6.8 CUBIC SPLINE INTERPOLATION

The name *spline function* is derived from a device known as mechanical spline used by draftsmen for drawing smooth curves. It consists of a flexible steel strip to which weights may be attached in such a way as to constrain it to pass through a given set of points. Mathematically, a *spline function* is one whose graph is a composite curve made up of a number of polynomial arcs of a given degree fitted together in such a way that the junctions of the successive arcs are as smooth as they could be made without going to a single polynomial over the entire range. Fitting to an empirical data by a spline function offers a numerical method for obtaining a curve similar to the one produced by a French curve. This technique has been used effectively in the areas of computer graphics, flow simulations and for smoothing of satellite data which is received at a tracking station with noise.

Definition 6.1 Suppose, we have $(n + 1)$ data points (x_i, y_i), $i = 0, 1, 2, \ldots, n$; where x_i may not be equally spaced and $x_0 = a$, $x_n = b$ and we wish to determine a cubic spline function $S(x)$, such that it has the following properties:

(i) The cubic spline function has the form

$$S(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

in each interval (x_i, x_{i+1}), $i = 0, 1, 2, \ldots, (n - 1)$.

(ii) $S(x_i) = y_i$, $(i = 0, 1, 2, \ldots, n)$.

(iii) The cubics are so joined that the function $S(x)$ and both its slope $S'(x)$ and curvature $S''(x)$ are continuous in (x_0, x_n). It means that the spline curve $S(x)$ will not have sharp corners and the radius of curvature is defined at each point.

Thus, the cubic spline function will have the form $S(x) = S_i(x)$ in the interval (x_i, x_{i+1}). To get $S(x)$, we have to put together the cubics $S_i(x)$ as shown in Fig. 6.1. A detailed account of the basic properties of the cubic spline can be found in Ahlberg et al. (1967).

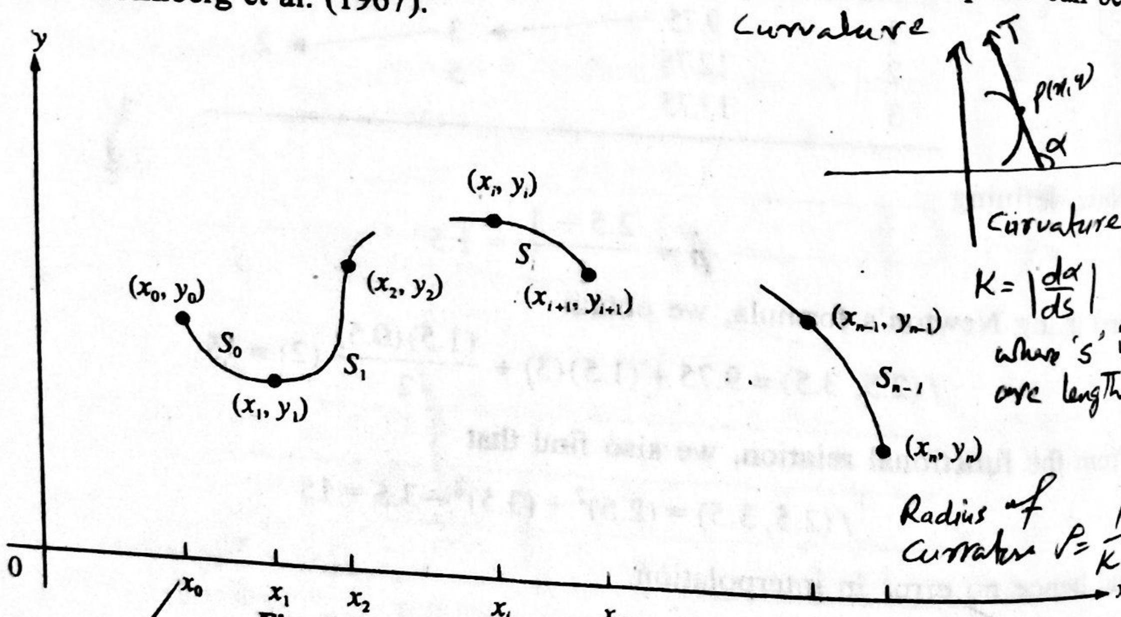

Fig. 6.1 Piecewise cubic spline interpolation.

6.8.1 Construction of Cubic Spline

Spath (1969) suggested a cubic spline $S(x)$ by the piecewise cubic polynomials of the form

$$S(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \tag{6.69}$$

in each interval (x_i, x_{i-1}), $i = 0, 1, 2, \ldots, (n - 1)$.

