Rel... differential operator and difference operator:

Show that $E = e^{hD}$, where $D = \frac{d}{dx}$

We know that:

$$Df(x) = \frac{d}{dx} f(x) = f'(x)$$

$$D^2 f(x) = \frac{d^2}{dx^2} f(x) = f''(x)$$

and so on

By Taylor's theorem:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots \quad (1)$$

We know that:

$$Ef(x) = f(x+h)$$

$$Ef(x) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots \quad \text{from (1)}$$

$$= f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \dots$$

$$Ef(x) = \left[1 + hD + \frac{h^2}{2!} D^2 + \dots\right] f(x)$$

$$E = 1 + hD + \frac{h^2}{2!} D^2 + \dots$$

$$\left\{ e^x = 1 + x + \frac{x^2}{2!} + \dots \right.$$

$$E = e^{hD} \quad \text{(by Maclarian)}$$

Also, $E = 1 + \Delta$

$$1 + \Delta = e^{hD}$$

or $$e^{hD} = 1 + \Delta$$

Following the definition of operators ∇ and E^{-1}, we have

$$\nabla y_x = y_x - y_{x-h} = y_x - E^{-1}y_x = (1 - E^{-1})\, y_x$$

Therefore,

$$\nabla = 1 - E^{-1} = \frac{E-1}{E} \tag{6.28}$$

The definition of operators δ and E gives

$$\delta y_x = y_{x+(h/2)} - y_{x-(h/2)} = E^{1/2}y_x - E^{-1/2}y_x = (E^{1/2} - E^{-1/2})\, y_x$$

Hence,

$$\delta = E^{1/2} - E^{-1/2} \tag{6.29}$$

The definition of μ and E similarly yields

$$\mu y_x = \frac{1}{2}[y_{x+(h/2)} + y_{x-(h/2)}] = \frac{1}{2}(E^{1/2} + E^{-1/2})y_x$$

Therefore,

$$\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right) \tag{6.30}$$

It is known that

$$Ey_x = y_{x+h} = f(x+h)$$

using Taylor series expansion, we have

$$Ey_x = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$

$$= f(x) + hDf(x) + \frac{h^2}{2!}D^2f(x) + \cdots$$

$$= \left(1 + \frac{hD}{1!} + \frac{h^2D^2}{2!} + \cdots\right) f(x) = e^{hD}\, y_x$$

Thus,

$$hD = \log E \tag{6.31}$$

Hence, all the operators are expressed in terms of E.

Example 6.5 Prove that

$$hD = \log(1 + \Delta) = -\log(1 - \nabla) = \sinh^{-1}(\mu\delta)$$

Solution Using the standard relations (6.27)–(6.31), we have

$$hD = \log E = \log(1 + \Delta) = -\log E^{-1} = -\log(1 - \nabla) \tag{1}$$

lso,

$$\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1}) = \frac{1}{2}(e^{hD} - e^{-hD}) = \sinh(hD)$$

Therefore,

$$hD = \sinh^{-1}(\mu\delta) \qquad (2)$$

Equations (1) and (2) constitute the required result.

Example 6.6 If Δ, ∇, δ denote forward, backward and central difference operators, E and μ are respectively the shift and average operators, in the analysis of data with equal spacing h, show that

(i) $1 + \delta^2\mu^2 = \left(1 + \dfrac{\delta^2}{2}\right)^2$ (ii) $E^{1/2} = \mu + \dfrac{\delta}{2}$

(iii) $\Delta = \dfrac{\delta^2}{2} + \delta\sqrt{1 + (\delta^2/4)}$ (iv) $\mu\delta = \dfrac{\Delta E^{-1}}{2} + \dfrac{\Delta}{2}$

(v) $\mu\delta = \dfrac{\Delta + \nabla}{2}$

Solutions (i) From the definition of operators, we have

$$\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1})$$

Therefore,

$$1 + \mu^2\delta^2 = 1 + \frac{1}{4}(E^2 - 2 + E^{-2}) = \frac{1}{4}(E + E^{-1})^2 \qquad (1)$$

Also,

$$1 + \frac{\delta^2}{2} = 1 + \frac{1}{2}(E^{1/2} - E^{-1/2})^2 = \frac{1}{2}(E + E^{-1}) \qquad (2)$$

From Eqs. (1) and (2), the first result follows.

