6.6 DIVIDED DIFFERENCES

When the function values are given at non-equispaced points, we have already developed the Lagrange's interpolation formula for interpolation in Section 6.5. Now, we shall introduce the concept of divided differences and then develop Newton's divided difference interpolation formula, whose accuracy is same as that of Lagrange's formula, but has the advantage of being computationally economical in the sense that it involves less number of arithmetic operations.

Let us assume that the function $y = f(x)$ is known for several values of x, (x_i, y_i), for $i = 0\ (1)\ n$. The divided differences of orders 0, 1, 2, ..., n are defined recursively as follows:

$$y\,[x_0] = y\,(x_0) = y_0$$

is the 0th order divided difference. The first order divided difference is defined as

$$y\,[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Similarly the higher order divided differences are defined in terms of lower order divided differences by the relations ([illegible], [illegible]) of the form

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0}$$

while

$$y[x_0, x_1, \cdots, x_n] = \frac{y[x_1, x_2, \cdots, x_n] - y[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0} \tag{6.46}$$

The standard format of the divided differences are displayed in Table 6.4.

Table 6.4 Divided Differences

x	$y(x)$	1st order	2nd order	3rd order	4th order
x_0	y_0				
		$y[x_1, x_0]$			
x_1	y_1		$y[x_0, x_1, x_2]$		
		$y[x_2, x_1]$		$y[x_0, x_1, x_2, x_3]$	
x_2	y_2		$y[x_1, x_2, x_3]$		$y[x_0, x_1, x_2, x_3, x_4]$
		$y[x_3, x_2]$		$y[x_1, x_2, x_3, x_4]$	
x_3	y_3		$y[x_2, x_3, x_4]$		
		$y[x_4, x_3]$			
x_4	y_4				

We can easily verify that the divided difference is a symmetric function of its arguments. That is,

$$y[x_1, x_0] = y[x_0, x_1] = \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0}$$

Now,

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0} = \frac{1}{x_2 - x_0}\left(\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\right)$$

Therefore,

$$y[x_0, x_1, x_2] = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

which is a symmetric form, hence suggests the general result as

$$y[x_0, \cdots, x_k] = \frac{y_0}{(x_0 - x_1)\cdots(x_0 - x_k)} + \frac{y_1}{(x_1 - x_0)\cdots(x_1 - x_k)} + \cdots$$

$$+ \frac{y_k}{(x_k - x_0)\cdots(x_k - x_{k-1})}$$

$$= \sum_{i=0}^{k} \frac{y_i}{\prod_{\substack{j=0 \\ i \neq j}}^{k} (x_i - x_j)} \tag{6.47}$$

In Eq. (6.47), it can be noted that zero factor $(x_i - x_i)$ is omitted in the denominator of each term of the sum.

6.6.1 Newton's Divided Difference Interpolation Formula

Let $y = f(x)$ be a function which takes values $y_0, y_1, \ldots, y_n$ corresponding to $x = x_i$, $i = 0, 1, \ldots, n$. We choose an interpolating polynomial, interpolating at $x = x_i$, $i = 0, 1, \ldots, n$ in the following convenient form

$$y = f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}) \tag{6.48}$$

Here, the coefficients a_k are so chosen as to satisfy Eq. (6.48) by the $(n + 1)$ pairs (x_i, y_i). Thus, we have

$$\left.\begin{aligned} y(x_0) &= f(x_0) = y_0 = a_0 \\ y(x_1) &= f(x_1) = y_1 = a_0 + a_1(x_1 - x_0) \\ y(x_2) &= f(x_2) = y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \\ &\vdots \\ y_n &= a_0 + a_1(x_n - x_0) + a_2(x_n - x_0)(x_n - x_1) + \cdots \\ &\quad + a_n(x_n - x_0) \cdots (x_n - x_{n-1}) \end{aligned}\right\} \tag{6.49}$$

The coefficients $a_0, a_1, \ldots, a_n$ can be easily obtained from the system of Eqs. (6.49), as they form a lower triangular matrix. The first equation of (6.49) gives

$$a_0 = y(x_0) = y_0 \tag{6.50}$$

The second equation of (6.49) and Eq. (6.50) gives

$$a_1 = \frac{y_1 - y_0}{x_1 - x_0} = y[x_0, x_1] \tag{6.51}$$

The third equation of (6.49) after using a_0 and a_1 as given in Eqs. (6.50) and (6.51) yields

$$a_2 = \frac{y_2 - y_0 - (x_2 - x_0)y[x_0, x_1]}{(x_2 - x_0)(x_2 - x_1)}$$

which can be rewritten as

$$a_2 = \frac{\left[y_2 - y_1 + \dfrac{y_1 - y_0}{x_1 - x_0}(x_1 - x_0)\right] - (x_2 - x_0)y[x_0, x_1]}{(x_2 - x_0)(x_2 - x_1)}$$