Since condition (ii) in Definition 6.1 implies that the cubic spline fits exactly at the two end points x_i and x_{i+1} of the i-th interval, we have

$$S(x_i) = y_i = a_i(x_i - x_i)^3 + b_i(x_i - x_i)^2 + c_i(x_i - x_i) + d_i = d_i \tag{6.70}$$

$$S(x_{i+1}) = y_{i+1} = a_i(x_{i+1} - x_i)^3 + b_i(x_{i+1} - x_i)^2 + c_i(x_{i+1} - x_i) + d_i \tag{6.71}$$

Now, introducing the notation $h_i = x_{i+1} - x_i$, Eq. (6.71) becomes

$$y_{i+1} = a_i h_i^3 + b_i h_i^2 + c_i h_i + d_i \tag{6.72}$$

To satisfy the third condition relating to the slope and curvature of the joining cubics, we obtain from Eq. (6.69) that

$$S'(x) = y'(x) = 3a_i(x - x_i)^2 + 2b_i(x - x_i) + c_i \tag{6.73}$$

$$S''(x) = y''(x) = 6a_i(x - x_i) + 2b_i \tag{6.74}$$

Using the notation $S_i'' = y''(x_i) = M_i$, we can determine a_i, b_i and c_i in terms of M_i. From Eq. (6.74), we have

$$M_i = 6a_i(x_i - x_i) + 2b_i = 2b_i \tag{6.75}$$

$$M_{i+1} = 6a_i(x_{i+1} - x_i) + 2b_i = 6a_i h_i + 2b_i \tag{6.76}$$

which gives us

$$b_i = \frac{M_i}{2}, \qquad a_i = \frac{M_{i+1} - M_i}{6h_i} \tag{6.77, 6.78}$$

Now, using the values of d_i, b_i and a_i given by Eqs. (6.70), (6,77) and (6.78), we get from Eq. (6.71) that

$$y_{i+1} = \frac{M_{i+1} - M_i}{6h_i} h_i^3 + \frac{M_i}{2} h_i^2 + c_i h_i + y_i$$

which gives

$$c_i = \frac{y_{i+1} - y_i}{h_i} - \frac{2h_i M_i + h_i M_{i+1}}{6} = y_i' \quad ① \tag{6.79}$$

At this stage, we shall recall the third condition that the slopes of the two cubics meeting at (x_i, y_i) are equal from $(i - 1)$th and ith intervals. Hence Eq. (6.73) at $x = x_i$ the left end in the ith interval is

$$y'(x_i) = y'_i = 3a_i(x_i - x_i)^2 + 2b_i(x_i - x_i) + c_i = c_i \tag{6.80}$$

while in the $(i - 1)$th interval (x_{i-1}, x_i), the slope at the right end, that is at $x = x_i$ given by

$$y'(x_i) = y'_i = 3a_{i-1}(x_i - x_{i-1})^2 + 2b_{i-1}(x_i - x_{i-1}) + c_{i-1}$$

$$= 3a_{i-1}h_{i-1}^2 + 2b_{i-1}h_{i-1} + c_{i-1} \tag{6.81}$$

Now, equating Eqs. (6.80) and (6.81) and using Eqs. (6.77)–(6.79) for b_{i-1}, a_{i-1} and c_{i-1} respectively, we get

first $y'(x) = 3a_{i-1}(x - x_{i-1})^2 + 2b_{i-1}(x - x_{i-1}) + c_{i-1}$

$y'(x_i) = y_i' = 3a_{i-1}(x_i - x_{i-1})^2 + 2b_{i-1}(x_i - x_{i-1}) + c_{i-1}$

$\Rightarrow$ $c_i = 3a_{i-1}(x_i - x_{i-1})$

using value of $c_i, a_{i-1}, b_{i-1}, c_{i-1}$ and $x_i - x_{i-1} = h$

$$\frac{y_{i+1} - y_i}{h_i} - \frac{2h_i M_i + h_i M_{i+1}}{6} = \left\{ 3\frac{M_i - M_{i-1}}{6h_{i-1}} h_{i-1}^2 + M_{i-1}h_{i-1} + \frac{y_i - y_{i-1}}{h_{i-1}} - \frac{2h_{i-1}M_{i-1} + h_{i-1}M_i}{6} \right\} \quad (6.82)$$

or it is $y_i' =$ (2)

