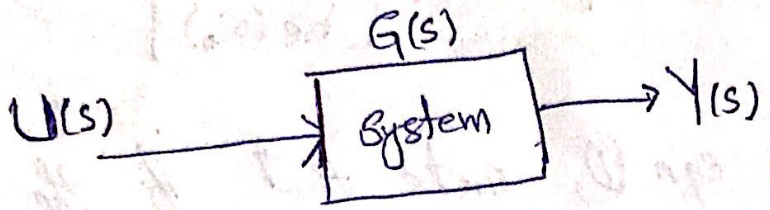
Frequency Response



Now what if $U(t)$ is a sinusoidal signal?

i.e., $\quad U(t) = A \sin(\omega_0 t)$

then $\quad U(s) = \dfrac{A\omega_0}{s^2 + \omega_0^2}$

and

$$Y(s) = G(s)\,U(s)$$

$$= G(s)\,\frac{A\omega_0}{s^2 + \omega_0^2}$$

$$Y(s) = \underbrace{\underbrace{\boxed{\frac{\alpha_1}{s-p_1} + \frac{\alpha_2}{s-p_2} + \cdots \frac{\alpha_n}{s-p_n}}}_{\text{Poles of the sys}} + \frac{\alpha_0}{s+j\omega_0} + \frac{\alpha_0^*}{s-j\omega_0}}_{\text{Partial fractions.}} \to (1)$$

then $\quad y(t) = \alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t} \cdots + \alpha_n e^{p_n t} + 2|\alpha_0| \sin(\omega_0 t + \phi)$

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where

$$\phi = \tan^{-1}\left[\frac{Im(\alpha_0)}{Re(\alpha_0)}\right]$$

⇒ In eqn 1, note that if the system $(G(s))$ is stable then the terms due to the poles of the sys. will decay to 0 $\left[\because \alpha_i e^{-2t} \searrow\right]$

⇒ What we will left with is

$$\boxed{Y(t) = AM \sin(\omega_0 t + \phi)} \rightarrow 2$$

↳ o/p in Steady State.

$$M = |G(s)|_{s=j\omega_0}$$

$$G(s)|_{s=j\omega_0} = M e^{+j\phi}$$

$$\phi = \tan^{-1}\left[\frac{Im(G(s))}{Re(G(s))}\right]_{s=j\omega_0}$$

$$U(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

Sinusoidal.

"The response of the sys. to sinusoidal i/Ps".

⇒ From 2

$$y(t) = AM \sin(\omega_0 t + \phi)$$

it is evident that

"When a linear system is feeded a sinusoidal i/P of frequency ω_0, then the o/p will always have the same frequency but different magnitude and phase.

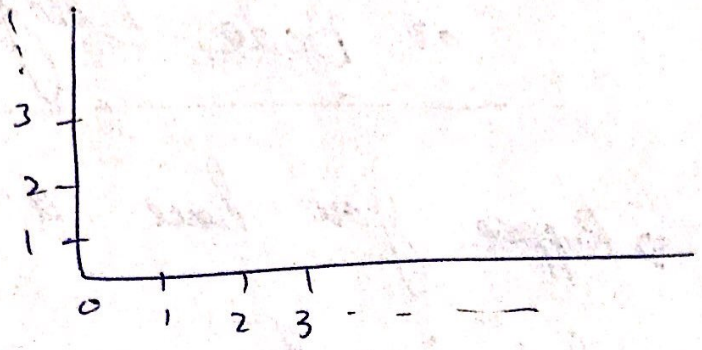
④

⇒ So what happens to a sys. or how the sys. responds over a wider range of frequencies is the study of this lecture.

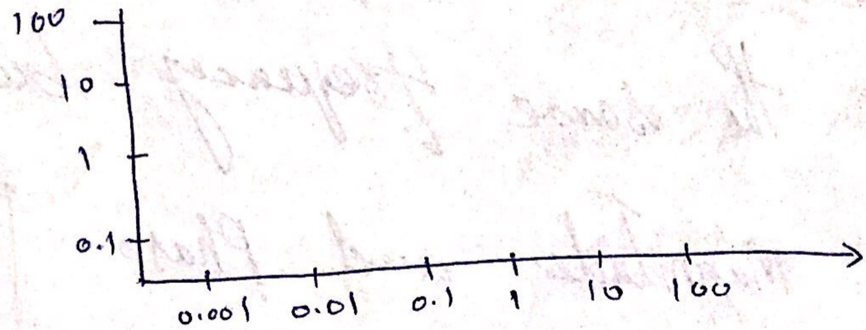
⇒ The most suitable technique to graphically show a system's frequency response over a wider range of frequencies is ~~called~~ known as the Bode plot.

⑤

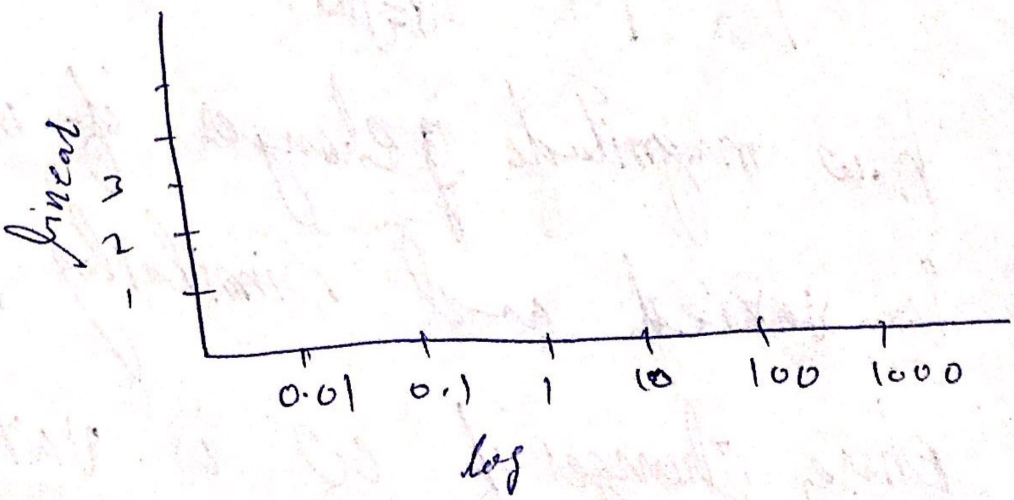
1→ Linea Scale.



