20 Jan

Shift Operator (E):

The operator E is defined as

$$E f(x) = f(x+h)$$

where,

h is an interval of difference

$$E^2 f(x) = E[E f(x)]$$

$$= E[f(x+h)]$$

$$E^2 f(x) = f(x+2h)$$ $E f(x) = f(x+h)$

Similarly,

$$E^3 f(x) = f(x+3h)$$

$$E^4 f(x) = f(x+4h)$$

$$\vdots \qquad \vdots$$

$$E^n f(x) = f(x+nh)$$

The operator E^{-1} is defined by,

$$E^{-1} f(x) = f(x-h)$$

$$E^{-2} f(x) = f(x-2h)$$

$$E^{-3} f(x) = f(x-3h)$$

$$\vdots \qquad \vdots$$

$$E^{-n} f(x) = f(x-nh)$$

Note:

$E^0 = 1 \rightarrow$ identity operator

$\Delta^0 = 1$

Relationship between the Operators:

Q: Prove that $E = 1 + \Delta$

We know that:

$$\Delta f(x) = f(x+h) - f(x)$$
$$= Ef(x) - f(x) \qquad \because Ef(x) = f(x+h)$$
$$= (E-1) f(x)$$
$$\Delta f(x) = (E-1) f(x)$$
$$\Delta = E - 1$$

Hence,

$$E = 1 + \Delta$$

Proved

Q: Prove that $E^{-1} = 1 - \nabla$

We know that:

$$\nabla f(x) = f(x) - f(x-h)$$
$$= f(x) - E^{-1} f(x)$$
$$\nabla f(x) = [1 - E^{-1}] f(x)$$
$$\nabla = 1 - E^{-1}$$
$$E^{-1} = 1 - \nabla$$

Proved

Q: Prove that $\Delta\nabla = \nabla\Delta$

L.H.S

$$\Delta\nabla = (E-1)(1-E^{-1})$$
$$= E - EE^{-1} - 1 + E^{-1}$$
$$= E - 1 + E^{-1}$$
$$\Delta\nabla = E + E^{-1} - 1$$

R.H.S

$$\nabla\Delta = (1-E^{-1})(E-1)$$
$$= E - 1 - E^{-1}E + E^{-1}$$
$$\nabla\Delta = E + E^{-1} - 1$$

L.H.S = R.H.S

Q: Prove that $\Delta E = E\Delta$

L.H.S

$\Delta E = (E-1)(1+\Delta)$ $\qquad E = 1+\Delta$, $\Delta = E-1$

$= E + E\Delta - 1 - \Delta$

$= E + E\Delta - 1 - (E-1)$

$= \cancel{E} + E\Delta \cancel{-1} - \cancel{E} + \cancel{1}$

$= E\Delta$

$=$ R.H.S

OR

L.H.S

$\Delta E = (E-1)(1+\Delta)$

$= E + E\Delta - 1 - \Delta$

$= E + E(E-1) - 1 - (E-1)$

$= \cancel{E} + E^2 - E \cancel{-1} - \cancel{E} + \cancel{1}$

$= E^2 - E$

$= E(E-1)$

$= E\Delta$

$=$ R.H.S

Proved

Q: Prove that $\Delta E^{-1} = E^{-1}\Delta$

L.H.S

$= \Delta E^{-1}$ $\qquad E = 1+\Delta$, $\Delta = E-1$, $E^{-1} = 1-\nabla$

$= (E-1)(1-\nabla)$

$= E - E\nabla - 1 + \nabla$

$= E - E(1-E^{-1}) - 1 + (1-E^{-1})$

$= \cancel{E} - \cancel{E} + EE^{-1} \cancel{-1} + \cancel{1} - E^{-1}$

$= EE^{-1} - E^{-1}$

$= E^{-1}(E-1)$

$= E^{-1}\Delta$

$=$ R.H.S

Proved

2nd Feb, 2015 Monday

3) Central Difference Operator (δ):

Prove $\delta = E^{1/2} - E^{-1/2}$

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

$E' f(x) = f(x+h)$

$$\delta f(x) = E^{1/2} f(x) - E^{-1/2} f(x)$$

$$\delta f(x) = (E^{1/2} - E^{-1/2}) f(x)$$

Hence,

$$\delta = E^{1/2} - E^{-1/2}$$

Average Difference Operator (μ):

Prove : $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

The average difference operator is defined as :

$$\mu f(x) = \frac{f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)}{2}$$

$$= \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\left[E^{1/2} f(x) + E^{-1/2} f(x)\right]$$

$$\mu f(x) = \frac{1}{2}(E^{1/2} + E^{-1/2}) f(x)$$

Hence,

$$\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$$

imp

Q: Prove that $\mu = \sqrt{1+\frac{\delta^2}{4}}$ or $\mu^2 = 1+\frac{\delta^2}{4}$