(ii) Now

$$\mu + \frac{\delta}{2} = \frac{1}{2}(E^{1/2} + E^{-1/2} + E^{1/2} - E^{-1/2}) = E^{1/2}$$

Thus, the second result is proved.

(iii) We can write

$$\frac{\delta^2}{2} + \delta\sqrt{1 + (\delta^2/4)} = \frac{\left(E^{1/2} - E^{-1/2}\right)^2}{2}$$

$$+ \frac{\left(E^{1/2} - E^{-1/2}\right)\sqrt{1 + \frac{1}{4}\left(E^{1/2} - E^{-1/2}\right)^2}}{1}$$

$$= \frac{E - 2 + E^{-1}}{2} + \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)\left(E^{1/2} + E^{-1/2}\right)$$

$$= \frac{E - 2 + E^{-1}}{2} + \frac{E - E^{-1}}{2}$$

$$= E - 1$$

Using Eq. (6.27), we get

$$E - 1 = \Delta$$

(iv) We have

$$\mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right) = \frac{1}{2}\left(E - E^{-1}\right)$$

Now, using Eq. (6.27), we get

$$= \frac{1}{2}\left(1 + \Delta - E^{-1}\right) = \frac{\Delta}{2} + \frac{1}{2}(1 - E^{-1})$$

$$= \frac{\Delta}{2} + \frac{1}{2}\left(\frac{E - 1}{E}\right) = \frac{\Delta}{2} + \frac{\Delta}{2E}$$

(v) We can write

$$\mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right) = \frac{1}{2}\left(E - E^{-1}\right)$$

Now using Eqs. (6.27) and (6.28), we have

$$\mu\delta = \frac{1}{2}(1 + \Delta - 1 + \nabla) = \frac{1}{2}(\Delta + \nabla)$$

Example 6.7 Show that the operators μ and E commute.

Solution From the definition of operators μ and E, we have

$$\mu E y_0 = \mu y_1 = \frac{1}{2}(y_{3/2} + y_{1/2}) \tag{1}$$

While

$$E\mu y_0 = \frac{1}{2}E\,(y_{1/2} + y_{-1/2}) = \frac{1}{2}(y_{3/2} + y_{1/2}) \tag{2}$$

Equating (1) and (2), we have

$$\mu E = E\mu$$

Therefore, the operators μ and E commute.

Theorem 6.1 (Differences of a polynomial). The nth differences of a polynomial of degree n is constant, when the values of the independent variable are given at equal intervals.

Proof Let us consider a polynomial of degree n in the form

$$y_x = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n,$$

where $a_0 \neq 0$ and $a_0, a_1, a_2, \ldots, a_n$ are constants. Let h be the interval of differencing. Then

$$y_{x+h} = a_0(x + h)^n + a_1(x + h)^{n-1} + a_2(x + h)^{n-2} + \ldots + a_{n-1}(x + h) + a_n$$

We now examine the differences of the polynomial:

$$\Delta y_x = y_{x+h} - y_x = a_0[(x + h)^n - x^n] + a_1[(x + h)^{n-1} - x^{n-1}]$$

$$+ a_2[(x + h)^{n-2} - x^{n-2}] + \ldots + a_{n-1}(x + h - x)$$

Binomial expansion yields

$$\Delta y_x = a_0(x^n + {}^nC_1x^{n-1}h + {}^nC_2x^{n-2}h^2 + \cdots + h^n - x^n)$$
$$+ a_1[x^{n-1} + {}^{(n-1)}C_1x^{n-2}h + {}^{(n-1)}C_2x^{n-3}h^2 + \cdots + h^{n-1} - x^{n-1}] + \cdots$$
$$+ a_{n-1}h$$
$$= a_0nhx^{n-1} + [a_0{}^nC_2h^2 + a_1{}^{(n-1)}C_1h]x^{n-2} + \cdots + a_{n-1}h$$

