$f(2) = F(1) - F(2) = 500426 - 441582.4325 = 58843.5675$

6.10 HERMITE INTERPOLATION

In Hermite interpolation, we use the expansion involving not only the function values but also its first derivative. To state the problem: given a set of data points (x_i, y_i, y'_i), $i = 0, 1, 2, \ldots, n$, we have to determine a polynomial $P(x)$ of degree $(2n + 1)$. Thus, keeping in mind the Lagrange's interpolation formula, we seek $P(x)$ in the form

$$P(x) = \sum_{i=0}^{n} U_i(x) y_i + \sum_{i=0}^{n} V_i(x) y'_i \tag{6.95}$$

where $U_i(x)$ and $V_i(x)$ are polynomials of degree $(2n + 1)$ that satisfy the relations

$$\left.\begin{aligned} & U_i(x_j) = \delta_{ij} \\ & \left.\frac{\partial U_i}{\partial x}\right|_{x = x_j} = 0 \\ \text{and} \quad & V_i(x_j) = 0 \\ & \left.\frac{\partial V_i}{\partial x}\right|_{x = x_j} = \delta_{ij} \end{aligned}\right\} \tag{6.96}$$

Here, δ_{ij} is a kronecker delta, whose value is unity if $i = j$, otherwise zero. Polynomials satisfying the above conditions are called *Hermite polynomials*. Now, we define

$$U_i = \left\{1 - 2(x - x_i)\left.\frac{dL_i}{dx}\right|_{x = x_i}\right\}[L_i(x)]^2$$

and

$$V_i = (x - x_i)[L_i(x)]^2 \tag{6.97}$$

which of course meets the requirements as defined in Eq. (6.96), where $L_i(x)$ is a Lagrange polynomial satisfying $L_i(x_j) = \delta_{ij}$

Substituting $x = x_i$ in Eq. (6.97), we find that

$$U_i(x_i) = [L_i(x_i)]^2 = 1$$

and

$$V_i(x_i) = 0$$

Now, differentiating Eqs. (6.97), we have

$$U_i'(x) = [1 - L_i'(x_i)(x - x_i)]2\, L_i(x)\, L_i'(x)$$
$$-2L_i'(x_i)[L_i(x)]^2$$

and $V_i'(x) = (x - x_i)\, 2L_i(x)\, L_i'(x) + L_i(x)^2$ observe that $U_i'(x_j) = 0$, $V_i'(x_j) = 0$ for $i \neq j$. Since $L_i(x_i) = 1$, we get

$$U_i'(x_i) = 2L_i'(x_i) - 2L_i'(x_i) = 0$$

and

$$V_i'(x_i) = [L_i(x_i)]^2 = 1$$

Hence, the Hermite Interpolation formula is given as

$$P(x) = \sum_{i=0}^{n} \left[1 - 2L_i'(x_i)(x - x_i)\right]\left[L_i(x)\right]^2 y_i + (x - x_i)\left[L_i(x)\right]^2 y_i' \tag{6.98}$$

For illustration, we consider the following example.

Example 6.25 Estimate the value of $y(1.05)$ using Hermite interpolation formula from the following data:

x	y	y'
1.00	1.00000	0.5000
1.10	1.04881	0.47673

Solution: At first we compute

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{1.05 - 1.10}{1.00 - 1.10} = 0.5$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{1.05 - 1.00}{1.10 - 1.00} = 0.5$$

$$L_0'(x) = \frac{1}{x_0 - x_1} = -\frac{1}{0.10}$$

$$L_1'(x) = \frac{1}{x_1 - x_0} = \frac{1}{0.1}$$

Substituting these expressions in Hermite formula,

$$P(x) = \sum_{i=0}^{n} [1 - 2L_i'(x_i)(x - x_i)][L_i(x)]^2 y_i + (x - x_i)[L_i(x)]^2 y_i'$$

$$= [1 - 2L_0'(x_0)(x - x_0)][L_0(x)]^2 y_0 + [1 - 2L_1'(x_1)(x - x_1)][L_1(x)]^2 y_1$$

we find

$$y(1.05) = \left[1 - 2\left(-\frac{1}{0.1}\right)(0.05)\right]\left(\frac{1}{2}\right)^2 (1) + (0.05)\left(\frac{1}{2}\right)^2 (0.5)$$

$$+ \left[1 - 2\left(\frac{1}{0.1}\right)(-0.05)\right]\left(\frac{1}{2}\right)^2 (1.04881)$$

$$+ (-0.05)\left(\frac{1}{2}\right)^2 (0.47673)$$

$$= 1.0247$$

EXERCISES

6.1 Express $\Delta^2 y_1$ and $\Delta^4 y_0$ in terms of the values of the function y.

6.2 Compute the missing values of y_n and Δy_n in the following table

y_n	Δy_n	$\Delta^2 y_n$
–	–	1
–	–	4
–	5	13
6	–	18
–	–	24
–	–	
–		

6.3 Show that $E\nabla = \Delta = \delta E^{1/2}$.

6.4 Prove that (i) $\delta = 2\sinh(hD/2)$ and, (ii) $\mu = 2\cosh(hD/2)$.

6.5 Show that the operators δ, μ, E, Δ and ∇ commute with one another.

6.6 Explain the concept of linear interpolation. Using linear interpolation, find $f(3)$ for $f(x) = 5^x$. Compare with the actual value. Comment on the result obtained.

