6.1 INTRODUCTION

Finite differences play an important role in numerical techniques, where tabulated values of the functions are available. For instance, consider a function $y = f(x)$. As x takes values $x_0, x_1, x_2, \ldots, x_n$, let the corresponding values of y be $y_0, y_1, y_2, \ldots, y_n$. That is, for a given table of values, (x_k, y_k), $k = 0, 1, 2, \ldots, n$; the process of estimating the value of y, for any intermediate value of x, is called *interpolation*. However, the method of computing the value of y, for a given value of x, lying outside the table of values of x is known as *extrapolation*. It may be noted that if the function $f(x)$ is known, the value of y corresponding to any x can be readily computed to the desired accuracy. But, in practice, it may be difficult or sometimes impossible to know the function $y = f(x)$ in its exact form.

To look at a practical example, let us consider the computation of trajectory of a rocket flight, where we solve the Euler's dynamical equations of motion to compute its position and velocity vectors at specified times during the flight. Under the same conditions, suppose, we require the position and velocity vector, at some other intermediate times; we need not compute the trajectory again by solving the dynamical equations. Instead, we can use the best known interpolation technique to get the desired values.

In general, for interpolation of a tabulated function, the concept of finite differences is important. The knowledge about various finite difference operators and their symbolic relations are very much needed to establish various interpolation formulae.

6.2 FINITE DIFFERENCE OPERATORS

6.2.1 Forward Differences

For a given table of values (x_k, y_k), $k = 0, 1, 2, \ldots, n$ with equally-spaced abscissas of a function $y = f(x)$, we define the *forward difference* operator Δ as follows: The first forward difference is usually expressed as

$$\Delta y_i = y_{i+1} - y_i, \qquad i = 0, 1, \ldots, (n-1) \tag{6.1}$$

To be explicit, we write

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\vdots \quad \vdots \quad \vdots$$
$$\Delta y_{n-1} = y_n - y_{n-1}$$

These differences are called *first differences of the function y* and are denoted by the symbol Δy_i. Here, Δ is called *forward difference operator*.

Similarly, the differences of the first differences are called *second differences*, defined by

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0, \qquad \Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

Thus, in general

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i \tag{6.2}$$

Here Δ^2 is called the *second difference operator*. Thus, continuing, we can define, *r*th difference of *y*, as

$$\Delta^r y_i = \Delta^{r-1} y_{i+1} - \Delta^{r-1} y_i \tag{6.3}$$

By defining a difference table as a convenient device for displaying various differences, the above defined differences can be written down systematically by constructing a difference table for values (x_k, y_k), $k = 0, 1, \ldots, 6$ as shown below:

Table 6.1 Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

This difference table is called *forward difference table* or *diagonal difference table*. Here, each difference is located in its appropriate column, mid-way between the elements of the previous column. It can be noted that the subscript remains constant along each diagonal of the table. The first term in the table, that is y_0 is called the *leading term*, while the differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots$ are called leading differences.

Example 6.1 Construct a forward difference table for the following values of *x* and *y*:

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
y	0.003	0.067	0.148	0.248	0.370	0.518	0.697

Solution

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.1	0.003					
		0.064				
0.3	0.067		0.017			
		0.081		0.002		
0.5	0.148		0.019		0.001	
		0.100		0.003		0.000
0.7	0.248		0.022		0.001	
		0.122		0.004		0.000
0.9	0.370		0.026		0.001	
		0.148		0.005		
1.1	0.518		0.031			
		0.179				
1.3	0.697					

Example 6.2 Express $\Delta^2 y_0$ and $\Delta^3 y_0$ in terms of the values of the function y.

Solution Noting that each higher order difference is defined in terms of the lower order difference, we have

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

and

$$\begin{aligned}\Delta^3 y_0 &= \Delta^2 y_1 - \Delta^2 y_0 = (\Delta y_2 - \Delta y_1) - (\Delta y_1 - \Delta y_0)\\ &= (y_3 - y_2) - (y_2 - y_1) - (y_2 - y_1) + (y_1 - y_0)\\ &= y_3 - 3y_2 + 3y_1 - y_0\end{aligned}$$

Hence, we observe that the coefficients of the values of y, in the expansion of $\Delta^2 y_0$, $\Delta^3 y_0$ are binomial coefficients. Thus, in general, we arrive at the following result

$$\Delta^n y_0 = y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - {}^nC_3 y_{n-3} + \cdots + (-1)^n y_0 \tag{6.4}$$

Example 6.3 Show that the value of y_n can be expressed in terms of the leading value y_0 and the leading differences $\Delta y_0, \Delta^2 y_0, \cdots, \Delta^n y_0$.

Solution We have from the forward difference table

$$\left.\begin{aligned} y_1 - y_0 = \Delta y_0 \quad &\text{or} \quad y_1 = y_0 + \Delta y_0\\ y_2 - y_1 = \Delta y_1 \quad &\text{or} \quad y_2 = y_1 + \Delta y_1\\ y_3 - y_2 = \Delta y_2 \quad &\text{or} \quad y_3 = y_2 + \Delta y_2 \end{aligned}\right\} \tag{6.5}$$

and so on. Similarly

$$\left.\begin{aligned} \Delta y_1 - \Delta y_0 = \Delta^2 y_0 \quad &\text{or} \quad \Delta y_1 = \Delta y_0 + \Delta^2 y_0\\ \Delta y_2 - \Delta y_1 = \Delta^2 y_1 \quad &\text{or} \quad \Delta y_2 = \Delta y_1 + \Delta^2 y_1 \end{aligned}\right\} \tag{6.6}$$

