6.5 LAGRANGE'S INTERPOLATION FORMULA

Newton's interpolation formulae developed in the earlier sections can be used only when the values of the independent variable x are equally spaced. Also the differences of y must ultimately become small. If the values of the independent variable are not given at equidistant intervals, then we have the basic formula associated with the name of *Lagrange* which is derived as follows:

Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x_0, x_1, x_2, \ldots, x_n$. Since there are $(n + 1)$ values of y corresponding to $(n + 1)$ values of x, we can represent the function $f(x)$ by a polynomial of degree n. Suppose we write this polynomial in the form

$$f(x) = A_0x^n + A_1x^{n-1} + \cdots + A_n$$

or, more conveniently, in the form

$$y = f(x) = a_0 (x - x_1) (x - x_2) \cdots (x - x_n) + a_1 (x - x_0) (x - x_2) \cdots (x - x_n) + a_2 (x - x_0) (x - x_1) \cdots (x - x_n) + \cdots + a_n(x - x_0) (x - x_1)\cdots(x - x_{n-1}) \tag{6.36}$$

Here, the coefficients a_k are so chosen as to satisfy Eq. (6.36) by the $(n + 1)$ pairs

(x_i, y_i). Thus, Eq. (6.36) yields

$$y_0 = f(x_0) = a_0 (x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)$$

Therefore,

$$a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)}$$

Similarly, we obtain

$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)}$$

$$a_i = \frac{y_i}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)}$$

and

$$a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})}$$

Now, substituting the values of $a_0, a_1, \dots, a_n$ into Eq. (6.36), we get

$$\begin{aligned} y = f(x) &= \frac{(x - x_1)(x - x_2) \cdots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)} y_0 \\ &+ \frac{(x - x_0)(x - x_2) \cdots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)} y_1 + \cdots \\ &+ \frac{(x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)} y_i + \cdots \\ &+ \frac{(x - x_0)(x - x_1)(x - x_2) \cdots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1)(x_n - x_2) \cdots (x_n - x_{n-1})} y_n \end{aligned} \tag{6.37}$$

Equation (6.37) is Lagrange's formula for interpolation. This formula can be used whether the values $x_0, x_1, x_2, \dots, x_n$ are equally spaced or not. Alternatively, Eq. (6.37) can also be written in compact form as

$$\begin{aligned} y = f(x) &= L_0(x)\, y_0 + L_1(x)\, y_1 + \dots + L_i(x)\, y_i + \dots + L_n(x)\, y_n \\ &= \sum_{k=0}^{n} L_k(x) y_k \\ &= \sum_{k=0}^{n} L_k(x) f(x_k) \end{aligned} \tag{6.38}$$

where,

$$L_i(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)} \tag{6.39}$$

we can easily observe that, $L_i(x_i) = 1$ and $L_i(x_j) = 0$, $i \neq j$. Thus introducing

Kronecker delta notation

$$L_i(x_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Further, if we introduce the notation

$$\Pi(x) = \prod_{i=0}^{n} (x - x_i) = (x - x_0)(x - x_1) \cdots (x - x_n) \tag{6.40}$$

that is, $\Pi(x)$ is a product of $(n + 1)$ factors. Clearly, its derivative $\Pi'(x)$ contains a sum of $(n + 1)$ terms in each of which one of the factors of $\Pi(x)$ will be absent. We also define,

$$P_k(x) = \prod_{i \neq k} (x - x_i) \tag{6.41}$$

which is same as $\Pi(x)$ except that the factor $(x - x_k)$ is absent. Then

$$\Pi'(x) = P_0(x) + P_1(x) + \cdots + P_n(x) \tag{6.42}$$

But, when $x = x_k$, all terms in the above sum vanishes except $P_k(x_k)$. Hence,

$$\Pi'(x_k) = P_k(x_k) = (x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n) \tag{6.43}$$

Therefore, using Eqs. (6.40)–(6.43), Eq. (6.39) can be rewritten as

$$L_k(x) = \frac{P_k(x)}{P_k(x_k)} = \frac{P_k(x)}{\Pi'(x_k)} = \frac{\Pi(x)}{(x - x_k)\Pi'(x_k)} \tag{6.44}$$

Finally, the Lagrange's interpolation polynomial of degree n can be written as

$$y(x) = f(x) = \sum_{k=0}^{n} \frac{\Pi(x)}{(x - x_k)\Pi'(x_k)} f(x_k) = \sum_{k=0}^{n} L_k(x) f(x_k) = \sum_{k=0}^{n} L_k(x) y_k \tag{6.45}$$

Lagrange's interpolation is illustrated through the following examples.