On simplification, we obtain

$$h_{i-1}M_{i-1} + (2h_{i-1} + 2h_i)\, M_i + h_i M_{i+1} = 6\left(\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}\right) \quad (6.83)$$

3

for $i = 1, 2, \ldots, (n-1)$. From Eq. (6.83), it may be observed that only M_i's are unknowns, while all other terms can be computed from the given data points. In fact, it represents a system of $(n-1)$ linear equations in $(n+1)$ unknowns M_0, $M_1, \ldots, M_n$. Hence, two more additional conditions are required, relating to the end points of the complete spline curve, to generate two more additional equations. Many types of end conditions are specified and discussed in the literature. However, we shall consider only two types of end conditions.

6.8.2 End Conditions

Type I: We specify $S_0 = S_n = 0$, which means $M_0 = 0$, $M_n = 0$. In this case, the end cubics linearly approach to their extremities. This is called *natural spline*. This form of specification of end conditions is very popular. In this case, Eq. (6.83) readily gives us the system

$$\left.\begin{aligned} &2(h_0 + h_1)M_1 + h_1 M_2 = 6\left(\frac{y_2 - y_1}{h_1} - \frac{y_1 - y_0}{h_0}\right) \\ &h_{i-1}M_{i-1} + (2h_{i-1} + 2h_i)M_i + h_i M_{i+1} = 6\left(\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}\right) \\ &\text{where } i = 2, 3, \ldots, (n-2) \\ &h_{n-2}M_{n-2} + 2(h_{n-2} + h_{n-1})M_{n-1} = 6\left(\frac{y_n - y_{n-1}}{h_{n-1}} - \frac{y_{n-1} - y_{n-2}}{h_{n-2}}\right) \end{aligned}\right\} \quad (6.84)$$

In compact matrix notation, we present it as

$$\begin{bmatrix} 2(h_0 + h_1) & h_1 & & & \\ h_1 & 2(h_1 + h_2) & h_2 & & \\ & h_2 & 2(h_2 + h_3) & h_4 & \\ & & & \vdots & \\ & & & h_{n-2} & 2(h_{n-2} + h_{n-1}) \end{bmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-1} \end{pmatrix}$$

$$= 6 \begin{pmatrix} \dfrac{y_2 - y_1}{h_1} - \dfrac{y_1 - y_0}{h_0} \\ \dfrac{y_3 - y_2}{h_2} - \dfrac{y_2 - y_1}{h_1} \\ \dfrac{y_4 - y_3}{h_3} - \dfrac{y_3 - y_2}{h_2} \\ \vdots \\ \dfrac{y_n - y_{n-1}}{h_{n-1}} - \dfrac{y_{n-1} - y_{n-2}}{h_{n-2}} \end{pmatrix} \tag{6.85}$$

This being an $(n - 1 \times n - 1)$ tridiagonal system can be solved economically using Crout's reduction method as explained in Section 3.4.

Type II: In this case, we specify the slopes at the end of entire spline curve; that is, we are given $y'(x_0) = A$ and $y'(x_n) = B$. This is called ***clamped cubic spline***. From Eqs. (6.79) and (6.80), we have

1

$$y_i' = -\frac{h_i}{3} M_i - \frac{h_i}{6} M_{i+1} + \frac{y_{i+1} - y_i}{h_i} \tag{6.86}$$

using the left-end condition for $i = 0$, we obtain

$$-\frac{h_0}{3} M_0 - \frac{h_0}{6} M_1 + \frac{y_1 - y_0}{h_0} = A$$

That is,

$$2M_0 + M_1 = \frac{6}{h_0}\left(\frac{y_1 - y_0}{h_0} - A\right) \tag{6.87}$$

2

Similarly from Eq. (6.82), we have

$$y_i' = \frac{M_i h_{i-1}}{3} + \frac{M_{i-1} h_{i-1}}{6} + \frac{y_i - y_{i-1}}{h_{i-1}}$$

Now, using the right-end condition for $i = n$, the above equation becomes

$$B = \frac{M_n h_{n-1}}{3} + \frac{M_{n-1} h_{n-1}}{6} + \frac{y_n - y_{n-1}}{h_{n-1}}$$

Further simplification yields

$$M_{n-1} + 2M_n = \frac{6}{h_{n-1}}\left(B - \frac{y_n - y_{n-1}}{h_{n-1}}\right) \tag{6.88}$$