We know that:

$$\mu f(x) = \frac{f(x+\frac{h}{2}) + f(x-\frac{h}{2})}{2}$$

$$= \frac{1}{2}\left[f(x+\tfrac{h}{2}) + f(x-\tfrac{h}{2})\right]$$

$$= \frac{1}{2}\left[E^{1/2} f(x) + E^{-1/2} f(x)\right]$$

$$\mu f(x) = \frac{1}{2}(E^{1/2} + E^{-1/2}) f(x)$$

$$\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$$

$$\mu^2 = \left[\frac{1}{2}(E^{1/2} + E^{-1/2})\right]^2$$

$$= \frac{1}{4}\left(E + E^{-1} + 2E^{1/2}\cdot E^{-1/2}\right)$$

$$= \frac{1}{4}(E + E^{-1} + 2)$$

$$= \frac{1}{4}(E + E^{-1} - 2 + 2 + 2)$$

$$= \frac{1}{4}\left[(E^{1/2} - E^{-1/2})^2 + 4\right]$$

$$= \frac{1}{4}(\delta^2 + 4) \qquad \because \delta = E^{1/2} - E^{-1/2}$$

$$= \frac{\delta^2}{4} + \frac{4}{4}$$

$$\mu^2 = 1 + \frac{\delta^2}{4}$$

$$\mu = \sqrt{1+\frac{\delta^2}{4}}$$

OR $\longrightarrow$

R.H.S

$$= \sqrt{1 + \frac{\delta^2}{4}}$$

$$= \sqrt{1 + \frac{(E^{1/2} - E^{-1/2})^2}{4}}$$

$$= \sqrt{\frac{4 + E + E^{-1} - 2E^{1/2} \cdot E^{-1/2}}{4}}$$

$$= \sqrt{\frac{4 + E + E^{-1} - 2}{4}}$$

$$= \sqrt{\frac{E + E^{-1} + 2}{4}}$$

$$= \sqrt{\frac{(E^{1/2} + E^{-1/2})^2}{4}}$$

$$= \frac{E^{1/2} + E^{-1/2}}{2}$$

$$= \mu$$

$$= \text{L.H.S}$$

Q: Prove that

$$\Delta = \nabla E = \delta E^{1/2} = E\nabla$$

$E = 1 + \Delta$

$\rightarrow \quad \Delta = E - 1$

$\rightarrow \quad \nabla E = (1 - E^{-1}) E$

$$= E - EE^{-1}$$

$$\cancel{= E(1 + E^{-1})}$$

$$\nabla E = E - 1$$

$\rightarrow \quad \delta E^{1/2} = (E^{1/2} - E^{-1/2}) E^{1/2}$

$$= E^{1/2} E^{1/2} - E^{1/2} \cdot E^{-1/2}$$

$$\delta E^{1/2} = E - 1$$

$\rightarrow \quad E\nabla = E(1 - E^{-1})$

$$= E - EE^{-1}$$

$$E\nabla = E - 1$$

Hence,

$$\Delta = \nabla E = \delta E^{1/2} = E\nabla$$

Relation between differential operator and difference operator:

or

Show that $E = e^{hD}$, where $D = \frac{d}{dx}$

We know that:

$$Df(x) = \frac{d}{dx} f(x) = f'(x)$$

$$D^2 f(x) = \frac{d^2}{dx^2} f(x) = f''(x)$$

and so on

By Taylor's theorem:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots \quad \text{—} (1)$$

We know that:

$$Ef(x) = f(x+h)$$

$$Ef(x) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots \quad \text{from } (1)$$

$$= f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \cdots$$

$$Ef(x) = \left[1 + hD + \frac{h^2}{2!} D^2 + \cdots\right] f(x)$$

$$E = 1 + hD + \frac{h^2}{2!} D^2 + \cdots \qquad \left\{ e^x = 1 + x + \frac{x^2}{2!} + \cdots \right.$$

$$E = e^{hD} \qquad \text{(by Maclarian)}$$

Also, $E = 1 + \Delta$

$$1 + \Delta = e^{hD}$$

or $$e^{hD} = 1 + \Delta$$

Q: Show that

i) $\Delta\nabla = \nabla\Delta = \Delta - \nabla = \delta^2$

$$\Delta\nabla = (E-1)(1-E^{-1})$$
$$= E - EE^{-1} - 1 + E^{-1}$$
$$= E - 1 - 1 + E^{-1}$$
$$= E + E^{-1} - 2$$
$$= (E^{1/2} - E^{-1/2})^2$$
$$\Delta\nabla = \delta^2$$