⇒ Log-Log Scale



Semi-Log Scale :→



⇒ ~~Now~~ Bode Plot

or Bode Diagram

⇒ Suppose you have a sys $G(s)$ which is when exposed to freq. (Sinusoidal i/ps) produce an o/p of the same frequency but different Magnitude and Phase $\boxed{y(t) = AM\sin(\omega t + \phi)}$

⇒ So what we require is to analyse $|G(s)|_{s=j\omega}$ i.e how magnitude changes if ω is varied and similarly how phase changes if ω is varied.

$\phi = \angle G(j\omega)$

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Example:-

$G(s) = 10$

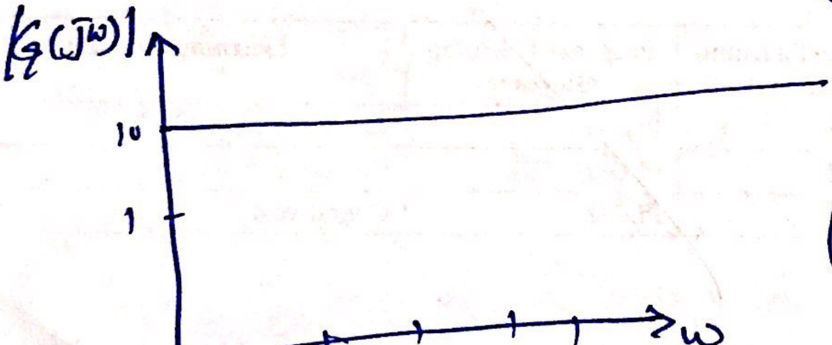
Bode plot?

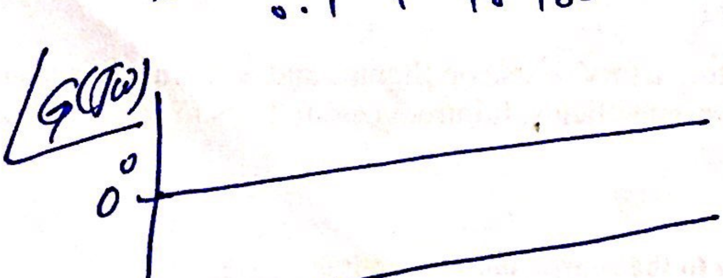
Solution

$G(j\omega) = 10$

$|G(j\omega)| = 10$

$\angle G(j\omega) = 0^\circ$





Example

$G(s) = -10$

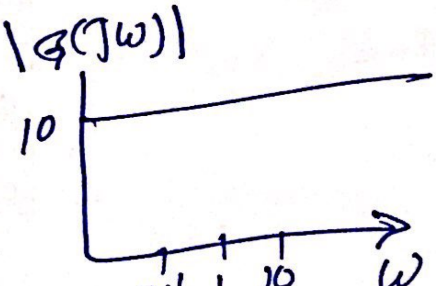
Bode plot?

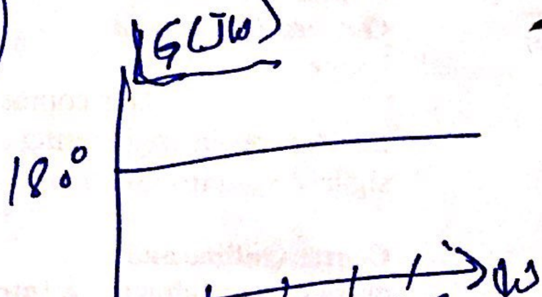
Solution.

$G(j\omega) = -10+0j$

$|G(j\omega)| = 10$

$\angle G(j\omega) = \pm 180^\circ$





⇒ So Bode Method is comprised of two plots: ① Magnitude plot

② Phase Plot

7* page.

Example

Let us assume a capicitor denated by

$$i(t) = C\frac{dV(t)}{dt}$$

Find the frequency response?