3

For $i = 1$, Eq. (6.83) gives

$$h_0 M_0 + (2h_0 + 2h_1) M_1 + h_1 M_2 = 6\left(\frac{y_2 - y_1}{h_1} - \frac{y_1 - y_0}{h_0}\right) \tag{6.89}$$

Eliminating M_0 from Eqs. (6.87) and (6.89), we get

$$\left(\frac{3}{2}h_0 + 2h_1\right)M_1 + h_1 M_2 = 6\,\frac{y_2 - y_1}{h_1} - 9\,\frac{y_1 - y_0}{h_0} + 3A \tag{6.90}$$

Also, for $i = n - 1$, Eq. (6.83) gives

$$h_{n-2}M_{n-2} + (2h_{n-2} + 2h_{n-1})M_{n-1} + h_{n-1}M_n = 6\left(\frac{y_n - y_{n-1}}{h_{n-1}} - \frac{y_{n-1} - y_{n-2}}{h_{n-2}}\right) \tag{6.91}$$

Eliminating M_n from Eqs. (6.88) and (6.91), we get

$$h_{n-2}M_{n-2} + \left(2h_{n-2} + \frac{3}{2}h_{n-1}\right)M_{n-1} = 9\,\frac{y_n - y_{n-1}}{h_{n-1}} - 6\,\frac{y_{n-1} - y_{n-2}}{h_{n-2}} - 3B \tag{6.92}$$

Hence, in type II, we have to solve Eqs. (6.83), (6.90) and (6.92). These equations together constitute an $(n - 1 \times n - 1)$ tridiagonal system in unknowns $M_1, M_2, \ldots, M_{n-1}$, which can be solved using Crout's reduction technique. In order to see the sequence of steps involved to construct a cubic spline $S(x)$ for a given data set, using cubic spline interpolation, we shall consider below a couple of simple examples.

x_0 y_0 x_1 y_1

Example 6.20 Fit a cubic spline curve that passes through (0, 1), (1, 4), (2, 0), (3, −2) with the natural end boundary conditions $S''(0) = S''(3) = 0.0$.

x_2 y_2 x_3 y_3

Solution From the given data, we observe that there are three intervals, in each of which we can construct a cubic spline function. These piecewise cubic spline polynomials when put together determine the cubic spline curve $S(x)$ in the entire interval (0, 3).

At the outset, we observe that $h_0 = h_1 = h_2 = 1$. For natural spline, we obtain Eqs. (6.84) as

$$\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = 6\begin{pmatrix} -4 & -3 \\ -2 & +4 \end{pmatrix} = \begin{pmatrix} -42 \\ 12 \end{pmatrix}$$

That is,

$$4M_1 + M_2 = -42$$

$$M_1 + 4\,M_2 = 12$$

Its solution is $M_1 = -12$, $M_2 = 6$. Natural-end conditions imply $M_0 = M_3 = 0.0$. Let the natural cubic spline is given by

$$S(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

where the coefficients are given by the relations

$$a_i = \frac{M_{i+1} - M_i}{6h_i}, \qquad b_i = \frac{M_i}{2}$$

$$c_i = \frac{y_{i+1} - y_i}{h_i} - \frac{2h_i M_i + h_i M_{i+1}}{6}, \qquad d_i = y_i$$

for $i = 0, 1, 2$. Using the data and the values of M_0 and M_3, we compute the coefficients as

$$a_0 = -2.0, \qquad a_1 = 3, \qquad a_2 = -1$$
$$b_0 = 0.0, \qquad b_1 = -6, \qquad b_2 = 3$$
$$c_0 = 3 + \frac{12}{6} = 5, \qquad c_1 = -1, \qquad c_2 = -4$$
$$d_0 = 1.0, \qquad d_1 = 4.0, \qquad d_2 = 0.0$$

Hence the required piecewise cubic splines in each interval is given by

$$S_0(x) = -2.0x^3 + 5x + 1 \qquad \text{for } 0 \le x \le 1$$
$$S_1(x) = 3(x-1)^3 - 6(x-1)^2 - (x-1) + 4 \qquad \text{for } 1 \le x \le 2$$
$$S_2(x) = -(x-2)^3 + 3(x-2)^2 - 4(x-2) \qquad \text{for } 2 \le x \le 3$$