and so on. Similarly, we can write

$$\left.\begin{aligned} \Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0 \quad &\text{or} \quad \Delta^2 y_1 = \Delta^2 y_0 + \Delta^3 y_0\\ \Delta^2 y_2 - \Delta^2 y_1 = \Delta^3 y_1 \quad &\text{or} \quad \Delta^2 y_2 = \Delta^2 y_1 + \Delta^3 y_1 \end{aligned}\right\} \tag{6.7}$$

and so on. Also, from Eqs. (6.6) and (6.7), we can rewrite Δy_2 as

$$\Delta y_2 = (\Delta y_0 + \Delta^2 y_0) + (\Delta^2 y_0 + \Delta^3 y_0)$$
$$= \Delta y_0 + 2\Delta^2 y_0 + \Delta^3 y_0 \tag{6.8}$$

From Eqs. (6.5)–(6.8), y_3 can be rewritten

$$\begin{aligned} y_3 &= y_2 + \Delta y_2 \\ &= (y_1 + \Delta y_1) + (\Delta y_1 + \Delta^2 y_1) \\ &= (y_0 + \Delta y_0) + 2(\Delta y_0 + \Delta^2 y_0) + (\Delta^2 y_0 + \Delta^3 y_0) \\ &= y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0 \\ &= (1 + \Delta)^3 y_0 \end{aligned} \tag{6.9}$$

Similarly, we can symbolically write

$$y_1 = (1 + \Delta) y_0, \qquad y_2 = (1 + \Delta)^2 y_0, \qquad y_3 = (1 + \Delta)^3 y_0$$

Continuing this procedure, we can show, in general

$$y_n = (1 + \Delta)^n y_0$$

Hence, we obtain

$$y_n = y_0 + {}^nC_1 \Delta y_0 + {}^nC_2 \Delta^2 y_0 + \cdots + \Delta^n y_0 \tag{6.10}$$

Equivalently, we can also write the result as

$$y_n = \sum_{i=0}^{n} {}^nC_i \, \Delta^i y_0 \tag{6.11}$$

6.2.2 Backward Differences

For a given table of values (x_k, y_k), $k = 0, 1, 2, \ldots, n$ of a function $y = f(x)$ with equally spaced abscissas, the first backward differences are usually expressed in terms of the backward difference operator ∇ as

$$\nabla y_i = y_i - y_{i-1}, \qquad i = n, (n - 1), \ldots, 1 \tag{6.12}$$

Thus,

$$\begin{aligned} \nabla y_1 &= y_1 - y_0 \\ \nabla y_2 &= y_2 - y_1 \\ &\vdots \\ \nabla y_n &= y_n - y_{n-1} \end{aligned}$$

The differences of these differences are called *second differences* and they are denoted by $\nabla^2 y_2, \nabla^2 y_3, \ldots, \nabla^2 y_n$. That is,

$$\begin{aligned} \nabla^2 y_2 &= \nabla y_2 - \nabla y_1 \\ \nabla^2 y_3 &= \nabla y_3 - \nabla y_2 \\ &\vdots \\ \nabla^2 y_n &= \nabla y_n - \nabla y_{n-1} \end{aligned}$$

Thus, in general, the second backward differences are

$$\nabla^2 y_i = \nabla y_i - \nabla y_{i-1}, \qquad i = n, (n-1), \ldots, 2 \tag{6.13}$$

while the *k*th backward differences are given as

$$\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1}, \qquad i = n, (n-1), \ldots, k \tag{6.14}$$

These backward differences can be systematically arranged for a table of values (x_k, y_k), $k = 0, 1, \ldots, 6$ as indicated in Table 6.2.

Table 6.2 Backward Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$	$\nabla^6 y$
x_0	y_0						
		∇y_1					
x_1	y_1		$\nabla^2 y_2$				
		∇y_2		$\nabla^3 y_3$			
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$		
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$	
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$		$\nabla^6 y_6$
		∇y_4		$\nabla^3 y_5$		$\nabla^5 y_6$	
x_4	y_4		$\nabla^2 y_5$		$\nabla^4 y_6$		
		∇y_5		$\nabla^3 y_6$			
x_5	y_5		$\nabla^2 y_6$				
		∇y_6					
x_6	y_6						

From this table, it can be observed that the subscript remains constant along every backward diagonal.

Example 6.4 Show that any value of y can be expressed in terms of y_n and its backward differences.

Solution From Eq. (6.12) we have

$$y_{n-1} = y_n - \nabla y_n \quad \text{and} \quad y_{n-2} = y_{n-1} - \nabla y_{n-1}$$

Also from the definition as given in Eq. (6.13), we get

$$\nabla y_{n-1} = \nabla y_n - \nabla^2 y_n$$

From these equations, we obtain

$$y_{n-2} = y_n - 2\nabla y_n + \nabla^2 y_n$$

Similarly, we can show that

$$y_{n-3} = y_n - 3\nabla y_n + 3\nabla^2 y_n - \nabla^3 y_n$$

Symbolically, these results can be rewritten as follows:

$$y_{n-1} = (1 - \nabla) y_n, \qquad y_{n-2} = (1 - \nabla)^2 y_n, \qquad y_{n-3} = (1 - \nabla)^3 y_n$$

Thus is general, we can write

$$y_{n-r} = (1 - \nabla)^r y_n$$

That is,

$$y_{n-r} = y_n - {}^rC_1 \nabla y_n + {}^rC_2 \nabla^2 y_n - \cdots + (-1)^r \nabla^r y_n \tag{6.15}$$