7.6.2 Simpson's Rules (Composite Forms)

In deriving Eq. (7.43), the Simpson's 1/3 rule, we have used two sub-intervals of equal width. In order to get a composite formula, we shall divide the interval of integration $[a, b]$ into an even number of sub-intervals say $2N$, each of width $(b - a)/2N$, thereby we have $x_0 = a, x_1, \ldots, x_{2N} = b$ and $x_k = x_0 + kh$, $k = 1, 2, \ldots, (2N - 1)$. Thus, the definite integral I can be written as

$$I = \int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{2N-2}}^{x_{2N}} f(x)\,dx \tag{7.50}$$

Applying Simpson's 1/3 rule as in Eq. (7.43) to each of the integrals on the right-hand side of Eq. (7.50), we obtain

$$I = \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots$$

$$+ (y_{2N-2} + 4y_{2N-1} + y_{2N})] - \frac{N}{90}h^5 y^{(iv)}(\xi)$$

That is,

$$\int_{x_0}^{x_{2N}} f(x)\,dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + \cdots + y_{2N-1})$$

$$+ 2(y_2 + y_4 + \cdots + y_{2N-2}) + y_{2N}] + \text{Error term} \tag{7.51}$$

This formula is called *composite Simpson's 1/3 rule*. The error term E, which is also called *global error*, is given by

$$E = -\frac{N}{90}h^5 y^{(iv)}(\xi) = -\frac{x_{2N} - x_0}{180}h^4 y^{(iv)}(\xi) \tag{7.52}$$

for some ξ in $[x_0, x_{2N}]$. Thus, in Simpson's 1/3 rule, the global error is of $O(h^4)$.

Similarly in deriving composite Simpson's 3/8 rule, we divide the interval of integration into n sub-intervals, where n is divisible by 3, and applying the integration formula (7.44) to each of the integral given below

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx + \cdots + \int_{x_{n-3}}^{x_n} f(x)\,dx$$

we obtain the composite form of Simpson's 3/8 rule as

$$\int_a^b f(x)\,dx = \frac{3}{8}h[y(a) + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \cdots$$

$$+ 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y(b)] \tag{7.53}$$

with the global error E given by

$$E = -\frac{x_n - x_0}{80}h^4 y^{(iv)}(\xi) \tag{7.54}$$

It may be noted from Eqs. (7.52) and (7.54), the global error in Simpson's 1/3 and 3/8 rules are of the same order. However, if we consider the magnitudes of the error terms, we notice that Simpson's 1/3 rule is superior to Simpson's 3/8 rule. For illustration, we consider few examples.

Example 7.6 Find the approximate value of

$$y = \int_0^{\pi} \sin x \, dx$$

using (i) trapezoidal rule, (ii) Simpson's 1/3 rule by dividing the range of integration into six equal parts. Calculate the percentage error from its true value in both the cases.

Solution We shall at first divide the range of integration $(0, \pi)$ into six equal parts so that each part is of width $\pi/6$ and write down the table of values:

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$y = \sin x$	0.0	0.5	0.8660	1.0	0.8660	0.5	0.0

Applying trapezoidal rule, we have

$$\int_0^{\pi} \sin x \, dx = \frac{h}{2} [y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

Here, h, the width of the interval is $\pi/6$. Therefore,

$$y = \int_0^{\pi} \sin x \, dx = \frac{\pi}{12} [0 + 0 + 2(3.732)] = \frac{3.1415}{6} \times 3.732 = 1.9540$$

Applying Simpson's 1/3 rule (7.41), we have

$$\int_0^{\pi} \sin x \, dx = \frac{h}{3} [y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{\pi}{18} [0 + 0 + (4 \times 2) + (2)(1.732)] = \frac{3.1415}{18} \times 11.464 = 2.0008$$

But the actual value of the integral is

$$\int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi} = 2$$

Hence, in the case of trapezoidal rule

$$\text{The percentage of error} = \frac{2 - 1.954}{2} \times 100 = 2.3$$

While in the case of Simpson's rule the percentage error is

$$\frac{2 - 2.0008}{2} \times 100 = 0.04 \quad \text{(sign ignored)}$$

Example 7.7 From the following data, estimate the value of

$$\int_1^5 \log x \, dx$$

using Simpson's 1/3 rule. Also, obtain the value of h, so that the value of the integral will be accurate up to five decimal places.

