6.2.3 Central Differences

In some applications, central difference notation is found to be more convenient to represent the successive differences of a function. Here, we use the symbol δ to represent central difference operator and the subscript of δy for any difference as the average of the subscripts of the two members of the difference. Thus, we write

$$\delta y_{1/2} = y_1 - y_0, \qquad \delta y_{3/2} = y_2 - y_1, \text{ etc.}$$

In general

$$\delta y_i = y_{i+(1/2)} - y_{i-(1/2)} \tag{6.16}$$

Higher order differences are defined as follows:

$$\delta^2 y_i = \delta y_{i+(1/2)} - \delta y_{i-(1/2)} \tag{6.17}$$

$$\delta^n y_i = \delta^{n-1} y_{i+(1/2)} - \delta^{n-1} y_{i-(1/2)} \tag{6.18}$$

These central differences can be systematically arranged as indicated in Table 6.3:

Table 6.3 Central Difference Table

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$	$\delta^6 y$
x_0	y_0						
		$\delta y_{1/2}$					
x_1	y_1		$\delta^2 y_1$				
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$		
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$		$\delta^6 y_3$
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		$\delta^5 y_{7/2}$	
x_4	y_4		$\delta^2 y_4$		$\delta^4 y_4$		
		$\delta y_{9/2}$		$\delta^3 y_{9/2}$			
x_5	y_5		$\delta^2 y_5$				
		$\delta y_{11/2}$					
x_6	y_6						

Thus, we observe that all the odd differences have a fractional suffix and all the even differences with the same subscript lie horizontally.

The following alternative notation may also be adopted to introduce finite difference operators. Let $y = f(x)$ be a functional relation between x and y, which is also denoted by y_x. Suppose, we are given consecutive values of x differing by h say $x, x + h, x + 2h, x + 3h$, etc. The corresponding values of y are $y_x, y_{x+h}, y_{x+2h}, y_{x+3h}$, etc. As before, we can form the differences of these values. Thus

$$\Delta y_x = y_{x+h} - y_x = f(x+h) - f(x) \tag{6.19}$$

Similarly

$$\Delta^2 y_x = \Delta y_{x+h} - \Delta y_x$$

and

$$\nabla y_x = y_x - y_{x-h} = f(x) - f(x-h) \tag{6.20}$$

$$\delta y_x = y_{x+(h/2)} - y_{x-(h/2)} = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) \tag{6.21}$$

Shift operator, E

Let $y = f(x)$ be a function of x, and let x takes the consecutive values x, $x + h$, $x + 2h$, etc. We then define an operator E having the property

$$E f(x) = f(x + h) \tag{6.22}$$

Thus, when E operates on $f(x)$, the result is the next value of the function. Here, E is called the *shift operator*. If we apply the operator E twice on $f(x)$, we get

$$E^2 f(x) = E[E f(x)] = E[f(x + h)] = f(x + 2h)$$

Thus, in general, if we apply the operator E n times on $f(x)$, we arrive at

$$E^n f(x) = f(x + nh)$$

In terms of new notation, we can write

$$E^n y_x = y_{x+nh}$$

or

$$E^n f(x) = f(x + nh) \tag{6.23}$$

for all real values of n. Also, if $y_0, y_1, y_2, y_3, \ldots$ are the consecutive values of the function y_x, then we can also write

$$Ey_0 = y_1, \quad E^2 y_0 = y_2, \quad E^4 y_0 = y_4, \quad \ldots, \quad E^2 y_2 = y_4$$

and so on. The inverse operator E^{-1} is defined as

$$E^{-1} f(x) = f(x - h)$$

and similarly

$$E^{-n} f(x) = f(x - nh) \tag{6.24}$$

Average operator, μ

The average operator μ is defined as

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right] = \frac{1}{2}[y_{x+(h/2)} + y_{x-(h/2)}] \tag{6.25}$$

Differential operator, D

It is known that D represents a differential operator having a property

$$\left.\begin{aligned} Df(x) &= \frac{d}{dx} f(x) = f'(x) \\ D^2 f(x) &= \frac{d^2}{dx^2} f(x) = f''(x) \end{aligned}\right\} \tag{6.26}$$

Having defined various difference operators Δ, ∇, δ, E, μ and D, we can obtain the following relations easily:

