1} & 2^{x+1}(x+1) \\ y_{x+2} & 2^{x+2} & 2^{x+2}(x+2) \end{vmatrix} = 0$$

$$2^x \cdot 2^x \begin{vmatrix} y_x & 1 & x \\ y_{x+1} & 2 & 2(x+1) \\ y_{x+2} & 4 & 4(x+2) \end{vmatrix} = 0$$

Since,

$$2^{2x} \neq 0$$

Therefore,

$$\begin{vmatrix} y_x & 1 & x \\ y_{x+1} & 2 & 2x+2 \\ y_{x+2} & 4 & 4x+8 \end{vmatrix} = 0$$

$$y_x\left[2(4x+8) - 4(2x+2)\right] - 1\left[(4x+8)y_{x+1} - (2x+2)y_{x+2}\right] + x\left[4y_{x+1} - 2y_{x+2}\right] = 0$$

$$8y_x - (4x+8)y_{x+1} + (2x+2)y_{x+2} + 4x\,y_{x+1} - 2x\,y_{x+2} = 0$$

$$2y_{x+2} - 8y_{x+1} + 8y_x = 0$$

$$y_{x+2} - 4y_{x+1} + 4y_x = 0$$

(iii) Find the difference equation generated by $y = \frac{a}{x} + b$

Given that:

$$y = \frac{a}{x} + b$$

$$y_x = \frac{a}{x} + b$$

$$y_{x+1} = \frac{a}{x+1} + b$$

$$y_{x+2} = \frac{a}{x+2} + b$$

which is the system of linear equation in a and b.

To eliminate the constant a and b we have:

$$\begin{vmatrix} y_x & \frac{1}{x} & 1 \\ y_{x+1} & \frac{1}{x+1} & 1 \\ y_{x+2} & \frac{1}{x+2} & 1 \end{vmatrix} = 0$$

$$y_x\left[\frac{1}{x+1} - \frac{1}{x+2}\right] - \frac{1}{x}\left[y_{x+1} - y_{x+2}\right] + 1\left[\frac{1}{x+2}y_{x+1} - \frac{1}{x+1}y_{x+2}\right] = 0$$

$$\left[\frac{1}{(x+1)(x+2)}\right]y_x - \frac{1}{x}y_{x+1} + \frac{1}{x}y_{x+2} + \frac{1}{x+2}y_{x+1} - \frac{1}{x+1}y_{x+2} = 0$$

$$\left[\frac{1}{x} - \frac{1}{x+1}\right]y_{x+2} + \left[\frac{1}{x+2} - \frac{1}{x}\right]y_{x+1} + \left[\frac{1}{(x+1)(x+2)}\right]y_x = 0$$

$$\frac{1}{x(x+1)}y_{x+2} - \frac{2}{x(x+2)}y_{x+1} + \frac{1}{(x+1)(x+2)}y_x = 0$$

$$\frac{(x+2)y_{x+2} - 2(x+1)y_{x+1} + xy_x}{x(x+1)(x+2)} = 0$$

$$(x+2)y_{x+2} - 2(x+1)y_{x+1} + xy_x = 0$$

Q: Form the difference equation from the Fibonacci number

OR

Derive the difference equation from the Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, ...)

OR

Develop the difference equation

$$y_0 = 1$$

$$y_1 = 1$$

$$y_x = y_{x-1} + y_{x-2} \quad ; \quad x = 2, 3, \ldots$$

Solution:

The Fibonacci sequence / numbers are

1, 1, 2, 3, 5, 8, 13, ...

we observe that any term is obtained by the sum of proceeding two terms with starting $y_0 = 1$, $y_1 = 1$.

Therefore,

$$y_2 = y_0 + y_1$$

$$y_3 = y_1 + y_2$$

$$y_4 = y_2 + y_3$$

$$\vdots$$

$$y_x = y_{x-1} + y_{x-2} \quad ; \quad x = 2, 3, 4, \ldots$$

Hence, the difference equation is:

$$y_x = y_{x-1} + y_{x-2} \quad , \quad y_0 = 1 \quad , \quad y_1 = 1$$

9th Feb, 2015 Monday

{1 Question from this chapter.} New Chapter

Solution of Difference Equation

Difference Equations:

The difference equations may be consider as discrete analouge of the differential equations.

Discrete functions occur in many physical and engineering problems.