Solution

$$I(s) = CsV(s)$$

$$\frac{I(s)}{V(s)} = sC = G(s)$$

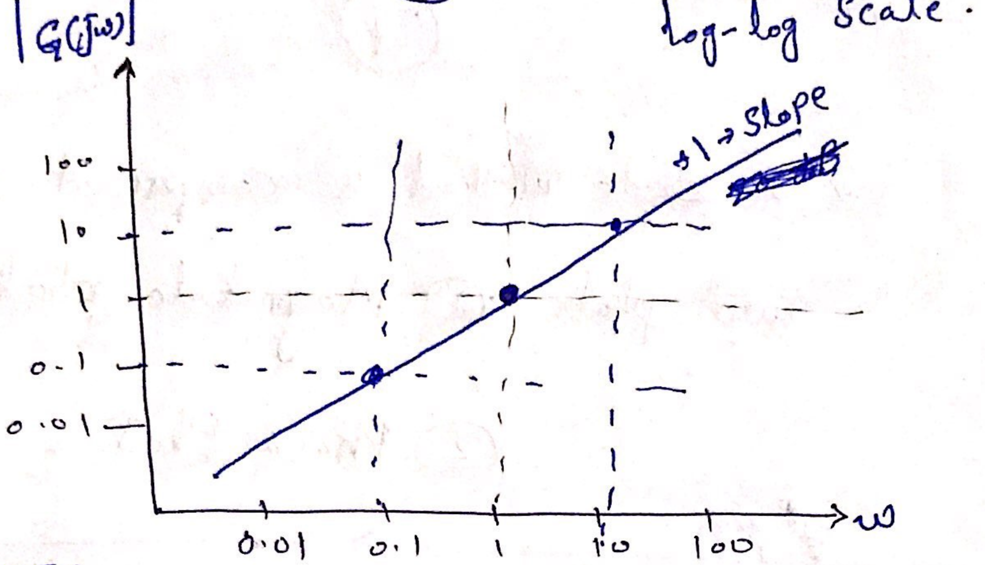
$G(s) = j\omega = s$

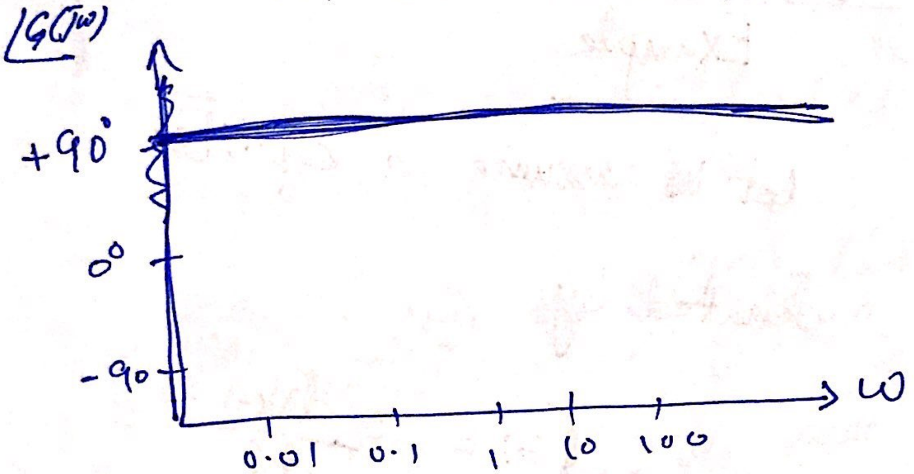
at $s = j\omega$

$$\angle G(j\omega) = +90^\circ$$

$$|G(j\omega)| = C\omega$$ → let us assume $C = 1$

⑧

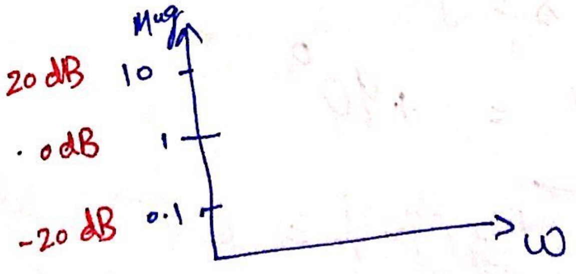




Discussion :→ By looking at the magnitude plot we see a slope of +1 on the log-log Scale.

⇒ Express in dBs

$$\boxed{20\log_{10}(\text{Mag}) \rightarrow \text{dBs}.}$$



(9)

Example ~~partta~~ 2

$\Rightarrow$ Now consider $\frac{V(s)}{I(s)} = \frac{1}{SC}$

and consider $C = 1$ (Capacitance = 1)

for simplicity.

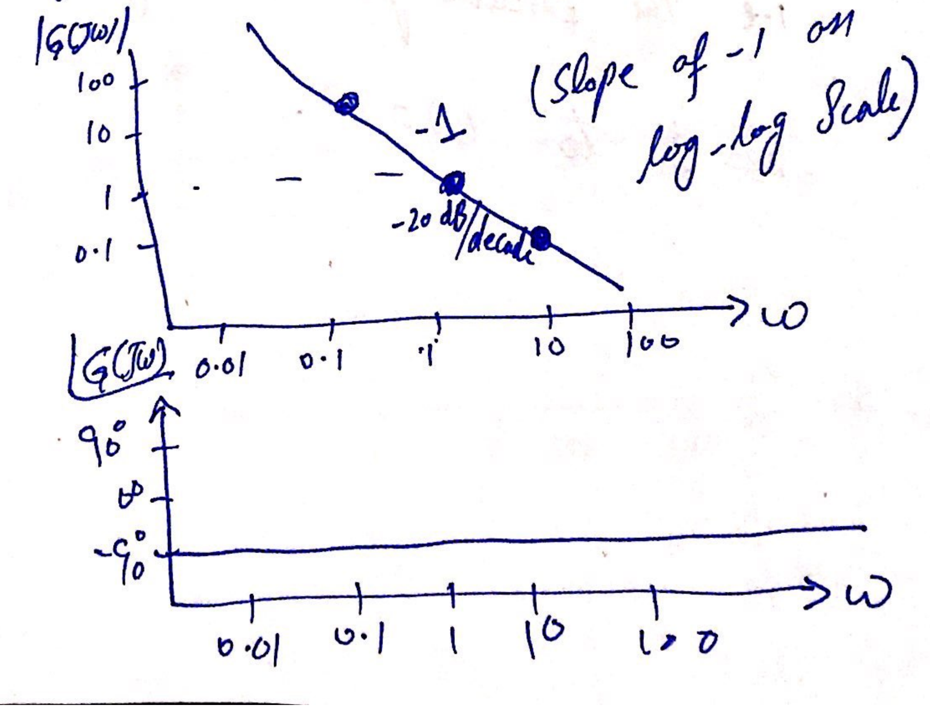
then

$$G(s) = \frac{1}{s} \quad \text{(Integrator)}$$

$$G(j\omega) = \frac{1+0j}{0+j\omega} \Rightarrow \tan^{-1}\left(\frac{0}{1}\right) - \tan^{-1}\left(\frac{\omega}{0}\right)$$
$$0^\circ - 90^\circ$$