From the definition of operators Δ and E, we have

$$\Delta y_x = y_{x+h} - y_x = Ey_x - y_x = (E - 1)y_x$$

Therefore,

$$\Delta = E - 1 \tag{6.27}$$

Following the definition of operators ∇ and E^{-1}, we have

$$\nabla y_x = y_x - y_{x-h} = y_x - E^{-1}y_x = (1 - E^{-1})\, y_x$$

Therefore,

$$\nabla = 1 - E^{-1} = \frac{E-1}{E} \tag{6.28}$$

The definition of operators δ and E gives

$$\delta y_x = y_{x+(h/2)} - y_{x-(h/2)} = E^{1/2}y_x - E^{-1/2}y_x = (E^{1/2} - E^{-1/2})\, y_x$$

Hence,

$$\delta = E^{1/2} - E^{-1/2} \tag{6.29}$$

The definition of μ and E similarly yields

$$\mu y_x = \frac{1}{2}[y_{x+(h/2)} + y_{x-(h/2)}] = \frac{1}{2}(E^{1/2} + E^{-1/2})y_x$$

Therefore,

$$\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right) \tag{6.30}$$

It is known that

$$Ey_x = y_{x+h} = f(x+h)$$

using Taylor series expansion, we have

$$Ey_x = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$

$$= f(x) + hDf(x) + \frac{h^2}{2!}D^2f(x) + \cdots$$

$$= \left(1 + \frac{hD}{1!} + \frac{h^2D^2}{2!} + \cdots\right) f(x) = e^{hD}\, y_x$$

Thus,

$$hD = \log E \tag{6.31}$$

Hence, all the operators are expressed in terms of E.

Example 6.5 Prove that

$$hD = \log(1 + \Delta) = -\log(1 - \nabla) = \sinh^{-1}(\mu\delta)$$

Solution Using the standard relations (6.27)–(6.31), we have

$$hD = \log E = \log(1 + \Delta) = -\log E^{-1} = -\log(1 - \nabla) \tag{1}$$

Also,

$$\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1}) = \frac{1}{2}(e^{hD} - e^{-hD}) = \sinh(hD)$$

Therefore,

$$hD = \sinh^{-1}(\mu\delta)$$

Equations (1) and (2) constitute the required result. (2)

Example 6.6 If Δ, ∇, δ denote forward, backward and central difference operators, E and μ are respectively the shift and average operators, in the analysis of data with equal spacing h, show that

(i) $1 + \delta^2\mu^2 = \left(1 + \dfrac{\delta^2}{2}\right)^2$ (ii) $E^{1/2} = \mu + \dfrac{\delta}{2}$

(iii) $\Delta = \dfrac{\delta^2}{2} + \delta\sqrt{1 + (\delta^2/4)}$ (iv) $\mu\delta = \dfrac{\Delta E^{-1}}{2} + \dfrac{\Delta}{2}$

(v) $\mu\delta = \dfrac{\Delta + \nabla}{2}$

Solutions (i) From the definition of operators, we have

$$\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1})$$

Therefore,

$$1 + \mu^2\delta^2 = 1 + \frac{1}{4}(E^2 - 2 + E^{-2}) = \frac{1}{4}(E + E^{-1})^2 \quad (1)$$

Also,

$$1 + \frac{\delta^2}{2} = 1 + \frac{1}{2}(E^{1/2} - E^{-1/2})^2 = \frac{1}{2}(E + E^{-1}) \quad (2)$$

From Eqs. (1) and (2), the first result follows

(ii) Now

$$\mu + \frac{\delta}{2} = \frac{1}{2}(E^{1/2} + E^{-1/2} + E^{1/2} - E^{-1/2}) = E^{1/2}$$

Thus, the second result is proved.

(iii) We can write

$$\frac{\delta^2}{2} + \delta\sqrt{1 + (\delta^2/4)} = \frac{\left(E^{1/2} - E^{-1/2}\right)^2}{2} + \frac{\left(E^{1/2} - E^{-1/2}\right)\sqrt{1 + \frac{1}{4}\left(E^{1/2} - E^{-1/2}\right)^2}}{1}$$

$$= \frac{E - 2 + E^{-1}}{2} + \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)\left(E^{1/2} + E^{-1/2}\right)$$

$$= \frac{E - 2 + E^{-1}}{2} + \frac{E - E^{-1}}{2}$$

$$= E - 1$$

Using Eq. (6.27), we get

$$E - 1 = \Delta$$

(iv) We have

$$\mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right) = \frac{1}{2}\left(E - E^{-1}\right)$$