Example 6.14 Find Lagrange's interpolation polynomial fitting the points $y(1) = -3$, $y(3) = 0$, $y(4) = 30$, $y(6) = 132$. Hence find $y(5)$.

Solution The given data can be arranged as follows:

x	1	3	4	6
$y = f(x)$	-3	0	30	132

using Lagrange's interpolation formula (6.37), we have

$$y(x) = f(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)}(-3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}(0)$$

$$+ \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)}(30) + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}(132)$$

$$= \frac{x^3 - 13x^2 + 54x - 72}{-30}(-3) + \frac{x^3 - 11x^2 + 34x - 24}{6}(0)$$

$$+ \frac{x^3 - 10x^2 + 27x - 18}{-6}(30) + \frac{x^3 - 8x^2 + 19x - 12}{30}(132)$$

On simplification, we get

$$y(x) = \frac{1}{10}\left(-5x^3 + 135x^2 - 460x + 300\right) = \frac{1}{2}(-x^3 + 27x^2 - 92x + 60)$$

which is the required Lagrange's interpolation polynomial. Now, $y(5) = 75$.

Example 6.15 Given the following data, evaluate $f(3)$ using Lagrange's interpolating polynomial.

x	1	2	5
$f(x)$	1	4	10

Solution Using Lagrange's interpolation formula given by Eq. (6.37), we have

$$f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Therefore,

$$f(3) = \frac{(3 - 2)(3 - 5)}{(1 - 2)(1 - 5)}(1) + \frac{(3 - 1)(3 - 5)}{(2 - 1)(2 - 5)}(4) + \frac{(3 - 1)(3 - 2)}{(5 - 1)(5 - 2)}(10) = 6.4999$$

EXERCISES

6.1 Express $\Delta^2 y_1$ and $\Delta^4 y_0$ in terms of the values of the function y.

6.2 Compute the missing values of y_n and Δy_n in the following table

y_n	Δy_n	$\Delta^2 y_n$
–		
	–	
–		1
	–	
–		4
	5	
6		13
	–	
–		18
	–	
–		24
	–	
–		

6.3 Show that $E\nabla = \Delta = \delta E^{1/2}$.

6.4 Prove that (i) $\delta = 2 \sinh (hD/2)$ and, (ii) $\mu = 2 \cosh (hD/2)$.

6.5 Show that the operators δ, μ, E, Δ and ∇ commute with one another.

6.6 Explain the concept of linear interpolation. Using linear interpolation, find $f(3)$ for $f(x) = 5^x$. Compare with the actual value. Comment on the result obtained.

6.7 The following table gives pressure of a steam at a given temperature. Using Newton's formula, compute the pressure for a temperature of 142°C.

Temperature, °C	140	150	160	170	180
Pressure, kgf/cm^2	3.685	4.854	6.302	8.076	10.225

6.8 Find Newton's backward interpolating polynomial for the following data:

x	1	2	3	4	5
y	1	–1	1	–1	1

6.9 A second degree polynomial passes through (0, 1) (1, 3), (2, 7), and (3, 13). Find the polynomial, using Newton's forward difference formula.

6.10 The following data gives the melting point of an alloy of lead and zinc; where T is the temperature in °C and P is the percentage of lead in the alloy. Find the melting point of the alloy containing 84% of lead using Newton's interpolation method.

P	60	70	80	90
T	226	250	276	304

6.11 Find the interpolating polynomial for the function $f(x)$ given by

x	0	1	2	5
$y = f(x)$	2	3	12	147

6.12 Find the interpolating polynomial for the following data using Lagrange's formula

x	1	2	–4
$y = f(x)$	3	–5	4

6.13 Starting from Newton's divided difference interpolation formula (6.54) and making use of Eq. (6.47) and recalling the definitions of $\Pi(x)$ from Eq. (6.40), show that it can be reduced to Lagrange's form given by Eq. (6.38).

6.14 Find the interpolating polynomial by (i) Newton's divided difference formula (ii) Lagrange's formula, for the following data and hence show that both the methods give raise to the same polynomial.

x	1	2	3	5
y	0	7	26	124