6.7 The following table gives pressure of a steam at a given temperature. Using Newton's formula, compute the pressure for a temperature of 142°C.

Temperature, °C	140	150	160	170	180
Pressure, kgf/cm^2	3.685	4.854	6.302	8.076	10.225

6.8 Find Newton's backward interpolating polynomial for the following data:

x	1	2	3	4	5
y	1	−1	1	−1	1

6.9 A second degree polynomial passes through (0, 1) (1, 3), (2, 7), and (3, 13). Find the polynomial, using Newton's forward difference formula.

6.10 The following data gives the melting point of an alloy of lead and zinc; where T is the temperature in °C and P is the percentage of lead in the alloy. Find the melting point of the alloy containing 84% of lead using Newton's interpolation method.

P	60	70	80	90
T	226	250	276	304

6.11 Find the interpolating polynomial for the function $f(x)$ given by

x	0	1	2	5
$y = f(x)$	2	3	12	147

6.12 Find the interpolating polynomial for the following data using Lagrange's formula

x	1	2	−4
$y = f(x)$	3	−5	4

6.13 Starting from Newton's divided difference interpolation formula (6.54) and making use of Eq. (6.47) and recalling the definitions of $\Pi(x)$ from Eq. (6.40), show that it can be reduced to Lagrange's form given by Eq. (6.38).

6.14 Find the interpolating polynomial by (i) Newton's divided difference formula (ii) Lagrange's formula, for the following data and hence show that both the methods give raise to the same polynomial.

x	1	2	3	5
y	0	7	26	124

6.15 Show that the nth differences of a polynomial of degree n is constant.

6.16 Tabulate the values of the function

$$u(x, y) = e^x \sin y + y - 0.1$$

for

$$x = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5$$

and

$$y = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$$

Hence, using this generated table and quadratic interpolation in x-direction and cubic in y-direction, compute $u(1.6, 0.33)$ by numerical two-dimensional interpolation.

6.17 Find the equation of a cubic curve which passes through the points (4, –43), (7, 83), (9, 327) and (12, 1053) using divided difference formula.

6.18 Using a polynomial of third degree, complete the record of the export of a certain commodity during five years, as given below:

Year, x	1985	1986	1987	1988	1989
Export in tons, y	443	384	–	397	467

6.19 Following is the table of values of x and y:

x	3	4	5	6	7	8
y	0.205	0.240	0.259	0.262	0.250	0.224

Find the value of x for which y is minimum using Newton's forward difference formula. Also find the minimum value of y.

6.20 Find the missing values in the following table:

x	0	1	2	3	4	5	6
y	–4	–2	–	–	220	546	1148

6.21 Fit a cubic spline curve that passes through (0, 0.0), (1, 0.5), (2, 2.0) and (3, 1.5) with the natural-end boundary conditions, $S''(0) = 0$, $S''(3) = 0$.

6.22 Fit a clamped cubic spline curve that passes through the points (0, 0.0), (1, 0.5), (2, 2.0) and (3, 1.5) with the end conditions $S'(0) = 0.2$, $S'(3) = -1$.

6.23 Fit a natural cubic spline curve that passes through (0.0, 2.0), (1.0, 4.4366), (1.5, 6.7134) and (2.25, 13.9130).

6.24 Using Newton's divided difference formula, evaluate $f(2)$ and $f(15)$ from the following table of values:

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

6.25 Using cubic spline interpolation, find the value of y at $x = 1/2$, for the following data:

(0, 1), (1, 2), (2, 9) and (3, 28).

6.26 Using Hermite interpolation, estimate the value of $y(1.05)$ from the following data:

x	0.9	1.0	1.1	1.2
$y = \sin x$	0.7833	0.8415	0.8912	0.9320
$y' = \cos x$	0.6216	0.5403	0.4536	0.3624

6.27 Using Hermite interpolation, estimate the value of $y(1.3)$ from the following data:

x	0.5	1.0	1.5	2.0
y	0.4794	0.8415	0.9975	0.9093
y'	0.8776	0.5403	0.7074	−0.4162

6.28 The natural logarithm and its derivative is given in the following table. Estimate the value of ln (0.6) using Hermite interpolation formula.

x	0.4	0.5	0.7	0.8
$\ln x$	−0.9163	−0.6932	−0.3567	−0.2231
$1/x$	2.5	2.0	1.43	1.25