Example 6.21 Fit a cubic spline curve that passes through points (0, 1), (1, 4), (2, 0) and (3, −2) with the given derivative boundary conditions

$$S'(0) = 2, \qquad S'(3) = 2$$

Solution In this example, we have three intervals, in each of which, we can construct a cubic spline functions denoted by S_0, S_1 and S_2. At the outset, we observe that $h_0 = h_1 = h_2 = 1$. For derivative boundary conditions, we use Eqs. (6.90) and (6.92) and get

$\because A = S'(0) = 2$

$$\frac{7}{2}M_1 + M_2 = 6(-4) - 9(3) + 3 \times 2 = -45$$

$$M_1 + \frac{7}{2}M_2 = 9(-2) - 6(-4) - 3 \times 2 = 0$$

using $n = 3$ in (6.92)

$B = S'(3) = 2$

Its solution is $M_1 = -14$, $M_2 = 4$. Now from Eq. (6.87), we obtain $2M_0 - 14 = 6$, which gives $M_0 = 10$. Also, Eq. (6.88) gives $M_2 + 2M_3 = 6(2 + 2) = 24$. Using the value of M_2, we get $M_3 = 10$.

→ $n = 3$ in (6.88)

Let the cubic spline in each interval is given by

$$S(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

The coefficients are computed as

$$a_0 = -4, \qquad a_1 = 3, \qquad a_2 = 1$$
$$b_0 = 5, \qquad b_1 = -7, \qquad b_2 = 2$$
$$c_0 = 2, \qquad c_1 = 0, \qquad c_2 = -5$$
$$d_0 = 1, \qquad d_1 = 4, \qquad d_2 = 0.0$$

Hence the required piecewise cubic spline polynomials in each interval is given by

$$S_0(x) = -4x^3 + 5x^2 + 2x + 1, \qquad \text{for } 0 \le x \le 1$$
$$S_1(x) = 3(x-1)^3 - 7(x-1)^2 + 4, \qquad \text{for } 1 \le x \le 2$$
$$S_2(x) = (x-2)^3 + 2(x-2)^2 - 5(x-2), \qquad \text{for } 2 \le x \le 3$$

6.9 MAXIMA AND MINIMA OF A TABULATED FUNCTION

The idea of finding the maxima and minima of a tabulated function is useful in many practical problems. Recalling the Newton's forward interpolation formula (6.33) as

$$y \equiv y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

and noting that $p = (x - x_0)/h$; its differentiation with respect to x yields

$\frac{dp}{dx} = \frac{1}{h}$

$\frac{d}{dx}(p^2 - p) = \frac{2p-1}{h}$

$$\frac{dy}{dx} \equiv \frac{1}{h}\left(\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \cdots\right) \tag{6.93}$$

From elementary calculus, it is known that the maxima and minima values of a function $y = f(x)$ are obtained by equating its first derivative with respect to x to zero and solving it for x. Same idea holds in the case of tabulated function, too. Thus, we have

$$\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \cdots = 0 \tag{6.94}$$

By retaining terms up to third difference only, we arrive at the expression

$$\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 = 0$$

Here, the first, second and third differences can be obtained from the difference table. Thus, Eq. (6.94) gives a polynomial in x, whose solution gives us those values of x, at which the given tabulated function may be maximum or minimum. This idea is illustrated in the following couple of examples.

Example 6.22 Find, for what value of x, y is minimum using the data given below:

x	3	4	5	6	7	8
y	0.205	0.240	0.259	0.262	0.250	0.224

Check values

Solution From the given data, it can be seen that the arguments are equally spaced and therefore, we construct the forward difference table as

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	0.205				
4	0.240	0.035			
5	0.259	0.019	−0.016	0.0	
6	0.262	0.003	−0.016	−0.001	−0.001
7	0.250	−0.012	−0.015	0.001	0.002
8	0.224	−0.026	−0.014		

In this example, $p = (x - x_0)/h = (x - 3)$, $\Delta y_0 = 0.035$, $\Delta^2 y_0 = -0.016$, $\Delta^3 y_0 = 0$, $\Delta^4 y_0 = -0.001$. Recalling Newton's forward difference formula given by Eq. (6.33) and retaining up to third differences only, we have

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

Substituting the values of p and the differences, the above equation becomes

$$y = 0.205 + (x-3)(0.035) + \frac{(x-3)(x-4)(-0.016)}{2}$$

That is,

$$y = -0.008x^2 + 0.091x + 0.004 \qquad (1)$$

For maxima or minima, we require

$$\frac{dy}{dx} = -0.016x + 0.091 = 0$$

$\frac{d^2y}{dx^2} = -0.016$ So y has maxi value ...

which gives $x = 5.6875$. Thus, the minimum value of y at $x = 5.6875$ is found from Eq. (1) as $y = -0.008\ (33.414) + 0.091(5.6875) + 0.004 = 0.25425$. Hence the minimum value at $x = 5.6875$ of the given tabulated function is $= 0.25425$.