Now, using Eq. (6.27), we get

$$= \frac{1}{2}\left(1 + \Delta - E^{-1}\right) = \frac{\Delta}{2} + \frac{1}{2}(1 - E^{-1})$$

$$= \frac{\Delta}{2} + \frac{1}{2}\left(\frac{E - 1}{E}\right) = \frac{\Delta}{2} + \frac{\Delta}{2E}$$

(v) We can write

$$\mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right) = \frac{1}{2}\left(E - E^{-1}\right)$$

Now using Eqs. (6.27) and (6.28), we have

$$\mu\delta = \frac{1}{2}(1 + \Delta - 1 + \nabla) = \frac{1}{2}(\Delta + \nabla)$$

Example 6.7 Show that the operators μ and E commute.

Solution From the definition of operators μ and E, we have

$$\mu E y_0 = \mu y_1 = \frac{1}{2}(y_{3/2} + y_{1/2}) \quad (1)$$

While

$$E\mu y_0 = \frac{1}{2}E\,(y_{1/2} + y_{-1/2}) = \frac{1}{2}(y_{3/2} + y_{1/2}) \quad (2)$$

Equating (1) and (2), we have

$$\mu E = E\mu$$

Therefore, the operators μ and E commute.

Theorem 6.1 (Differences of a polynomial). The nth differences of a polynomial of degree n is constant, when the values of the independent variable are given at equal intervals.

Proof Let us consider a polynomial of degree n in the form

$$y_x = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n,$$

where $a_0 \neq 0$ and $a_0, a_1, a_2, \dots, a_n$ are constants. Let h be the interval of differencing. Then

$$y_{x+h} = a_0(x + h)^n + a_1(x + h)^{n-1} + a_2(x + h)^{n-2} + \dots + a_{n-1}(x + h) + a_n$$

We now examine the differences of the polynomial:

$$\Delta y_x = y_{x+h} - y_x = a_0[(x + h)^n - x^n] + a_1[(x + h)^{n-1} - x^{n-1}]$$
$$+ a_2[(x + h)^{n-2} - x^{n-2}] + \dots + a_{n-1}(x + h - x)$$

Binomial expansion yields

$$\Delta y_x = a_0(x^n + {}^nC_1x^{n-1}h + {}^nC_2x^{n-2}h^2 + \cdots + h^n - x^n)$$
$$+ a_1[x^{n-1} + {}^{(n-1)}C_1x^{n-2}h + {}^{(n-1)}C_2x^{n-3}h^2 + \cdots + h^{n-1} - x^{n-1}] + \cdots$$
$$+ a_{n-1}h$$
$$= a_0nhx^{n-1} + [a_0{}^nC_2h^2 + a_1{}^{(n-1)}C_1h]x^{n-2} + \cdots + a_{n-1}h$$

Therefore,

$$\Delta y_x = a_0nhx^{n-1} + b'x^{n-2} + c'x^{n-3} + \cdots + k'x + l'$$

where b', c', ..., k', l' are constants involving h but not x. Thus, the first difference of a polynomial of degree n is another polynomial of degree $(n - 1)$. Similarly

$$\Delta^2 y_x = \Delta(\Delta y_x) = \Delta y_{x+h} - \Delta y_x$$
$$= a_0nh[(x + h)^{n-1} - x^{n-1}] + b'[(x + h)^{n-2} - x^{n-2}] + \cdots$$
$$+ k'(x + h - x)$$

On simplification, it reduces to the form

$$\Delta^2 y_x = a_0n(n - 1)\, h^2x^{n-2} + b''x^{n-3} + c''x^{n-4} + \cdots + q''$$

Therefore, $\Delta^2 y_x$ is a polynomial of degree $(n - 2)$ in x. Similarly, we can form the higher order differences, and every time we observe that the degree of the polynomial is reduced by one. After differencing n times, we are left with only the first term in the form

$$\Delta^n y_x = a_0n(n - 1)\,(n - 2) \ldots (2)(1)\, h^n = a_0\,(n!)h^n = \text{Constant}$$

This constant is independent of x. Since $\Delta^n y_x$ is a constant, $\Delta^{n+1} y_x = 0$. Hence the $(n + 1)$th and higher order differences of a polynomial of degree n are zero.