That is,

$$a_2 = \frac{y_2 - y_1 + y[x_0, x_1](x_1 - x_2)}{(x_2 - x_0)(x_2 - x_1)} = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0}$$

Thus, in terms of second order divided differences, we have

$$a_2 = y[x_0, x_1, x_2] \tag{6.52}$$

Similarly, we can show that

$$a_n = y[x_0, x_1, \ldots, x_n] \tag{6.53}$$

Hence, Newton's divided difference interpolation formula can be written as

$$y = f(x) = y_0 + (x - x_0)\, y[x_0, x_1] + (x - x_0)(x - x_1)\, y[x_0, x_1, x_2] + \cdots$$
$$+ (x - x_0)(x - x_1) \ldots (x - x_{n-1})\, y[x_0, x_1, \ldots, x_n] \tag{6.54}$$

Newton's divided differences can also be expressed in terms of forward, backward and central differences. They can be easily derived as follows: Assuming equispaced values of abscissa, we have

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0} = \frac{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}}{2h} = \frac{\Delta^2 y_0}{2!h^2}$$

By induction, we can in general arrive at the result

$$y[x_0, x_1, \cdots, x_n] = \frac{\Delta^n y_0}{n!h^n} \tag{6.55}$$

Similarly

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\nabla y_1}{h}$$

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0} = \frac{\frac{\nabla y_2}{h} - \frac{\nabla y_1}{h}}{2h} = \frac{\nabla^2 y_2}{2!h^2}$$

In general, we have

$$y[x_0, x_1, \cdots, x_n] = \frac{\nabla^n y_n}{n!h^n} \tag{6.56}$$

Also, in terms of central differences, we have

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\delta y_{1/2}}{h}$$

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0} = \frac{\frac{\delta y_{3/2}}{h} - \frac{\delta y_{1/2}}{h}}{2h} = \frac{\delta^2 y_1}{2!h^2}$$

In general, the following pattern is arrived:

$$y\,[x_0, x_1, \ldots, x_{2m}] = \frac{\delta^{2m} y_m}{(2m)!h^{2m}}$$

or

$$y\,[x_0, x_1, \ldots, x_{2m+1}] = \frac{\delta^{2m+1} y_{m+(1/2)}}{(2m+1)!h^{2m+1}} \tag{6.57}$$

We present below few examples for illustration.

Example 6.16 Find the interpolating polynomial by (i) Lagrange's formula, and (ii) Newton's divided difference formula for the following data, and hence show that they represent the same interpolating polynomial.

x	0	1	2	4
y	1	1	2	5

Solution The divided difference table for the given data is constructed as follows:

x	y	1st divided difference	2nd divided difference	3rd divided difference
0	1			
		0		
1	1		1/2	
		1		−1/12
2	2		1/6	
		3/2		
4	5			

(i) Lagrange's interpolation formula (6.37) gives

$$y = f(x) = \frac{(x-1)(x-2)(x-4)}{(-1)(-2)(-4)}(1) + \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)}(1)$$

$$+ \frac{(x-0)(x-1)(x-4)}{(2)(2-1)(2-4)}(2) + \frac{(x-0)(x-1)(x-2)}{4(4-1)(4-2)}(5)$$

$$= \frac{-(x^3 - 7x^2 + 14x - 8)}{8} + \frac{x^3 - 6x^2 + 8x}{3} - \frac{x^3 - 5x^2 + 4x}{2}$$

$$+ \frac{5(x^3 - 3x^2 + 2x)}{24}$$

$$= -\frac{x^3}{12} + \frac{3x^2}{4} - \frac{2}{3}x + 1 \qquad (1)$$

(ii) Newton's divided difference formula gives

$$y = f(x) = 1 + (x-0)(0) + (x-0)(x-1)\left(\frac{1}{2}\right) + (x-0)(x-1)(x-2)\left(-\frac{1}{12}\right)$$

$$= -\frac{x^3}{12} + \frac{3x^2}{4} - \frac{2}{3}x + 1 \qquad (2)$$

From Eqs. (1) and (2) we observe that the interpolating polynomial by both Lagrange's and Newton's divided difference formulae is one and the same. Also Newton's formula involves less number of arithmetic operations than that of Lagrange's.

Example 6.17 Using Newton's divided difference formula, find the quadratic equation for the following data. Hence find $y(2)$.

x	0	1	4
y	2	1	4

Solution The divided difference table for the given data is constructed as follows:

x	y	1st divided difference	2nd divided difference
0	2		
		−1	
1	1		1/2
		1	
4	4		

Now, using Newton's divided difference formula, we have

$$y = 2 + (x - 0)(-1) + (x - 0)(x - 1)\left(\frac{1}{2}\right) = \frac{1}{2}(x^2 - 3x + 4)$$

Hence, $y(2) = 1$.