Example 6.23 Find, for what value of x, y is maximum, from the following data:

x	-1	1	2	3
y	-21	15	12	3

Solution From the given data, it can be seen that the arguments are not equally-spaced, and therefore, we can use either Lagrange's interpolation or Newton's divided difference interpolation formula. We choose the former and hence the Lagrange's interpolation formula for the given data given as

$$y = \frac{(x-1)(x-2)(x-3)}{(-1-1)(-1-2)(-1-3)}(-21) + \frac{(x+1)(x-2)(x-3)}{(1+1)(1-2)(1-3)}(15)$$

$$+ \frac{(x+1)(x-1)(x-3)}{(2+1)(2-1)(2-3)}(12) + \frac{(x+1)(x-1)(x-2)}{(3+1)(3-1)(3-2)}(3)$$

$$= \frac{21}{24}(x^3 - 6x^2 + 11x - 6) + \frac{15}{4}(x^3 - 4x^2 + x + 6)$$

$$- \frac{12}{3}(x^3 - 3x^2 - x + 3) + \frac{3}{8}(x^3 - 2x^2 - x + 2)$$

which simplifies to

$$y = x^3 - 9x^2 + 17x + 6 \qquad (1)$$

For maxima or minima, it is required that

$$\frac{dy}{dx} = 3x^2 - 18x + 17 = 0$$

which is a quadratic equation, whose solution is given by

$$x = \frac{18 \pm \sqrt{18^2 - 12 \times 17}}{6} = 4.8257 \quad \text{or} \quad 1.1743$$

Here, we note that $x = 4.8257$ is outside the considered range. However,

$$\left.\frac{d^2y}{dx^2}\right|_{x=1.1743} = (6x - 18)\Big|_{x=1.1743} = \text{negative}$$

Hence, the maximum value of y at $x = 1.1743$ is given as

$$y = (1.1743)^3 - 9(1.1743)^2 + 17(1.1743) + 6 = 15.1716$$

Example 6.24 Given $\Sigma_1^{10} f(x) = 500426$, $\Sigma_4^{10} f(x) = 329240$, $\Sigma_7^{10} f(x) = 175212$ and $f(10) = 40365$, find $f(2)$.

Solution In this example, we are given the cumulative values of the function and therefore, we adopt the following notation:

$$F(1) = \sum_{1}^{10} f(x) = 500426$$

$$F(4) = \sum_{4}^{10} f(x) = 329240$$

$$F(7) = \sum_{7}^{10} f(x) = 175212$$

$$F(10) = f(10) = 40365$$

$\because F(10) = \sum^{10} f(x) = f(10) = 40365$

$F(10)$

and construct the forward difference table as

x	$F(x)$	$\Delta F(x)$	$\Delta^2 F(x)$	$\Delta^3 F(x)$
1	500426			
4	329240	–171186		
7	175212	–154028	17158	2023
10	40365	–134847	19181	

Now, to find $F(2) = \Sigma_2^{10} f(x)$, we may recall Newton's forward difference formula given by Eq. (6.33) as

$F(x) = F(x_0) + p\Delta F(x_0) + \frac{p(p-1)}{2!}\Delta^2 F(x_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 F(x_0)$

$$F(2) = F(1) + p\Delta F(1) + \frac{p(p-1)}{2!}\Delta^2 F(1) + \frac{p(p-1)(p-2)}{3!}\Delta^3 F(1)$$

where

$$p = \frac{x - x_0}{h} = \frac{2-1}{3} = \frac{1}{3}$$

Also, from the table we note that $\Delta F(1) = -171186$, $\Delta^2 F(1) = 17158$ and $\Delta^3 F(1) = -2023$. Substituting these values, the above equation gives.

$$F(2) = 500426 - \frac{171186}{3} - \frac{17158}{9} + \frac{5(2023)}{81} = 441582.4325$$

Therefore, we finally have

$$f(2) = F(1) - F(2) = 500426 - 441582.4325 = 58843.5675$$

HERMITE INTERPOLATION