Note: First we find the value of ω at which $|G(j\omega)| = 1$

$\Rightarrow$ We Know that $\angle G(j\omega) = -90^\circ$



Now in continuation to this example

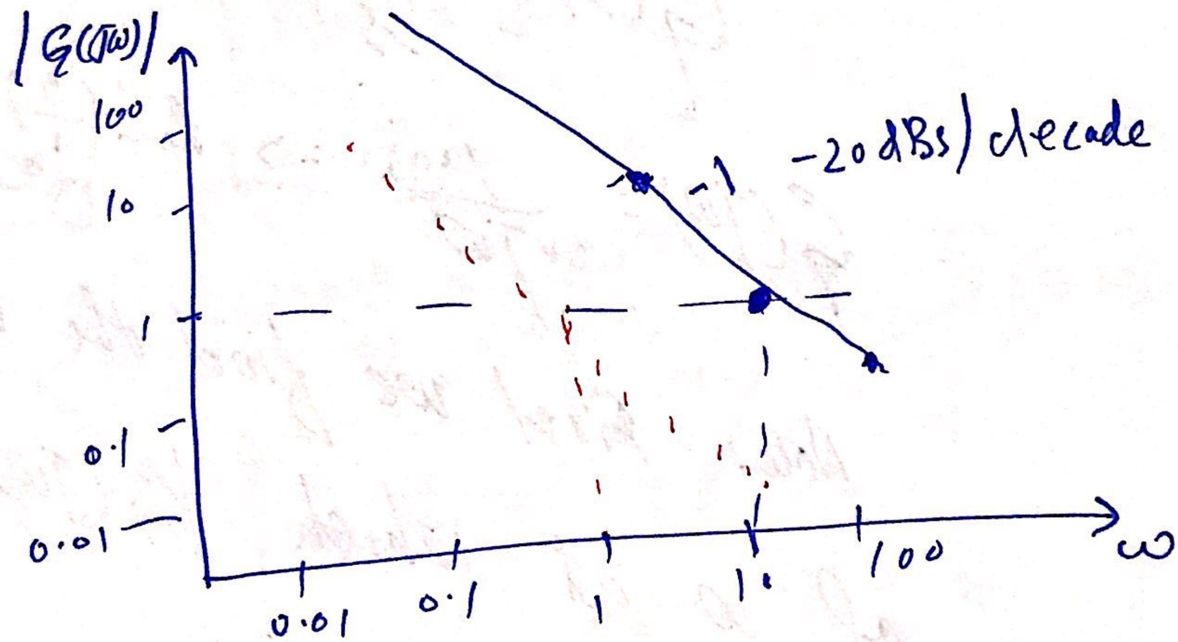
let $G(s) = \frac{10}{s}$ then

$|G(j\omega)| = \frac{10}{\omega}$ $\qquad \therefore 10 \frac{1}{\sqrt{0^2+\omega^2}} = \frac{10}{\omega}$

$\angle G(j\omega) = -90^0$

$\frac{10}{\omega} = 1$

$\frac{10}{100} = 0.1$



i.e the Break point changed from

$W = 1$ to $\omega = 10$.

Example:->

$$L(s) = s^2$$

Bode plot?

$$1 + K\,L(s) = 0$$

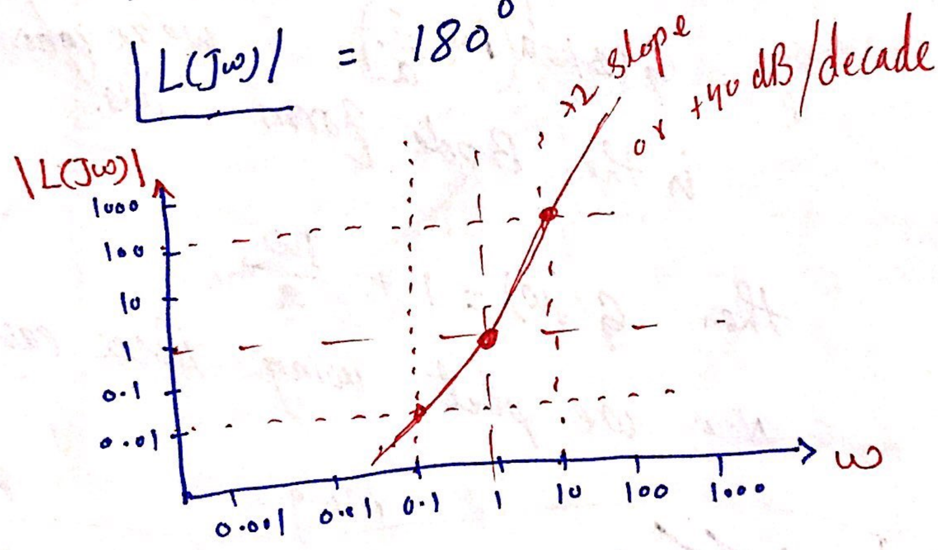
Solution

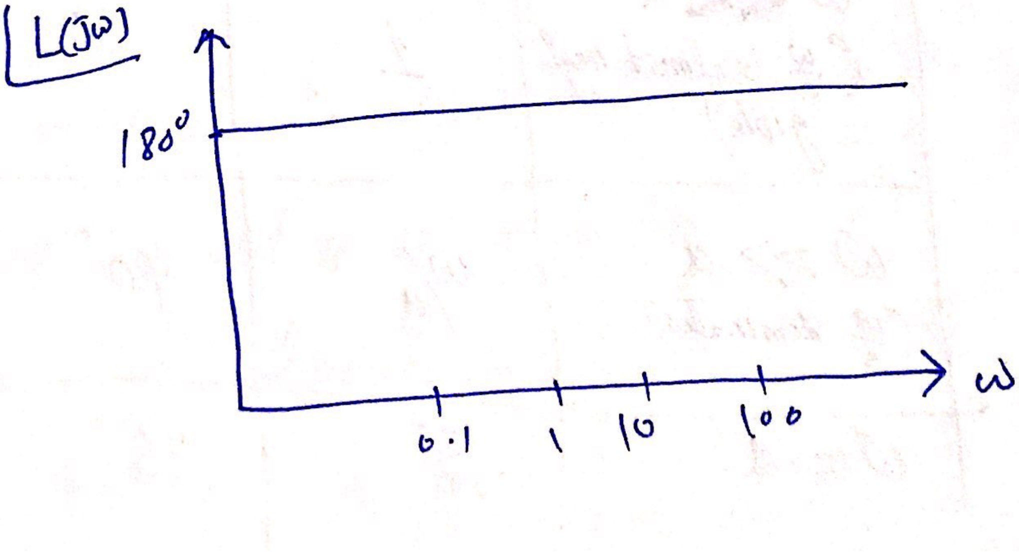
$$\cancel{|L(J\omega)|}$$

$$L(J\omega) = J^2\omega^2 = -\omega^2$$

$$|L(J\omega)| = \omega^2$$

$$\angle L(J\omega) = 180^0$$





Example :-> A simple zero.

$$G(s) = s + a$$

Bode plot?