Therefore,

$$\Delta y_x = a_0nhx^{n-1} + b'x^{n-2} + c'x^{n-3} + \cdots + k'x + l'$$

where b', c', ..., k', l' are constants involving h but not x. Thus, the first difference of a polynomial of degree n is another polynomial of degree $(n - 1)$. Similarly

$$\Delta^2 y_x = \Delta(\Delta y_x) = \Delta y_{x+h} - \Delta y_x$$
$$= a_0nh[(x + h)^{n-1} - x^{n-1}] + b'[(x + h)^{n-2} - x^{n-2}] + \cdots$$
$$+ k'(x + h - x)$$

On simplification, it reduces to the form

$$\Delta^2 y_x = a_0n(n - 1)\, h^2x^{n-2} + b''x^{n-3} + c''x^{n-4} + \cdots + q''$$

Therefore, $\Delta^2 y_x$ is a polynomial of degree $(n - 2)$ in x. Similarly, we can form the higher order differences, and every time we observe that the degree of the polynomial is reduced by one. After differencing n times, we are left with only the first term in the form

$$\Delta^n y_x = a_0n(n - 1)\,(n - 2)\,\ldots\,(2)(1)\, h^n = a_0\,(n!)h^n = \text{Constant}$$

This constant is independent of x. Since $\Delta^n y_x$ is a constant, $\Delta^{n+1} y_x = 0$. Hence the $(n + 1)$th and higher order differences of a polynomial of degree n are zero.

6.3 NEWTON'S FORWARD DIFFERENCE INTERPOLATION FORMULA

Let $y = f(x)$ be a function which takes values $f(x_0), f(x_0 + h), f(x_0 + 2h), \ldots,$ corresponding to various equispaced values of x with spacing h, say x_0, $x_0 + h$, $x_0 + 2h$, ... Suppose, we wish to evaluate the function $f(x)$ for a value $x_0 + ph$, where p is any real number, then for any real number p, we have the operator E such that $E^pf(x) = f(x + ph)$. Therefore, using Eq. (6.27) we have

$$f(x_0 + ph) = E^pf(x_0) = (1+ \Delta)^pf(x_0)$$
$$= \left[1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots\right]f(x_0)$$

That is,

$$f(x_0 + ph) = f(x_0) + p\Delta f(x_0) + \frac{p(p - 1)}{2!}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 f(x_0) + \cdots$$
$$+ \frac{p(p - 1)\cdots(p - n + 1)}{n!}\Delta^n f(x_0) + \text{Error} \tag{6.32}$$

This is known as *Newton's forward difference formula for interpolation*, which gives the value of $f(x_0 + ph)$ in terms of $f(x_0)$ and its leading differences. This formula is also known as *Newton–Gregory forward difference interpolation formula*. Here, $p = (x - x_0)/h$. Equation (6.32) can also be written in another alternate form as

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$+ \frac{p(p-1)(p-n+1)}{n!}\Delta^n y_0 + \text{Error} \qquad (6.33)$$

If we retain $(r + 1)$ terms in Eq. (6.33), we obtain a polynomial of degree r agreeing with y_x at $x_0, x_1, \ldots, x_r$.

This formula is mainly used for interpolating the values of y near the beginning of a set of tabular values and for extrapolating values of y, a short distance backward from y_0. We shall illustrate these formulae by considering the following simple examples.

Example 6.8 Evaluate $f(15)$, given the following table of values:

x	10	20	30	40	50
$y = f(x)$	46	66	81	93	101

Solution We may note that $x = 15$ is very near to the beginning of the table. Hence, we use Newton's forward difference interpolation formula. The forward differences are calculated and tabulated as given below:

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	46				
		20			
20	66		−5		
		15		2	
30	81		−3		−3
		12		−1	
40	93		−4		
		8			
50	101				

We have Newton's forward difference interpolation formula as

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 \qquad (1)$$

In this example, from the above table, we have

$x_0 = 10,\quad y_0 = 46,\quad \Delta y_0 = 20,\quad \Delta^2 y_0 = -5,\quad \Delta^3 y_0 = 2,\quad \Delta^4 y_0 = -3$