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$y = \log x$	0.0000	0.4055	0.6931	0.9163	1.0986	1.2528	1.3863	1.5041	1.6094

Solution We have from the data, $n = 0, 1, \ldots, 8$, and $h = 0.5$. Now using Simpson's 1/3 rule,

$$\int_1^5 \log x \, dx = \frac{h}{3}[y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{0.5}{3}[(0 + 1.6094) + 4(4.0787) + 2(3.178)]$$

$$= \frac{0.5}{3}(1.6094 + 16.3148 + 6.356)$$

$$= 4.0467$$

The error in Simpson's rule is given by

$$E = \frac{x_{2N} - x_0}{180} h^4 y^{(iv)}(\xi) \quad \text{(ignoring the sign)}$$

Since

$$y = \log x, \quad y' = \frac{1}{x}, \quad y'' = -\frac{1}{x^2}, \quad y''' = \frac{2}{x^3}, \quad y^{(iv)} = -\frac{6}{x^4}$$

$$\max_{1 \le x \le 5} y^{(iv)}(x) = 6, \qquad \min_{1 \le x \le 5} y^{(iv)}(x) = 0.0096$$

Therefore, the error bounds are given by

$$\frac{(0.0096)(4)h^4}{180} < E < \frac{(6)(4)h^4}{180}$$

If the result is to be accurate up to five decimal places, then

$$\frac{24h^4}{180} < 10^{-5}$$

That is, $h^4 < 0.000075$ or $h < 0.09$. It may be noted that the actual value of integral is

$$\int_1^5 \log x \, dx = [x \log x - x]_1^5 = 5 \log 5 - 4$$

Example 7.8 Evaluate the integral

$$I = \int_0^1 \frac{dx}{1+x^2}$$

using (i) trapezoidal rule, (ii) Simpson's 1/3 rule by taking $h = 1/4$. Hence, compute the approximate value of π.

Solution At first, we shall tabulate the function as

x	0	1/4	1/2	3/4	1
$y = \frac{1}{1+x^2}$	1	0.9412	0.8000	0.6400	0.5000

using trapezoidal rule, and taking $h = 1/4$

$$I = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)] = \frac{1}{8}[1.5 + 2(2.312)] = 0.7828 \quad (1)$$

using Simpson's 1/3 rule, and taking $h = 1/4$, we have

$$I = \frac{h}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2y_2] = \frac{1}{12}[1.5 + 4(1.512) + 1.6] = 0.7854 \quad (2)$$

But the closed form solution to the given integral is

$$\int_0^1 \frac{dx}{1+x^2} + [\tan^{-1} x]_0^1 = \frac{\pi}{4} \quad (3)$$

Equating (2) and (3), we get $\pi = 3.1416$.

Example 7.9 Compute the integral

$$I = \sqrt{\frac{2}{\pi}} \int_0^1 e^{-x^2/2} \, dx$$

using Simpson's 1/3 rule, taking $h = 0.125$.

Solution At the outset, we shall construct the table of the function as required.

x	0	0.125	0.250	0.375	0.5	0.625	0.750	0.875	1.0
$y = \sqrt{\frac{2}{\pi}} e^{-x^2/2}$	0.7979	0.7917	0.7733	0.7437	0.7041	0.6563	0.6023	0.5441	0.4839

Using Simpson's 1/3 rule, we have

$$I = \frac{h}{3}[y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{0.125}{3}[0.7979 + 0.4839 + 4(0.7917 + 0.7437 + 0.6563 + 0.5441)$$

$$+ 2(0.7733 + 0.7041 + 0.6023)]$$

$$= \frac{0.125}{3}(1.2818 + 10.9432 + 4.1594) = 0.6827$$

Hence, $I = 0.6827$.

Example 7.10 A missile is launched from a ground station. The acceleration during its first 80 seconds of flight, as recorded, is given in the following table:

t (s)	0	10	20	30	40	50	60	70	80
a (m/s^2)	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

compute the velocity of the missile when t = 80 s, using Simpson's 1/3 rule.

Solution Since acceleration is defined as the rate of change of velocity, we have

$$\frac{dv}{dt} = a \quad \text{or} \quad v = \int_0^{80} a\,dt$$

Using Simpson's 1/3-rule, we have

$$v = \frac{h}{3}[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{10}{3}[(30 + 50.67) + 4(31.63 + 35.47 + 40.33 + 46.69)$$

$$+ 2(33.34 + 37.75 + 43.25)]$$

$$= 3086.1 \text{ m/s}$$

Therefore, the required velocity is given by v = 3.0861 km/s

7.8 DOUBLE INTEGRATION

To evaluate numerically a double integral of the form

$$I = \int \left[\int f(x, y)\, dx \right] dy \qquad (7.56)$$

over a rectangular region bounded by the lines $x = a$, $x = b$, $y = c$, $y = d$ we shall employ either trapezoidal rule or Simpson's rule, developed in Section 7.6, repeatedly with respect to one variable at a time. Noting that, both the integrations are just a linear combination of values of the given function at different values of the independent variable, we divide the interval $[a, b]$ into N equal sub-intervals of size h, such that $h = (b - a)/N$; and the interval (c, d) into M equal sub-intervals of size k, so that $k = (d - c)/M$. Thus, we have

$$x_i = x_0 + ih, \quad x_0 = a, \quad x_N = b, \qquad \text{for } i = 1, 2, \ldots, N - 1$$

$$y_i = y_0 + ik, \quad y_0 = c, \quad y_M = d, \qquad \text{for } i = 1, 2, \ldots, M - 1$$

Thus, we can generate a table of values of the integrand, and the above procedure of integration is illustrated by considering a couple of examples.