Example 6.18 A function $y = f(x)$ is given at the sample points $x = x_0, x_1$ and x_2. Show that the Newton's divided difference interpolation formula and the corresponding Lagrange's interpolation formula are identical.

Solution For the function $y = f(x)$, we have the data (x_i, y_i), $i = 0, 1, 2$. The interpolation polynomial using Newton's divided difference formula is given as

$$y = f(x) = y_0 + (x - x_0)\, y[x_0, x_1] + (x - x_0)(x - x_1)\, y[x_0, x_1, x_2] \qquad (1)$$

Using the definition of divided differences and Eq. (6.47), we can rewrite Eq. (1) in the form

$$y = y_0 + (x - x_0)\frac{(y_1 - y_0)}{(x_1 - x_0)} + (x - x_0)(x - x_1)\left[\frac{y_0}{(x_0 - x_1)(x_0 - x_2)}\right.$$

$$\left. + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}\right]$$

$$= \left[1 - \frac{(x_0 - x)}{(x_0 - x_1)} + \frac{(x - x_0)(x - x_1)}{(x_0 - x_1)(x_0 - x_2)}\right] y_0$$

$$+ \left[\frac{(x - x_0)}{(x_1 - x_0)} + \frac{(x - x_0)(x - x_1)}{(x_1 - x_0)(x_1 - x_2)}\right] y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}\, y_2$$

On simplification, it reduces to

$$y = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2 \quad (2)$$

which is the Lagrange's form of interpolation polynomial. Hence Eqs. (1) and (2) are identical.

6.6.2 Newton's Divided Difference Formula with Error Term

Following the basic definition (6.46) of divided differences, we have for any x the Newton's divided difference formula for the interpolating polynomial that fits a divided difference table at $x = x_0, x_1, x_2, \ldots, x_n$ following Eq. (6.54) can be rewritten with error term $\varepsilon(x)$ as

$$y(x) = y_0 + (x - x_0)\, y[x_0, x_1] + (x - x_0)(x - x_1)\, y[x_0, x_1, x_2] + \cdots$$

$$+ (x - x_0)(x - x_1) \cdots (x - x_{n-1})\, y[x_0, x_1, \ldots, x_n] + \varepsilon(x) \qquad (6.58)$$

where

$$\varepsilon(x) = (x - x_0)(x - x_1) \cdots (x - x_n)\, y[x, x_0, \ldots, x_n] \qquad (6.59)$$

It may be noted that for $x = x_0, x_1, \ldots, x_n$, the error term $\varepsilon(x)$ vanishes.

6.6.3 Error Term in Interpolation Formulae

We have seen in Section 6.6.2 that if $y(x)$ is approximated by a polynomial $P_n(x)$ of degree n then the error is given by

$$\varepsilon(x) = y(x) - P_n(x), \qquad (6.60)$$

where,

$$\varepsilon(x) = (x - x_0)(x - x_1) \ldots (x - x_n) y\, [x, x_0, \ldots, x_n]$$

Alternatively it is also expressed as

$$\varepsilon(x) = \Pi(x)\, y\, [x, x_0, \ldots, x_n] = K\Pi(x) \qquad (6.61)$$

Now, consider a function $F(x)$, such that

$$F(x) = y(x) - P_n(x) - K\Pi(x) \qquad (6.62)$$

and determine the constant K in such a way that $F(x)$ vanishes for $x = x_0, x_1, \ldots, x_n$ and also for an arbitrarily chosen point $\bar{x}$, which is different from the given $(n + 1)$ points. Let I denotes the closed interval spanned by the values $x_0, \ldots, x_n, \bar{x}$. Then $F(x)$ vanishes $(n + 2)$ times in the interval I. By Rolle's theorem $F'(x)$ vanishes at least $(n + 1)$ times in the interval I, $F''(x)$ vanishes at least n times, and so on. Eventually, we can show that $F^{(n+1)}(x)$ vanishes at least once in the interval I, say at $x = \xi$. Thus, we obtain

$$0 = y^{(n+1)}(\xi) - P_n^{(n+1)}(\xi) - K\Pi^{(n+1)}(\xi) \qquad (6.63)$$

Since $P_n(x)$ is a polynomial of degree n, its $(n + 1)$th derivative is zero. Also, from the definition of $\Pi(x)$, $\Pi^{(n+1)}(x) = (n + 1)!$. Therefore, Eq. (6.63) gives