Solution :->

In such cases i.e., 1st order zero or a pole anywhere other than origin, first we've to convert it into Bode form. i.e.,

$$G(s) = a\left(1 + \frac{s}{a}\right)$$ → In this example we're ignoring this.

is the Bode form

then $G(j\omega) = 1 + \frac{j\omega}{a}$

Now we plot using three cases

	Mag	Phas
$\omega \ll a$ ($\frac{\omega}{a}$ is almost negligible)	1	0°
$\omega \gg a$ ($\frac{\omega}{a}$ dominates)	ω/a	90°
$\omega = a$	$\sqrt{2}$	45°

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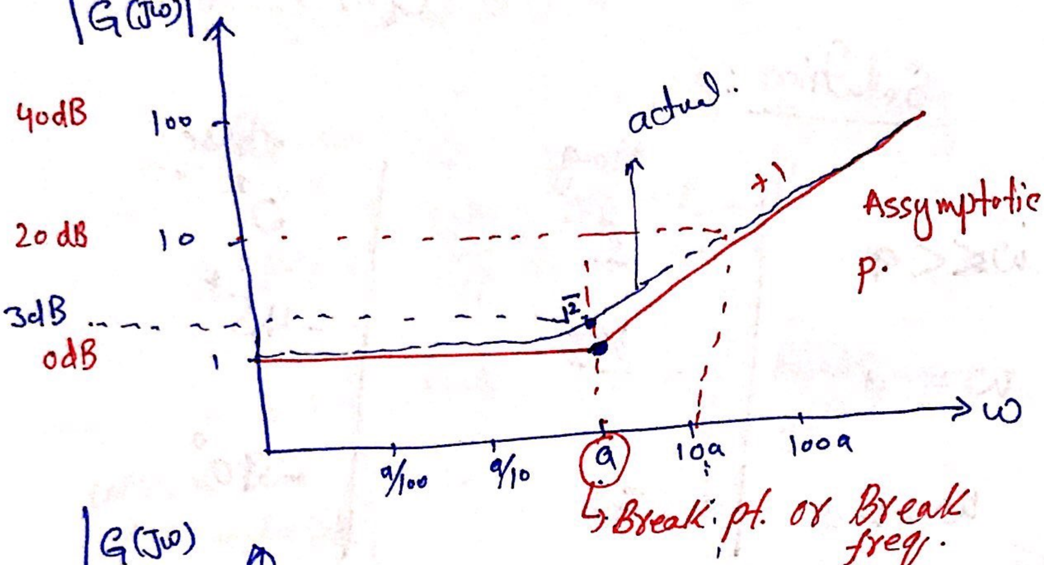
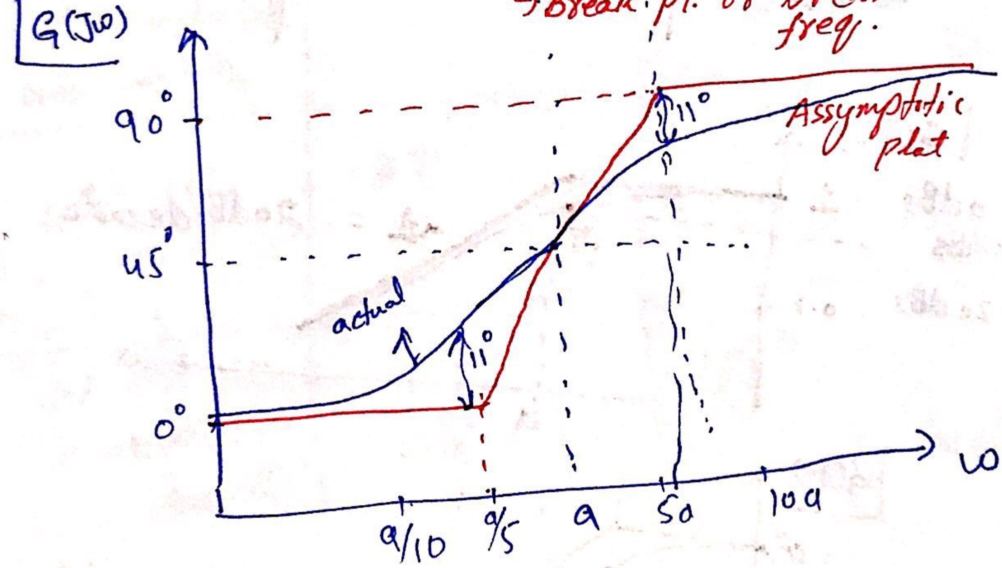
Using the above information we can ~~plot~~ draw the Bode plot for G(s)





The exact phase is $\angle G(j\omega) = \tan^{-1}\left(\frac{\omega}{a}\right)$

⇒ The red phase plot is an assymptotic approximation which is very close to the actual one (Blue plot)

Example :- (Simple pole)

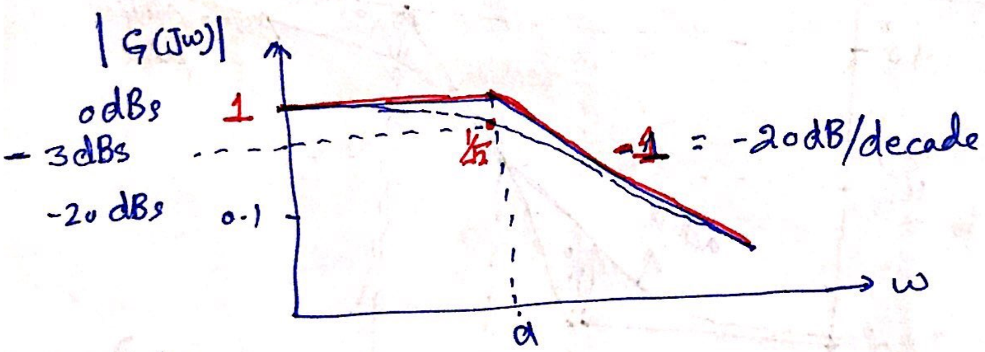
$$G(s) = \frac{1}{1 + s/a}$$

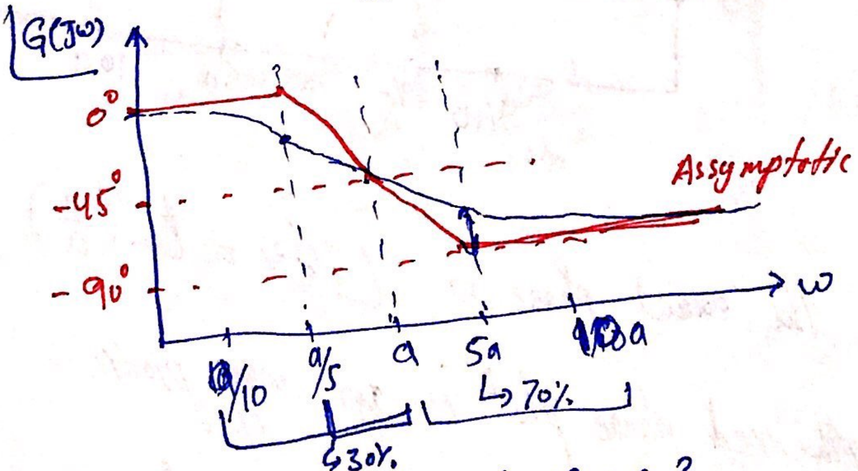
Bode plot?

$$G(j\omega) = \frac{1}{1 + j\omega/a}$$

Solution :-

	Mag	Phase
$\omega << a$	1	$0°$
$\omega = a$	$1/\sqrt{2}$	$-45°$
$\omega >> a$	a/ω	$-90°$





$\log 2 = 0.3$

$\log 5 = 0.7$

actual and assymptotic having max. error of $11°$.