Example 7.12 Evaluate the double integral

$$I = \int_1^2 \int_1^2 \frac{dx\, dy}{x + y}$$

by using trapezoidal rule, with $h = k = 0.25$.

Solution Taking $x = 1, 1.25, 1.50, 1.75, 2.0$ and $y = 1, 1.25, 1.50, 1.75, 2.0$, the following table is generated using the integrand

$$f(x, y) = \frac{1}{x + y}$$

x	y				
	1.00	1.25	1.50	1.75	2.00
1.00	0.5	0.4444	0.4	0.3636	0.3333
1.25	0.4444	0.4	0.3636	0.3333	0.3077
1.50	0.4	0.3636	0.3333	0.3077	0.2857
1.75	0.3636	0.3333	0.3077	0.2857	0.2667
2.00	0.3333	0.3077	0.2857	0.2667	0.25

Keeping one variable say x fixed and varying the variable y, the application of trapezoidal rule to each row in the above table gives

$$\int_1^2 f(1, y)\, dy = \frac{0.25}{2}[0.5 + 2(0.4444 + 0.4 + 0.3636) + 0.3333]$$
$$= 0.4062 \qquad (1)$$

$$\int_1^2 f(1.25, y)\, dy = \frac{0.25}{2}[0.4444 + 2(0.4 + 0.3636 + 0.3333) + 0.3077]$$
$$= 0.3682 \qquad (2)$$

$$\int_1^2 f(1.5, y)\, dy = \frac{0.25}{2}[0.4 + 2(0.3636 + 0.3333 + 0.3077) + 0.2857]$$
$$= 0.3369 \qquad (3)$$

$$\int_1^2 f(1.75, y)\, dy = \frac{0.25}{2}[0.3636 + 2(0.3333 + 0.3077 + 0.2857) + 0.2667]$$
$$= 0.3105 \qquad (4)$$

and

$$\int_1^2 f(2, y)\, dy = \frac{0.25}{2}[0.3333 + 2(0.3077 + 0.2857 + 0.2667) + 0.25]$$
$$= 0.2879 \qquad (5)$$

Therefore,

$$I = \int_1^2 \int_1^2 \frac{dx\, dy}{x + y} = \frac{h}{2}\{f(1, y) + 2[f(1.25, y) + f(1.5, y) + f(1.75, y)] + f(2, y)\} \qquad (6)$$

Substituting Eqs. (1)–(5) into Eq. (6), we get the required result as

$$I = \frac{0.25}{2}[0.4062 + 2(0.3682 + 0.3369 + 0.3105) + 0.2879] = 0.3407$$

Example 7.13 Evaluate

$$\int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\sin (x + y)}\, dx\, dy$$

by numerical double integration.

Solution Taking $x = y = 0, \pi/8, \pi/4, 3\pi/8, \pi/2$, we can generate the following table of the integrand

$$f(x, y) = \sqrt{\sin(x + y)}$$

x	y: 0	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$
0	0.0	0.6186	0.8409	0.9612	1.0
$\pi/8$	0.6186	0.8409	0.9612	1.0	0.9612
$\pi/4$	0.8409	0.9612	1.0	0.9612	0.8409
$3\pi/8$	0.9612	1.0	0.9612	0.8409	0.6186
$\pi/2$	1.0	0.9612	0.8409	0.6186	0.0

Keeping one variable say x as fixed and y as variable, and applying trapezoidal rule to each row of the above table, we get

$$\int_0^{\pi/2} f(0, y)\, dx = \frac{\pi}{16}[0.0 + 2(0.6186 + 0.8409 + 0.9612) + 1.0] = 1.1469$$

$$\int_0^{\pi/2} f\left(\frac{\pi}{8}, y\right) dx = \frac{\pi}{16}[0.6186 + 2(0.8409 + 0.9612 + 1.0) + 0.9612] = 1.4106$$

Similarly, we get

$$\int_0^{\pi/2} f\left(\frac{\pi}{4}, y\right) dx = 1.4778, \int_0^{\pi/2}\left(\frac{3\pi}{8}, y\right) dx = 1.4106,$$

and

$$\int_0^{\pi/2}\left(\frac{\pi}{2}, y\right) dx = 1.1469$$

Using these results, we finally obtain

$$\int_0^{\pi/2}\int_0^{\pi/2} \sqrt{\sin\,(x+y)}\, dx\, dy = \frac{\pi}{16}\left\{f(0, y) + 2\left[f\left(\frac{\pi}{8}, y\right) + f\left(\frac{\pi}{4}, y\right)\right.\right.$$

$$\left.\left. + f\left(\frac{3\pi}{8}, y\right)\right] + f\left(\frac{\pi}{2}, y\right)\right\}$$

$$= \frac{\pi}{16}[1.1469 + 2(1.4106 + 1.4778 + 1.4106)$$

$$+ 1.1469] = 2.1386$$