Let y_{15} be the value of y when $x = 15$, then

$$p = \frac{x - x_0}{h} = \frac{15 - 10}{10} = 0.5$$

we can also find 'p' as
Put $x_0 + ph = 15$
$ph = 15 - x_0$
$p = \frac{15 - x_0}{h}$
'p' $= \frac{15-10}{10} = 0.5$

Substituting these values in Eq. (1), we get

$$f(15) = y_{15} = 46 + (0.5)(20) + \frac{(0.5)(0.5 - 1)}{2}(-5)$$

$$+ \frac{(0.5)(0.5 - 1)(0.5 - 2)}{6}(2) + \frac{(0.5)(0.5 - 1)(0.5 - 2)(0.5 - 3)}{24}(-3)$$

$$= 46 + 10 + 0.625 + 0.125 + 0.1172$$

Therefore, $f(15) = 56.8672$ correct to four decimal places.

Example 6.9 Find Newton's forward difference interpolating polynomial for the following data:

x	0.1	0.2	0.3	0.4	0.5
$y = f(x)$	1.40	1.56	1.76	2.00	2.28

Solution We shall first construct the forward difference table to the given data as indicated below:

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.20		0.00	
0.3	1.76		0.04		0.00
		0.24		0.00	
0.4	2.00		0.04		
		0.28			
0.5	2.28				

Since, third and fourth leading differences are zero, we have Newton's forward difference interpolating formula as

$$y = y_0 + p\Delta y_0 + \frac{p(p - 1)}{2}\Delta^2 y_0 \qquad (1)$$

In this problem, $x_0 = 0.1$, $y_0 = 1.40$, $\Delta y_0 = 0.16$, $\Delta^2 y_0 = 0.04$, and

$$p = \frac{x - 0.1}{0.1} = 10x - 1$$

Substituting these values in Eq. (1), we obtain

$$y = f(x) = 1.40 + (10x - 1)(0.16) + \frac{(10x - 1)(10x - 2)}{2}(0.04)$$

That is, $y = 2x^2 + x + 1.28$. This is the required Newton's interpolating polynomial.

Example 6.10 Estimate the missing figure in the following table:

x	1	2	3	4	5
$y = f(x)$	2	5	7	–	32

Solution Since we are given four entries in the table, the function $y = f(x)$ can be represented by a polynomial of degree three. Using Theorem 6.1, we have

$$\Delta^3 f(x) = \text{Constant} \quad \text{and} \quad \Delta^4 f(x) = 0$$

for all x. In particular, $\Delta^4 f(x_0) = 0$. Equivalently, $(E-1)^4 f(x_0) = 0$. Expanding, we have

$$(E^4 - 4E^3 + 6E^2 - 4E + 1)\, f(x_0) = 0$$

That is,

$$f(x_4) - 4f(x_3) + 6f(x_2) - 4f(x_1) + f(x_0) = 0$$

Using the values given in the table, we obtain

$$32 - 4f(x_3) + 6 \times 7 - 4 \times 5 + 2 = 0$$

which gives $f(x_3)$, the missing value equal to 14.

Example 6.11 Find a cubic polynomial in x which takes on the values –3, 3, 11, 27, 57 and 107, when x = 0, 1, 2, 3, 4 and 5 respectively.

Solution Here, the observations are given at equal intervals of unit width. To determine the required polynomial, we first construct the difference table as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	–3			
		6		
1	3		2	
		8		6
2	11		8	
		16		6
3	27		14	
		30		6
4	57		20	
		50		
5	107			

Since the fourth and higher order differences are zero, we have the required Newton's interpolation formula in the form

$$f(x_0 + ph) = f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{6}\Delta^3 f(x_0) \tag{1}$$

Here,

$$p = \frac{x - x_0}{h} = \frac{x - 0}{1} = x, \quad \Delta f(x_0) = 6, \quad \Delta^2 f(x_0) = 2. \quad \Delta^3 f(x_0) = 6$$