$$K = \frac{y^{(n+1)}(\xi)}{(n+1)!}$$

Substituting the value of constant K into Eq. (6.62) gives

$$\varepsilon(x) = y(x) - P_n(x) = \frac{y^{(n+1)}(\xi)}{(n+1)!}\Pi(x) \tag{6.64}$$

for some $\xi = \xi(x)$ in the interval I. Incidentally, by equating Eqs. (6.61) and (6.64), we observe that

$$y[x, x_0, \ldots, x_n] = \frac{y^{(n+1)}(\xi)}{(n+1)!} \tag{6.65}$$

Thus, the error committed in replacing $y(x)$ by either Newton's divided difference formula or by an identical Lagrange's formula is given by

$$\varepsilon(x) = \Pi(x)y[x, x_0, \cdots, x_n] = \Pi(x)\frac{y^{(n+1)}(\xi)}{(n+1)!} \tag{6.66}$$

6.7 INTERPOLATION IN TWO DIMENSIONS

Let u be a polynomial function in two variables, say x and y, in particular quadratic in x and cubic in y, which in general can be written as

$$\begin{aligned} u = f(x, y) = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6y^3 \\ + a_7y^2x + a_8yx^2 + a_9y^3x + a_{10}y^2x^2 + a_{11}y^3x^2 \end{aligned} \tag{6.67}$$

This relation involves many terms. If we have to write a relation involving three or more variables, even low degree polynomials give rise to prohibitatively long expressions. If necessary, we can certainly write, but such complications can be avoided by handling each variable separately.

If we let x, a constant, say $x = c$, Eq. (6.67) immediately simplifies to the form

$$u\,|_{x=c} = b_0 + b_1y + b_2y^2 + b_3y^3 \tag{6.68}$$

Now, we adopt the following procedure to interpolate at a point (l, m) in a table of two variables, by treating one variable a constant say $x = x_1$. The problem reduces to that of a single variable interpolation. Any one of the methods discussed in preceding sections can then be applied to get $f(x_1, m)$. Then we repeat this procedure for various values of x say $x = x_2, x_3, \ldots, x_n$ keeping y constant. Thus, we get a new table with y constant at the value $y = m$ and with x varying. We can then interpolate at $x = l$. We shall illustrate this procedure by considering the following example.

Example 6.19 Tabulate the values of the function

$$f(x, y) = x^2 + y^2 - y$$

for $x = 0, 1, 2, 3, 4$ and $y = 0, 1, 2, 3, 4$. Using this table of values, compute $f(2.5, 3.5)$ by numerical double interpolation.

Solution The values of the function for the given values of x and y are given in the following table:

x	y = 0	1	2	3	4
0	0	0	2	6	12
1	1	1	3	7	13
2	4	4	6	10	16
3	9	9	11	15	21
4	16	16	18	22	28

Using quadratic interpolation in both x and y directions we need to consider three points in x and y directions. To start with, we have to treat one variable constant, say x. Keeping $x = 2.5$, $y = 3.5$ as the near centre of the set, we choose the table of values corresponding to $x = 1, 2, 3$ and $y = 2, 3, 4$. The region of fit for the construction of our interpolation polynomial is shown with dots in the above table.

Thus, using Newton's forward difference formula, we have

At $x = 1$

y	f	Δf	$\Delta^2 f$
2	3		
		4	
3	7		2
		6	
4	13		

At $x = 2$

y	f	Δf	$\Delta^2 f$
2	6		
		4	
3	10		2
		6	
4	16		

At $x = 3$

y	f	Δf	$\Delta^2 f$
2	11		
		4	
3	15		2
		6	
4	21		

with

$$p = \frac{y - y_0}{h} = \frac{3.5 - 2}{1} = 1.5$$

We obtain,

$$f(1, 3.5) = f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 = 3 + (1.5)(4) + \frac{(1.5)(0.5)}{2}(2) = 9.75$$

$$f(2, 3.5) = 6 + (1.5)(4) + \frac{(1.5)(0.5)}{2}(2) = 12.75$$

$$f(3, 3.5) = 11 + (1.5)(4) + \frac{(1.5)(0.5)}{2}(2) = 17.75$$

Therefore, we arrive at the following result

At $y = 3.5$

x	f	Δf	$\Delta^2 f$
1	9.75		
		3	
2	12.75		2
		5	
3	17.75		

Now, defining

$$p = \frac{2.5 - 1}{1} = 1.5$$

and using Newton's formula, we obtain

$$f(2.5, 3.5) = 9.75 + (1.5)(3) + \frac{(1.5)(0.5)}{2}(2) = 15$$

From the functional relation, we also find that

$$f(2.5, 3.5) = (2.5)^2 + (3.5)^2 - 3.5 = 15$$

and hence no error in interpolation.

6.8