Example :- (Complex Conjugate zeros)

$$G(s) = \left(\frac{s}{\omega_n}\right)^2 + 2\xi\frac{s}{\omega_n} + 1$$

Bode plot ?

Solution :-

$$G(j\omega) = -\frac{\omega^2}{\omega_n^2} + 2\xi\frac{j\omega}{\omega_n} + 1$$

	Mag	Phase
$\omega << \omega_n$ ($\frac{\omega}{\omega_n}$ neglegible)	1	0°
$\omega = \omega_n$	2ξ	90°
$\omega >> \omega_n$ (The square term $\frac{\omega^2}{\omega_n^2}$ dominates)	$\frac{\omega^2}{\omega_n^2}$	180°

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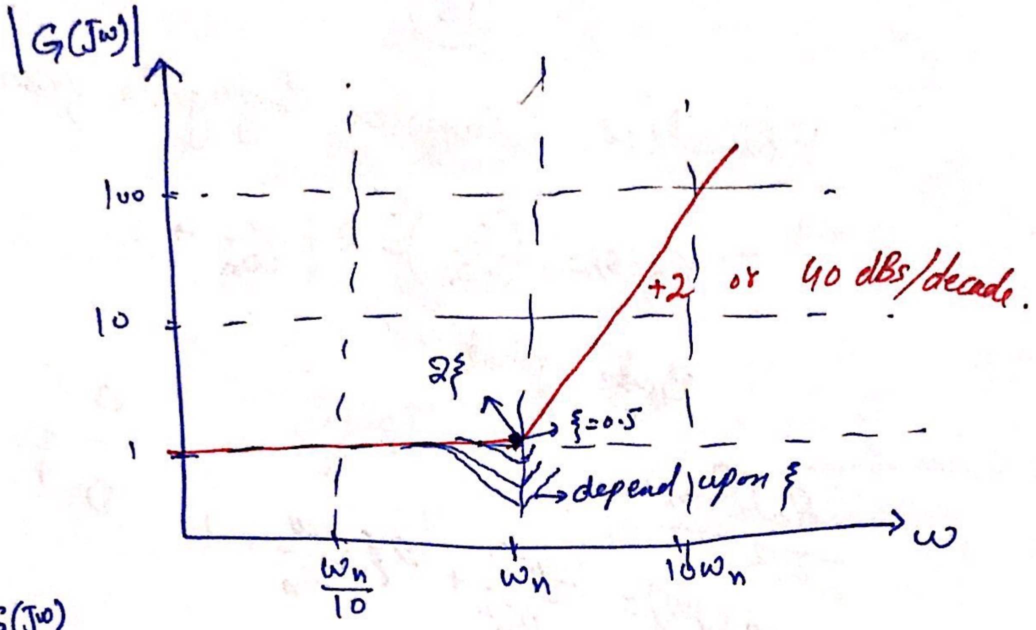

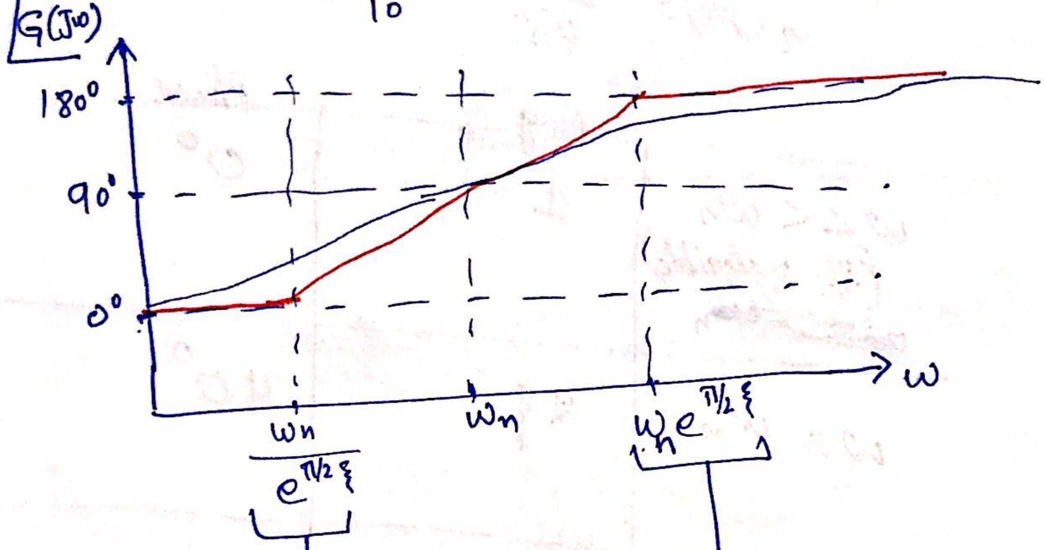

Now this should be fun" of

ξ.

$e^{\pi/2\,\xi} \approx 3$ if $\xi = 0.707$

≈ 1.17 " $\xi = 0.1$

etc.

$\Rightarrow$ In this case the error b/w the exact and the asymptotic phase curve will be $22^\circ - 32^\circ$.

$\boxed{\phi = \tan^{-1}\left(\frac{2\xi\, \omega/\omega_n}{1-\omega^2/\omega_n^2}\right)}$ can be used to plot exact plot.