Substituting these values into Eq. (1), we have

If one wishes to interpolate the value of the function $y = f(x)$ near the end of table of values, and to extrapolate value of the function a short distance forward from y_n, Newton's backward interpolation formula is used, which can be derived as follows:

Let $y = f(x)$ be a function which takes on values $f(x_n), f(x_n - h), f(x_n - 2h), \ldots, f(x_0)$ corresponding to equispaced values $x_n, x_n - h, x_n - 2h, \ldots, x_0$. Suppose, we wish to evaluate the function $f(x)$ at $(x_n + ph)$, where p is any real number, then we have the shift operator E, such that

$$f(x_n + ph) = E^p f(x_n) = (E^{-1})^{-p} f(x_n) = (1 - \nabla)^{-p} f(x_n)$$

Binomial expansion yields,

$$f(x_n + ph) = \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \cdots + \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n + \text{Error}\right] f(x_n)$$

That is,

$$f(x_n + ph) = f(x_n) + p\nabla f(x_n) + \frac{p(p+1)}{2!}\nabla^2 f(x_n) + \frac{p(p+1)(p+2)}{3!}\nabla^3 f(x_n) + \cdots + \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n f(x_n) + \text{Error} \qquad (6.34)$$

This formula is known as *Newton's backward interpolation formula*. This formula is also known as *Newton–Gregory backward difference interpolation formula*. If we retain $(r + 1)$ terms in Eq. (6.34), we obtain a polynomial of degree r agreeing with $f(x)$ at $x_n, x_{n-1}, \ldots, x_{n-r}$. Alternatively, this formula can also be written as

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots + \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n y_n + \text{Error} \qquad (6.35)$$

where

$$p = \frac{x - x_n}{h}$$

Here follows a couple of examples for illustration.

Example 6.12 For the following table of values, estimate $f(7.5)$.

x	1	2	3	4	5	6	7	8
$y = f(x)$	1	8	27	64	125	216	343	512

Solution The value to be interpolated is at the end of the table. Hence, it is appropriate to use Newton's backward interpolation formula. We shall first construct the backward difference table for the given data:

x	$y = f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		0
		91		6	
6	216		36		0
		127		6	
7	343		42		
		169			
8	512				

Since the fourth and higher order differences are zero, the required Newton's backward interpolation formula is

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

In this problem,

$$p = \frac{x - x_n}{h} = \frac{7.5 - 8.0}{1} = -0.5$$

and

$$\nabla y_n = 169, \quad \nabla^2 y_n = 42, \quad \nabla^3 y_n = 6$$

Therefore,

$$y_{7.5} = 512 + (-0.5)(169) + \frac{(-0.5)(0.5)}{2}(42) + \frac{(-0.5)(0.5)(1.5)}{6}(6)$$

$$= 512 - 84.5 - 5.25 - 0.375$$

$$= 421.875$$

Example 6.13 The sales in a particular department store for the last five years is given in the following table:

Year	1974	1976	1978	1980	1982
Sales (in lakhs)	40	43	48	52	57

Estimate the sales for the year 1979.

Solution At the outset, we shall construct Newton's backward difference table for the given data as

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1974	40				
		3			
1976	43		2		
		5		−3	
1978	48		−1		5
		4		2	
1980	52		1		
		5			
1982	57				

In this example,

$$p = \frac{1979 - 1982}{2} = -1.5$$

and

$$\nabla y_n = 5, \quad \nabla^2 y_n = 1, \quad \nabla^3 y_n = 2, \quad \nabla^4 y_n = 5$$

Newton's backward interpolation formula gives

$$y_{1979} = 57 + (-1.5)5 + \frac{(-1.5)(-0.5)}{2}(1)$$

$$+ \frac{(-1.5)(-0.5)(0.5)}{6}(2) + \frac{(-1.5)(-0.5)(0.5)(1.5)}{24}(5)$$

$$= 57 - 7.5 + 0.375 + 0.125 + 0.1172$$

Therefore,

$$y_{1979} = 50.1172$$