6.3 NEWTON'S FORWARD DIFFERENCE INTERPOLATION FORMULA

Let $y = f(x)$ be a function which takes values $f(x_0)$, $f(x_0 + h)$, $f(x_0 + 2h)$, ..., corresponding to various equispaced values of x with spacing h, say x_0, $x_0 + h$, $x_0 + 2h$, ... Suppose, we wish to evaluate the function $f(x)$ for a value $x_0 + ph$, where p is any real number, then for any real number p, we have the operator E such that $E^pf(x) = f(x + ph)$. Therefore, using Eq. (6.27) we have

$$f(x_0 + ph) = E^pf(x_0) = (1 + \Delta)^pf(x_0)$$

$$= \left[1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots\right]f(x_0)$$

That is,

$$f(x_0 + ph) = f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 f(x_0) + \cdots$$
$$+ \frac{p(p-1)\cdots(p-n+1)}{n!}\Delta^n f(x_0) + \text{Error} \qquad (6.32)$$

This is known as *Newton's forward difference formula for interpolation*, which gives the value of $f(x_0 + ph)$ in terms of $f(x_0)$ and its leading differences. This formula is also known as *Newton–Gregory forward difference interpolation formula*. Here, $p = (x - x_0)/h$. Equation (6.32) can also be written in another alternate form as

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$+ \frac{p(p-1)(p-n+1)}{n!}\Delta^n y_0 + \text{Error} \qquad (6.33)$$

If we retain $(r + 1)$ terms in Eq. (6.33), we obtain a polynomial of degree r agreeing with y_x at $x_0, x_1, \ldots, x_r$.

This formula is mainly used for interpolating the values of y near the beginning of a set of tabular values and for extrapolating values of y, a short distance backward from y_0. We shall illustrate these formulae by considering the following simple examples.

Example 6.8 Evaluate $f(15)$, given the following table of values:

x	10	20	30	40	50
$y = f(x)$	46	66	81	93	101

Solution We may note that $x = 15$ is very near to the beginning of the table. Hence, we use Newton's forward difference interpolation formula. The forward differences are calculated and tabulated as given below:

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	46				
		20			
20	66		−5		
		15		2	
30	81		−3		−3
		12		−1	
40	93		−4		
		8			
50	101				

We have Newton's forward difference interpolation formula as

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

In this example, from the above table, we have

$$x_0 = 10, \quad y_0 = 46, \quad \Delta y_0 = 20, \quad \Delta^2 y_0 = -5, \quad \Delta^3 y_0 = 2, \quad \Delta^4 y_0 = -3$$

Let y_{15} be the value of y when $x = 15$, then

$$p = \frac{x - x_0}{h} = \frac{15 - 10}{10} = 0.5$$

we can also find 'p' as
Put $x_0 + ph = 15$
$ph = 15 - x_0$
$p = \frac{15 - x_0}{h}$
$'p' = \frac{15-10}{10} = 0.5$

Substituting these values in Eq. (1), we get

$$f(15) = y_{15} = 46 + (0.5)(20) + \frac{(0.5)(0.5 - 1)}{2}(-5)$$

$$+ \frac{(0.5)(0.5 - 1)(0.5 - 2)}{6}(2) + \frac{(0.5)(0.5 - 1)(0.5 - 2)(0.5 - 3)}{24}(-3)$$

$$= 46 + 10 + 0.625 + 0.125 + 0.1172$$

Therefore, $f(15) = 56.8672$ correct to four decimal places.

Example 6.9 Find Newton's forward difference interpolating polynomial for the following data:

x	0.1	0.2	0.3	0.4	0.5
$y = f(x)$	1.40	1.56	1.76	2.00	2.28

Solution We shall first construct the forward difference table to the given data as indicated below:

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.20		0.00	
0.3	1.76		0.04		0.00
		0.24		0.00	
0.4	2.00		0.04		
		0.28			
0.5	2.28				

Since, third and fourth leading differences are zero, we have Newton's forward difference interpolating formula as

$$y = y_0 + p\Delta y_0 + \frac{p(p - 1)}{2}\Delta^2 y_0 \qquad (1)$$

In this problem, $x_0 = 0.1$, $y_0 = 1.40$, $\Delta y_0 = 0.16$, $\Delta^2 y_0 = 0.04$, and

$$p = \frac{x - 0.1}{0.1} = 10x - 1$$

Substituting these values in Eq. (1), we obtain

$$y = f(x) = 1.40 + (10x - 1)(0.16) + \frac{(10x - 1)(10x - 2)}{2}(0.04)$$

That is, $y = 2x^2 + x + 1.28$. This is the required Newton's interpolating polynomial.