So, difference equations are natural choice for dealing with discrete situations. While differential equation deals with continuous functions.

Definition:

A difference equation is an equation involving an independent variable x, depending variable y_x and successive differences of y_x.

Examples:

(i) $y_{x+2} - 5y_{x+1} + 6y_x = 3^x$

(ii) $E^2 y_x - 5Ey_x + 6y_x = \cos x$

(iii) $\Delta^2 y_x + 5\Delta y_x - y_x = \cos x$

(iv) $\Delta^2 y_x + 5\Delta y_x = x^2$

Note:

$$\Delta y_x = y_{x+1} - y_x$$

$$\Delta^2 y_x = y_{x+2} - 2y_{x+1} + y_x$$

$$E^2 y_x = y_{x+2}$$

Order of Difference Equations:

$$y_{x+2} - 5y_{x+1} + 6y_x = 5$$

Order is $x+2-x = 2$

Degree of Difference Equations:

1) $(y_{x+2})^1 - 5y_{x+1} + 6y_x = 5$

The degree is 1 power of highest derivative

2) $(y_{x+2})^2 + y_{x+1} = 5$

The degree is 2

Linear Homogeneous Difference Equations with constant co-efficients:

The general form of linear homogeneous difference equations with constant coefficients is

$$a_1 y_{x+k} + a_2 y_{x+k-1} + a_3 y_{x+k-2} + \dots + a_{k-1} y_{x+1} + a_k y_x = 0$$

The differential equation is:

$$(a_1 m^{x+k} + a_2 m^{x+k-1} + a_3 m^{x+k-2} + \dots + a_k) y_x = 0$$

The characteristic equation is:

$$a_1 m^{x+k} + a_2 m^{x+k-1} + \dots + a_k = 0$$

In differential equations:

In roots are real and distinct. If

1) If roots are real and distint

m_1, m_2, m_3 are real and distint

then

$$y_x = c_1 m_1^x + c_2 m_2^x + c_3 m_3^x$$

2) If roots are real and repeated.

If m_1, m_2, m_3 are real and repeated

then

$$y_x = (c_1 + c_2 x) m_1^x + c_3 m_2^x + c_4 m_3^x$$

3) If roots are complex & non-repeated

let $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$.

then

$$y_x = (c_1 \cos\theta x + c_2 \sin\theta x) r^x$$

where,

$$r = \sqrt{\alpha^2 + \beta^2}$$

$$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

4) If roots are complex and repeated

If m_1, m_2, m_3 are complex and repeated

then

$$y_x = (c_1 + c_2 x) m_1^x + c_3 m_2^x + c_4 m_3^x$$

5) If roots are complex. If $m_1, m_2, \ldots$ are roots

Then

$$y_x = c_1 \cos m_1 x + c_2 \sin m_2 x$$

Q. Obtain the solution of difference equations.

(i) $y_{x+1} - 5y_x = 0$

The differential equation is :

$$y_{x+1} - 5y_x = 0$$

or

$$E'y_x - 5y_x = 0$$

$$(E-5)\,y_x = 0$$

The characteristics equ. is :

$$m - 5 = 0$$

$$m = 5$$

The general solution is:

$$y_x = c_1 5^x$$

(ii) $3U_{x+1} + 2U_x = 0$

The differential equation is :

$$3U_{x+1} + 2U_x = 0$$

or

$$E' 3U_x + 2Ux = 0$$

$$(3E + 2)\,Ux = 0$$

The characteristic equ. is :

$$3m + 2 = 0$$

$$m = -\frac{2}{3}$$

Hence, the general solution is :

$$y_x = c_1 \left(\frac{-2}{3}\right)^x$$

(iii) $y_{x+2} - 6y_{x+1} + 8y_x = 0$

The differential equation is:

$$y_{x+2} - 6y_{x+1} + 8y_x = 0$$

or

$$E^2 y_x - 6E^1 y_x + 8y_x = 0$$

$$(E^2 - 6E + 8) y_x = 0$$

The characteristic equ. is:

$$m^2 - 6m + 8 = 0$$

$$m^2 - 4m - 2m + 8 = 0$$

$$m(m-4) - 2(m-4) = 0$$

$$(m-2)(m-4) = 0$$

$$m - 2 = 0 \quad , \quad m - 4 = 0$$

$$m = 2 \quad , \quad m = 4$$

Hence, The general solution is:

$$y_x = c_1 2^x + c_2 4^x$$

(iv) $9y_{x+2} - 6y_{x+1} + y_x = 0$

The differential equation is:

$$9y_{x+2} - 6y_{x+1} + y_x = 0$$

or

$$9E^2 y_x - 6E^1 y_x + y_x = 0$$

$$(9E^2 - 6E + 1) y_x = 0$$

The characteristic equ. is:

$$9m^2 - 6m + 1 = 0$$

$$9m^2 - 3m - 3m + 1 = 0$$

$$3m(3m-1) - 1(3m-1) = 0$$

$$(3m-1)(3m-1) = 0$$

$$3m - 1 = 0 \quad , \quad 3m - 1 = 0$$

$$m = \frac{1}{3} \quad , \quad m = \frac{1}{3}$$

Hence, the general solution is:

$$y_x = (C_1 + C_2 x)\left(\frac{1}{3}\right)^x$$

(v) $y_{x+2} - 4y_{x+1} + 13y_x = 0$

The differential equation is:

$$y_{x+2} - 4y_{x+1} + 13y_x = 0$$

or

$$E^2 y_x - 4E y_x + 13y_x = 0$$

$$(E^2 - 4E + 13)y_x = 0$$

The characteristic equation is:

$$m^2 - 4m + 13 = 0$$

$$m = \frac{4 \pm \sqrt{16 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm 6i}{2}$$

$$m = 2 \pm 3i$$

$$m = 2 + 3i, \quad m = 2 - 3i$$

Hence, the general solution is:

$$y_x = (c_1 \cos\theta x + c_2 \sin\theta x) r^x$$

where,

$$r = \sqrt{\alpha^2 + \beta^2} = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$$

$$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \tan^{-1}\left(\frac{3}{2}\right) = 56.3$$

Then

$$y_x = [c_1 \cos(56.3)x + c_2 \sin(56.3)x](\sqrt{13})^x$$

$$= [c_1(0.55)x + c_2(0.83)x](\sqrt{13})^x$$

Q: Solve:

(i) $2y_{x+2} - 5y_{x+1} + 2y_x = 0$

with condition $y_0 = 0$

$y_1 = 1$

The differential equation is:

$2y_{x+2} - 5y_{x+1} + 2y_x = 0$

or

$2E^2 y_x - 5E^1 y_x + 2y_x = 0$

$(2E^2 - 5E + 2)y_x = 0$

The characteristic equ. is:

$2m^2 - 5m + 2 = 0$

$2m^2 - 4m - m + 2 = 0$

$2m(m-2) - 1(m-2) = 0$

$(2m-1)(m-2) = 0$

$2m - 1 = 0$, $m - 2 = 0$

$m = \frac{1}{2}$, $m = 2$

The general solution is:

$y_x = c_1\left(\frac{1}{2}\right)^x + c_2(2)^x$ — (A)

By applying giving conditions:

$y_0 = 0$ $\qquad$ $y_1 = 1$

$y_0 = c_1\left(\frac{1}{2}\right)^0 + c_2(2)^0$ $\qquad$ $y_1 = c_1\left(\frac{1}{2}\right)^1 + c_2(2)^1$

$0 = c_1 + c_2$ $\qquad$ $1 = c_1 \frac{1}{2} + c_2 2$

$c_1 = -c_2$ — (1) $\qquad$ or

$\frac{1}{2}c_1 + 2c_2 = 1$ — (2)

Put $c_1 = -c_2$ in (2)

$\frac{1}{2}(-c_2) + 2c_2 = 1$

$\frac{3}{2}c_2 = 1$

$c_2 = \frac{2}{3}$

Put c_2 in ①

$$c_1 = -\frac{2}{3}$$

Equ. ① becomes.

$$y_c = -\frac{2}{3}\left(\frac{1}{2}\right)^x + \frac{2}{3}(2)^x$$

or

$$y_c = \frac{2}{3}\cdot 2^x - \frac{2}{3}\left(\frac{1}{2}\right)^x$$

(ii) $y_{x+3} - 3y_{x+1} + 2y_x = 0$

with conditions

$$y_1 = 0 \quad , y_2 = 8 \quad , y_3 = -2$$

The differential equation is:

$$y_{x+3} - 3y_{x+1} + 2y_x = 0$$

or

$$E^3 y_x - 3E^1 y_x + 2y_x = 0$$

$$(E^3 - 3E^1 + 2)y_x = 0$$

The characteristic equ. is:

$$m^3 - 3m + 2 = 0$$

By using Synthetic division:

1	1	0	-3	2
		1	1	-2
	1	1	2	0

$2 = 1, 2 \rightarrow$ divisors of 2

and 1 satisfy the equ. So, 1 is the root of given equ.

$$m^2 + m - 2 = 0 \quad , m = 1$$

$$m^2 + 2m - m - 2 = 0$$

$$m(m+2) - 1(m+2) = 0$$

$$(m-1)(m+2) = 0$$

$$m - 1 = 0 \ ; \ m + 2 = 0 \quad , m = 1$$

$$m = 1, \quad m = -2 \quad , m = 1$$

The general solution is:

$$y_x = (c_1 + c_2 x)\,1^x + c_3(-2)^x \quad \text{— (A)}$$

By applying the given conditions.

$\Rightarrow y_1 = 0$

$$y_1 = [c_1 + c_2(1)](1)^1 + c_3(-2)^1$$

$$0 = c_1 + c_2 - 2c_3$$

or

$$c_1 + c_2 - 2c_3 = 0 \quad \text{— (1)}$$

$\Rightarrow y_2 = 8$

$$y_2 = (c_1 + c_2(2))(1)^2 + c_3(-2)^2$$

$$8 = c_1 + 2c_2 + 4c_3$$

$$c_1 + 2c_2 + 4c_3 = 8 \quad \text{— (2)}$$

$\Rightarrow y_3 = -2$

$$y_3 = [c_1 + c_2(3)](1)^3 + c_3(-2)^3$$

$$-2 = c_1 + 3c_2 - 8c_3$$

$$c_1 + 3c_2 - 8c_3 = -2 \quad \text{— (3)}$$

from (1)

$$c_1 = 2c_3 - c_2$$

Put c_1 in (2) & (3)

$2c_3 - c_2 + 2c_2 + 4c_3 = 8$; $2c_3 - c_2 + 3c_2 - 8c_3 = -2$

$c_2 + 6c_3 = 8$ — (4) ; $2c_2 - 6c_3 = -2$ — (5)

Adding equ. (4) & (5)

$$c_2 + \cancel{6c_3} = 8$$

$$2c_2 - \cancel{6c_3} = -2$$

$$3c_2 = 6$$

$$\boxed{c_2 = 2}$$

Put c_2 in equ. (1) & (2)

$c_1 + 2 - 2c_3 = 0$; $c_1 + 2(2) + 4c_3 = 8$

$c_1 - 2c_3 = -2$ — (6) ; $c_1 + 4c_3 = 4$ — (7)

Subtracting (6) from (7)

$$\cancel{c_1} + 4c_3 = 4$$
$$\underline{\pm \cancel{c_1} \mp 2c_3 = \mp 2}$$
$$\cancel{6}c_3 = \cancel{6}$$

$$\boxed{c_3 = 1}$$

Put c_2 & c_3 in (1)

$$c_1 + 2 - 2(1) = 0$$
$$c_1 + 2 - 2 = 0$$
$$\boxed{c_1 = 0}$$

Putting values of c_1, c_2, c_3 in (A)

$$y_c = (0 + 2x)(1)^x + (1)(-2)^x$$

$$y_c = 2x(1)^x - 2^x$$

* ——— *

Non-homogeneous linear differential equations with constant co-efficients:-

Consider,

$$\phi(E)\, y_x = F(x)$$

Its solution consist of three steps:

Step-1

Complementary function (C.F):

Consider,

$$\phi(E)\, y_x = 0$$

Step-2

Particular function/integral: (P.I)

$$P.I = \frac{F(x)}{\phi(E)}$$

Step-3

Complete Solution:

$$y_x = C.F + P.I$$

Types Of Non-homogeneous Differential Equation :-

Type-I :

When $F(x) = a^x$

Q. Solve:

Q1) $y_{x+1} - 5y_x = 2^x$

The differential equation is:

$$y_{x+1} - 5y_x = 2^x$$

or

$$E^1 y_x - 5y_x = 2^x$$

$$(E-5)y_x = 2^x$$

1) For complementary function:

Consider,

$$(E-5)y_x = 0$$

The characteristic equ. is:

$$m - 5 = 0$$

$$m = 5$$

Thus, $y_c = c_1 5^x$

2) For particular solution:

$$y_p = \frac{2^x}{E-5}$$

$$= \frac{2^x}{2-5} \quad \text{(Replace E by 2)}$$

$$y_p = -\frac{2^x}{3}$$

3) Complete solution:

$$y_x = C.F + P.I$$

$$= c_1 5^x - \frac{2^x}{3}$$

(2) $y_{x+2} - 2y_{x+1} + y_x = 2^x$, $y_0 = 2$, $y_1 = 1$

The differential equation is,

$$y_{x+2} - 2y_{x+1} + y_x = 2^x$$

or

$$E^2 y_x - 2E^1 y_x + y_x = 2^x$$

$$(E^2 - 2E + 1) y_x = 2^x$$

1) For complementary function:

Consider,

$$(E^2 - 2E + 1) y_x = 0$$

the characteristic equ. is:

$$m^2 - 2m + 1 = 0$$

$$m^2 - m - m + 1 = 0$$

$$m(m-1) - 1(m-1) = 0$$

$$(m-1)(m-1) = 0$$

$$m-1 = 0 \quad ; \quad m-1 = 0$$

$$m = 1 \quad ; \quad m = 1$$

Thus,

$$y_c = (C_1 + C_2 x)(1)^x$$

2) For particular solution:

$$y_p = \frac{2^x}{E^2 - 2E + 1}$$

Replace E by '2'

$$y_p = \frac{2^x}{(2)^2 - 2(2) + 1}$$

$$= \frac{2^2}{\cancel{4} - \cancel{4} + 1}$$

$$y_p = 2^x$$

3) Complete Solution.

$y_x = y_c + y_p$

$y_x = (c_1 + c_2 x) 1^x + 2^x$ ——— (1)

By applying the given conditions:

$y_0 = 2$; $\Rightarrow y_1 = 1$

$y_0 = [c_1 + c_2(0)](1)^0 + 2^0$; $y_1 = [c_1 + c_2(1)](1)^1 + 2^1$

$2 = c_1 + 1$

$\boxed{c_1 = 1}$

$1 = c_1 + c_2 + 2$

$c_1 + c_2 = -1$

$1 + c_2 = -1$

$c_2 = -1 - 1$

$\boxed{c_2 = -2}$

Put c_1 & c_2 in (1)

$y_x = [1 + (-2)x](1)^x + 2^x$

$y_x = (1 - 2x)(1)^x + 2^x$

* ——— *

Type - II :

12 Feb, 2015

Thursday

Failure Case:

$$P.I = \frac{a^x}{(x-a)^r} = \frac{x(x-1)(x-2)\cdots(x-(r-1))}{r!} \cdot a^{x-r}$$

Q: Solve the Difference Equations:

(i) $y_{x+2} - 8y_{x+1} + 16y_x = 4^x$

The differential equation is:

$$y_{x+2} - 8y_{x+1} + 16y_x = 4^x$$

or

$$E^2 y_x - 8E y_x + 16 y_x = 4^x$$

$$(E^2 - 8E + 16) y_x = 4^x$$

1) For complementary function:

Consider,

$$(E^2 - 8E + 16) y_x = 0$$

The characteristic equation is:

$$m^2 - 8m + 16 = 0$$

$$m^2 - 4m - 4m + 16 = 0$$

$$m(m-4) - 4(m-4) = 0$$

$$(m-4)(m-4) = 0$$

$$m-4 = 0 \quad ; \quad m-4 = 0$$

$$m = 4 \quad , \quad m = 4$$

Thus,

$$y_c = (c_1 + c_2 x) 4^x$$

2) For particular solution: $\frac{a^x}{(x-a)^r}$

$$y_p = \frac{4^x}{E^2 - 8E + 16} = \frac{4^x}{(E-4)^2}$$

$4^2 - 8(4) + 16 = 16 - 32 + 16 = 0$

(failure case)

$$y_p = \frac{x(x-1)\,4^{x-2}}{2!} \qquad \frac{x(x-1)}{r!} \cdot a^{x-r}$$

3) Complete solution:

$$y_x = y_c + y_p$$

$$= (c_1 + c_2 x)4^x + \frac{x(x-1) \cdot 4^{x-2}}{2!}$$

(ii) $U_{x+2} - 2U_{x+1} + U_x = 7$

The differential equation is:

$$U_{x+2} - 2U_{x+1} + U_x = 7$$

or

$$E^2 U_x - 2EU_x + U_x = 7$$

$$(E^2 - 2E + 1)U_x = 7$$

1) For complementary function:

Consider,

$$(E^2 - 2E + 1)U_x = 0$$

The characteristic equation is:

$$m^2 - 2m + 1$$

$$m^2 - m - m + 1 = 0$$

$$m(m-1) - 1(m-1) = 0$$

$$(m-1)(m-1) = 0$$

$$m - 1 = 0 \quad ; \quad m - 1 = 0$$

$$m = 1 \quad ; \quad m = 1$$

Thus,

$$y_c = (c_1 + c_2 x)(1)^x$$

2) For particular solution:

$$y_p = \frac{7 \cdot 1^x}{E^2 - 2E + 1}$$

$$= 7 \cdot \frac{1^x}{(E-1)^2}$$

(failure case)

$$y_p = 7 \cdot \frac{x(x-1)}{2!} 1^{x-2}$$

$$y_p = \frac{7x(x-1)}{2!} \qquad \because 1^x = 1, \quad \because 1^{x-2} = 1$$

3) Complete Solution:

$$y_x = y_c + y_p$$

$$= (c_1 + c_2 x) 1^x + \frac{7x(x-1)}{2!}$$

$$y_x = c_1 + c_2 x + \frac{7x(x-1)}{2!}$$

(iii) $U_{x+3} - 5U_{x+2} + 8U_{x+1} - 4U_x = 3 \cdot 2^x$

The differential equation is:

$$U_{x+3} - 5U_{x+2} + 8U_{x+1} - 4U_x = 3 \cdot 2^x$$

or

$$E^3 U_x - 5E^2 U_x + 8E U_x - 4U_x = 3 \cdot 2^x$$

$$(E^3 - 5E^2 + 8E - 4) U_x = 3 \cdot 2^x$$

i) For complementary function:

Consider,

$$(E^3 - 5E^2 + 8E - 4) U_x = 0$$

The characteristic equation is:

$$m^3 - 5m^2 + 8m - 4 = 0 \rightarrow m = 1 \text{ satisfy the equ.}$$

By using Synthetic division:

1	1	-5	8	-4
		1	-4	4
	1	-4	4	0

$$m=1, \quad m^2 - 4m + 4 = 0$$
$$m^2 - 2m - 2m + 4 = 0$$
$$m(m-2) - 2(m-2) = 0$$
$$(m-2)(m-2) = 0$$
$$m=1 \; ; \quad m=2 \; , \quad m=2$$

Thus,

$$y_c = (c_1 + c_2 x)\, 2^x + c_3 (1)^x$$

2) <u>For particular solution:</u>

$$y_p = \frac{3 \cdot 2^x}{E^3 - 5E^2 + 8E - 4}$$

$$y_p = \frac{3 \cdot 2^x}{(E-1)(E-2)^2}$$

(failure case)

$$= 3 \cdot \frac{x(x-1)\, 2^{x-2}}{2!\,(2-1)}$$

$$= \frac{3x(x-1)\, 2^x}{2^2 \cdot 2!}$$

$$y_p = \frac{3x(x-1)\, 2^x}{8}$$

3) <u>Complete solution:</u>

$$y_x = y_c + y_p$$
$$= (c_1 + c_2 x)\, 2^x + c_3 + \frac{3x(x-1)\, 2^x}{8}$$

Type - III :

If $f(x) = a^x g(x)$
where $g(x)$ is polynomial in /
Then

$$P.I = \frac{a^x g(x)}{\phi(E)}$$

(Replace E by aE)

$$P.I = \frac{a^x g(x)}{\phi(aE)}$$

Q: Solve the differential equations

(i) $y_{x+1} - a y_x = (2x+1) a^x$

The differential equation is:

$$y_{x+1} - a y_x = (2x+1) a^x$$

or

$$E y_x - a y_x = (2x+1) a^x$$

$$(E - a) y_x = (2x+1) a^x$$

(1) For complementary function:

Consider, $(E-a) y_x = 0$

The characteristic equation is:

$$m - a = 0$$

$$m = a$$

Thus,

$$y_c = c_1 a^x$$

(2) For particular solution:

$$y_p = \frac{(2x+1) a^x}{E - a}$$

Replace E by aE

$$y_p = \frac{(2x+1)a^x}{aE - a}$$

$$= \frac{a^x(2x+1)}{a(E-1)}$$

$$= a^{x-1} \cdot \frac{(2x+1)}{\Delta}$$

$\because E = 1 + \Delta$

$$= a^{x-1} \cdot \Delta^{-1}(2x+1)$$

$$= a^{x-1}\left(2\Delta^{-1}(x) + \Delta^{-1}(1)\right)$$

Δ^{-1} behaves as integration

$$= a^{x-1}\left(\frac{\cancel{2}x^2}{\cancel{2}} + x\right)$$

$$=$$

(iii) $y_{x+2} + y_{x+1} - 56y_x = 2^x(x^2-2)$

The differential equation is:

$y_{x+2} + y_{x+1} - 56y_x = 2^x(x^2-2)$

or

$E^2 y_x + E y_x - 56 y_x = 2^x(x^2-2)$

$\Rightarrow (E^2+E-56)y_x = 2^x(x^2-2)$

(1) For complementary function:

Consider,

$$(E^2+E-56)y_x = 0$$

The characteristic equation is:

$$m^2+m-56=0$$
$$m^2+8m-7m-56=0$$
$$m(m+8)-7(m+8)=0$$
$$(m-7)(m+8)=0$$
$$m-7=0, \quad m+8=0$$
$$m=7, \quad m=-8$$
$$y_c = c_1 7^x + c_2(-8)^x$$
$$y_c = c_1 7^x - c_2 8^x$$

(2) For particular solution:

$$y_p = \frac{2^x(x^2-2)}{E^2+E-56}$$

Replace E by 2E:

$$y_p = \frac{2^x(x^2-2)}{(2E)^2+2E-56}$$
$$= \frac{2^x(x^2-2)}{4E^2+2E-56}$$

$\because E=1+\Delta$

$$= \frac{2^x(x^2-2)}{4(1+\Delta)^2+2(1+\Delta)-56}$$

$$= \frac{2^x (x^2-2)}{4(1+\Delta^2+2\Delta)+2(3+\Delta)-56}$$

$$= \frac{2^x}{2} \frac{(x^2-2)}{2(1+\Delta^2+2\Delta)+1+\Delta-28}$$

$$= 2^{x-1} \frac{(x^2-2)}{2+2\Delta^2+4\Delta+1+\Delta-28}$$

$$= 2^{x-1} \frac{(x^2-2)}{2\Delta^2+5\Delta-25}$$

$$= 2^{x-1} \frac{(x^2-2)}{-25\left(-\frac{2}{25}\Delta^2-\frac{5}{25}\Delta+1\right)}$$

$$= -\frac{2^{x-1}}{25} \frac{(x^2-2)}{\left[1-\left(\frac{2}{25}\Delta^2+\frac{1}{5}\Delta\right)\right]}$$

$$= -\frac{2^{x-1}}{25}(x^2-2)\left[1-\left(\frac{2}{25}\Delta^2+\frac{1}{5}\Delta\right)\right]^{-1}$$

$$= -\frac{2^{x-1}}{25}(x^2-2)\left[1+(-1)\left(-\frac{2}{25}\Delta^2-\frac{1}{5}\Delta\right)+\frac{(-1)(-1-1)}{2!}\left(-\frac{2}{25}\Delta^2-\frac{1}{5}\Delta\right)^2\right]$$

$$= -\frac{2^{x-1}}{25}(x^2-2)\left[1+\frac{2}{25}\Delta^2+\frac{1}{5}\Delta+\left(\frac{2}{25}\Delta^2-\frac{1}{5}\right)^2\right]$$

$$= -\frac{2^{x-1}}{25}(x^2-2)\left[1+\frac{2}{25}\Delta^2+\frac{1}{5}\Delta+\frac{1}{25}+\frac{4\Delta^2}{125}\right]$$

$$= -\frac{2^{x-1}}{25}(x^2-2)\left(\frac{14}{125}\Delta^2+\frac{1}{5}\Delta+\frac{26}{25}\right)$$

$$= -\frac{2^{x-1}}{25}\left(\frac{14}{125}\Delta^2x^2+\frac{1}{5}\Delta x^2+\frac{26}{25}x^2-\frac{28}{125}\Delta^2-\frac{2}{5}\Delta-\frac{52}{25}\right)$$