$$\nabla\Delta = (1-E^{-1})(E-1)$$
$$= E - 1 - EE^{-1} + E^{-1}$$
$$= E - 1 - 1 + E^{-1}$$
$$= E - 2 + E^{-1}$$
$$= (E^{1/2} - E^{-1/2})^2$$
$$\nabla\Delta = \delta^2$$

$$\Delta - \nabla = (E-1) - (1-E^{-1})$$
$$= E - 1 - 1 + E^{-1}$$
$$= E + E^{-1} - 2$$
$$= (E^{1/2} - E^{-1/2})^2$$
$$\Delta - \nabla = \delta^2$$

Hence,

$$\Delta\nabla = \nabla\Delta = \Delta - \nabla = \delta^2$$

(ii) $\Delta + \nabla = \dfrac{\Delta}{\nabla} - \dfrac{\nabla}{\Delta}$

R.H.S

$$= \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$$
$$= \frac{\Delta^2 - \nabla^2}{\nabla\Delta}$$
$$= \frac{\cancel{(\Delta - \nabla)}(\Delta + \nabla)}{\cancel{(\Delta - \nabla)}} \qquad \because \nabla\Delta = \Delta - \nabla$$
$$= \Delta + \nabla = \text{L.H.S}$$

Friday

(vi) $\Delta = \frac{\delta^2}{2} + \delta\sqrt{1+\frac{\delta^2}{4}}$

R.H.S

$= \frac{\delta^2}{2} + \delta\sqrt{1+\frac{\delta^2}{4}}$

$= \frac{(E^{1/2}-E^{-1/2})^2}{2} + (E^{1/2}-E^{-1/2})\sqrt{1+\frac{(E^{1/2}-E^{-1/2})^2}{4}}$

$= \frac{E+E^{-1}-2}{2} + (E^{1/2}-E^{-1/2})\sqrt{\frac{4+E+E^{-1}-2}{4}}$

$= \frac{E+E^{-1}-2}{2} + (E^{1/2}-E^{-1/2})\sqrt{\frac{E+E^{-1}+2}{4}}$

$= \frac{E+E^{-1}-2}{2} + (E^{1/2}-E^{-1/2})\sqrt{\left(\frac{E^{1/2}+E^{-1/2}}{2}\right)^2}$

$= \frac{E+E^{-1}-2}{2} + \frac{(E^{1/2}-E^{-1/2})(E^{1/2}+E^{-1/2})}{2}$

$= \frac{E+E^{-1}-2}{2} + \frac{E+\cancel{1}-\cancel{1}-E^{-1}}{2}$

$= \frac{E+\cancel{E^{-1}}-2+E-\cancel{E^{-1}}}{2}$

$= \frac{2E-2}{2}$

$= \frac{\cancel{2}(E-1)}{\cancel{2}}$

$= \Delta \qquad \because E = 1+\Delta$

$= L.H.S$

Proved.

(vii) $\Delta = \frac{1}{2}\delta^2 + \delta$

or

R.H.S

$$= \frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}}$$

$$= \frac{1}{2}\delta^2 + \delta\mu \qquad \because \mu = \sqrt{1+\frac{\delta^2}{4}}$$

$$= \frac{\Delta-\nabla}{2} + \frac{\Delta+\nabla}{2} \qquad \because \delta^2 = \Delta-\nabla, \quad \delta\mu = \frac{1}{2}(\Delta+\nabla)$$

$$= \frac{\Delta-\cancel{\nabla}+\Delta+\cancel{\nabla}}{2}$$

$$= \frac{\cancel{2}\Delta}{\cancel{2}}$$

$$= \Delta$$

$= L.H.S$

- Proved.

(vii) $\mu = \frac{2+\Delta}{2\sqrt{1+\Delta}}$

R.H.S $= \frac{2+\Delta}{2\sqrt{1+\Delta}}$

$$= \frac{2+E-1}{2\sqrt{\cancel{1}+E-\cancel{1}}} \qquad \because E = 1+\Delta$$

$$= \frac{1+E}{2\sqrt{E}}$$

$$= \frac{1}{2\sqrt{E}} + \frac{E}{2\sqrt{E}}$$

$$= \frac{E^{-1/2}}{2} + \frac{E^{1/2}}{2}$$

$$= \frac{E^{-1/2}+E^{1/2}}{2}$$

$= \mu$

$= L.H.S$