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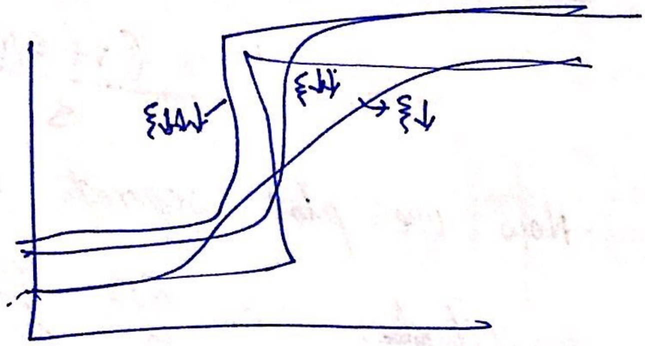
⇒ Note that the range

$$\frac{\omega_n}{e^{\pi/2\,\xi}} \longrightarrow \omega_n e^{\pi/2\,\xi}$$

Squeezes as $\xi \downarrow$.

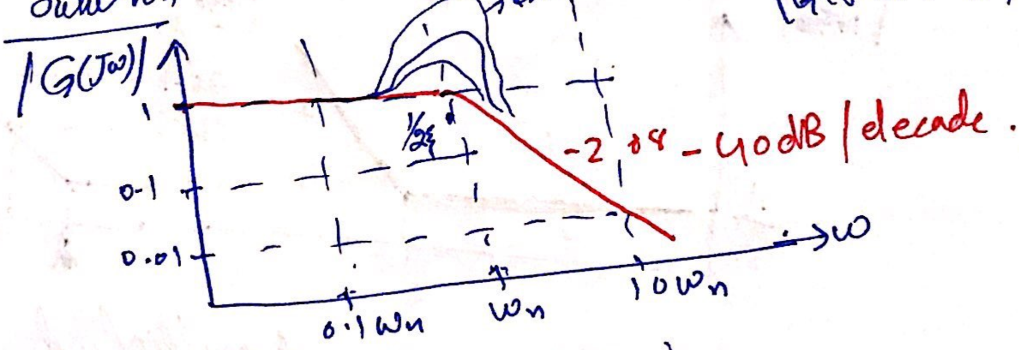


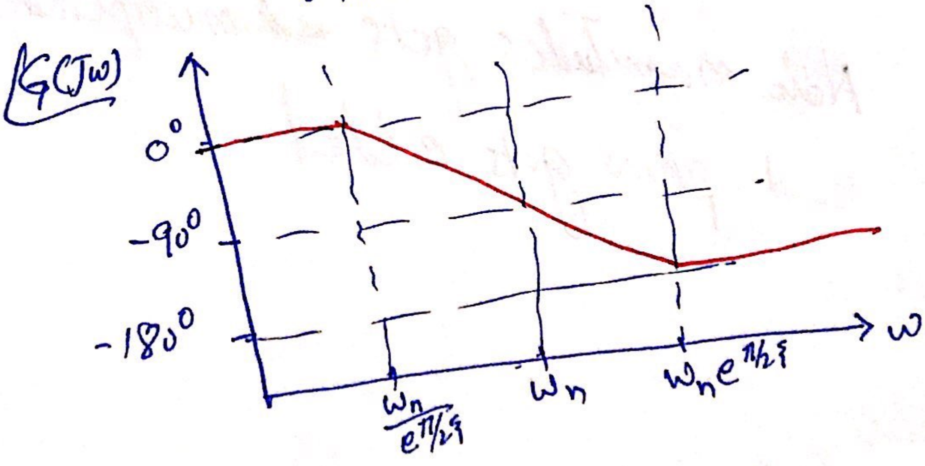
Example :→ (2^{nd} order complex poles)

$$G(s) = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\xi \, s/\omega_n + 1}$$

Solution

$|G(j\omega_n)| = \frac{1}{2\xi}$





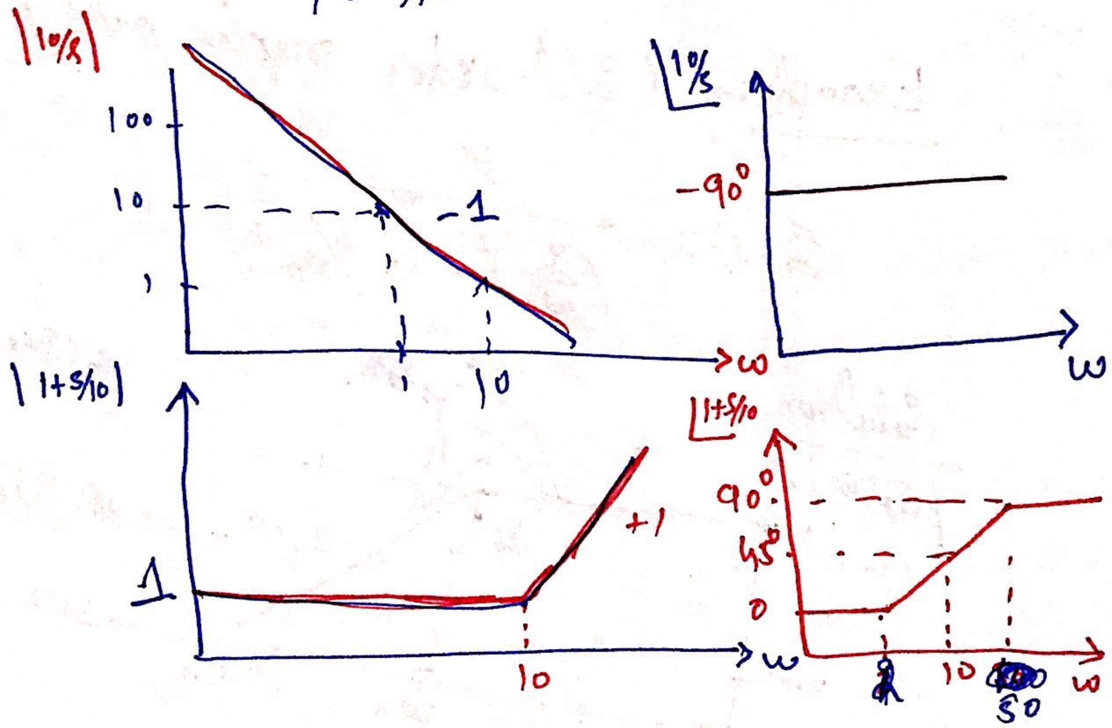
Example :→ $G(s) = \frac{s+10}{s}$

Bode plot?