Example 6.10 Estimate the missing figure in the following table:

x	1	2	3	4	5
$y = f(x)$	2	5	7	–	32

Solution Since we are given four entries in the table, the function $y = f(x)$ can be represented by a polynomial of degree three. Using Theorem 6.1, we have

$$\Delta^3 f(x) = \text{Constant} \quad \text{and} \quad \Delta^4 f(x) = 0$$

for all x. In particular, $\Delta^4 f(x_0) = 0$. Equivalently, $(E - 1)^4 f(x_0) = 0$. Expanding, we have

$$(E^4 - 4E^3 + 6E^2 - 4E + 1) f(x_0) = 0$$

That is,

$$f(x_4) - 4f(x_3) + 6f(x_2) - 4f(x_1) + f(x_0) = 0$$

Using the values given in the table, we obtain

$$32 - 4f(x_3) + 6 \times 7 - 4 \times 5 + 2 = 0$$

which gives $f(x_3)$, the missing value equal to 14.

Example 6.11 Find a cubic polynomial in x which takes on the values −3, 3, 11, 27, 57 and 107, when $x = 0, 1, 2, 3, 4$ and 5 respectively.

Solution Here, the observations are given at equal intervals of unit width. To determine the required polynomial, we first construct the difference table as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	−3			
		6		
1	3		2	
		8		6
2	11		8	
		16		6
3	27		14	
		30		6
4	57		20	
		50		
5	107			

Since the fourth and higher order differences are zero, we have the required Newton's interpolation formula in the form

$$f(x_0 + ph) = f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{6}\Delta^3 f(x_0) \tag{1}$$

Here,

$$p = \frac{x - x_0}{h} = \frac{x - 0}{1} = x, \quad \Delta f(x_0) = 6, \quad \Delta^2 f(x_0) = 2, \quad \Delta^3 f(x_0) = 6$$

Substituting these values into Eq. (1), we have

$$f(x) = -3 + 6x + \frac{x(x-1)}{2}(2) + \frac{x(x-1)(x-2)}{6}(6)$$

That is, $f(x) = x^3 - 2x^2 + 7x - 3$, is the required cubic polynomial.

6.4 NEWTON'S BACKWARD DIFFERENCE INTERPOLATION FORMULA

If one wishes to interpolate the value of the function $y = f(x)$ near the end of table of values, and to extrapolate value of the function a short distance forward from y_n, Newton's backward interpolation formula is used, which can be derived as follows:

Let $y = f(x)$ be a function which takes on values $f(x_n), f(x_n - h), f(x_n - 2h), \ldots, f(x_0)$ corresponding to equispaced values $x_n, x_n - h, x_n - 2h, \ldots, x_0$. Suppose, we wish to evaluate the function $f(x)$ at $(x_n + ph)$, where p is any real number, then we have the shift operator E, such that

$$f(x_n + ph) = E^p f(x_n) = (E^{-1})^{-p} f(x_n) = (1 - \nabla)^{-p} f(x_n)$$

Binomial expansion yields,

$$f(x_n + ph) = \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \cdots + \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n + \text{Error}\right] f(x_n)$$

That is,

$$f(x_n + ph) = f(x_n) + p\nabla f(x_n) + \frac{p(p+1)}{2!}\nabla^2 f(x_n) + \frac{p(p+1)(p+2)}{3!}\nabla^3 f(x_n) + \cdots + \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n f(x_n) + \text{Error} \quad (6.34)$$

This formula is known as *Newton's backward interpolation formula*. This formula is also known as *Newton–Gregory backward difference interpolation formula*. If we retain $(r + 1)$ terms in Eq. (6.34), we obtain a polynomial of degree r agreeing with $f(x)$ at $x_n, x_{n-1}, \ldots, x_{n-r}$. Alternatively, this formula can also be written as

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots + \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n y_n + \text{Error} \quad (6.35)$$

where

$$p = \frac{x - x_n}{h}$$

Here follows a couple of examples for illustration.