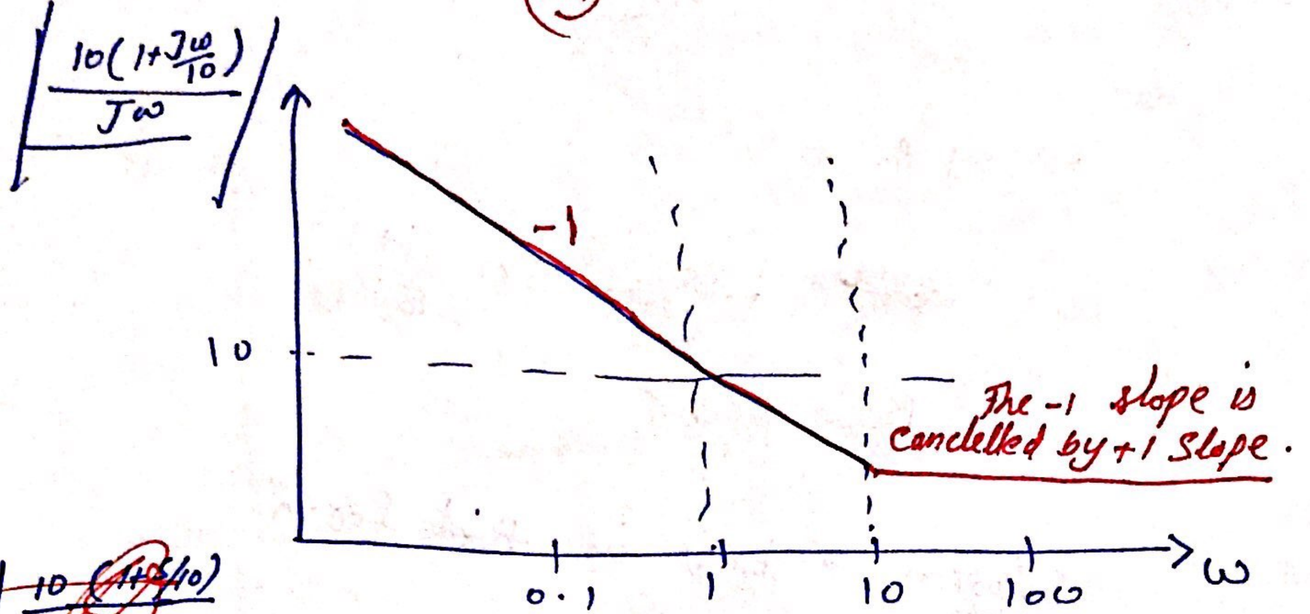
Solution :→ Writing G(s) in Bode form

$$G(s) = \frac{10\,(1+s/10)}{s}$$

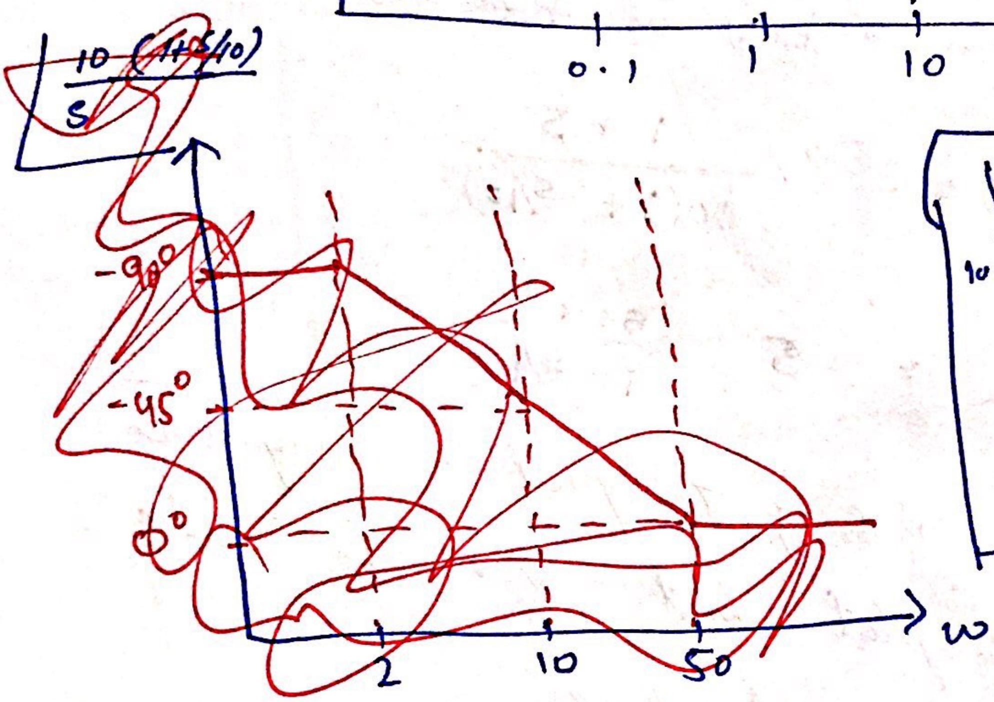
Now we plot separate plots for magnitudes i.e. for $\frac{10}{s}$ and for $1+s/10$.



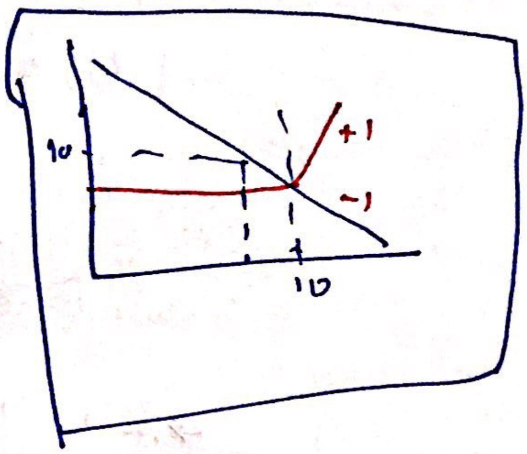
Note magnitudes gets ~~ad~~ multiplied and phase gets added.



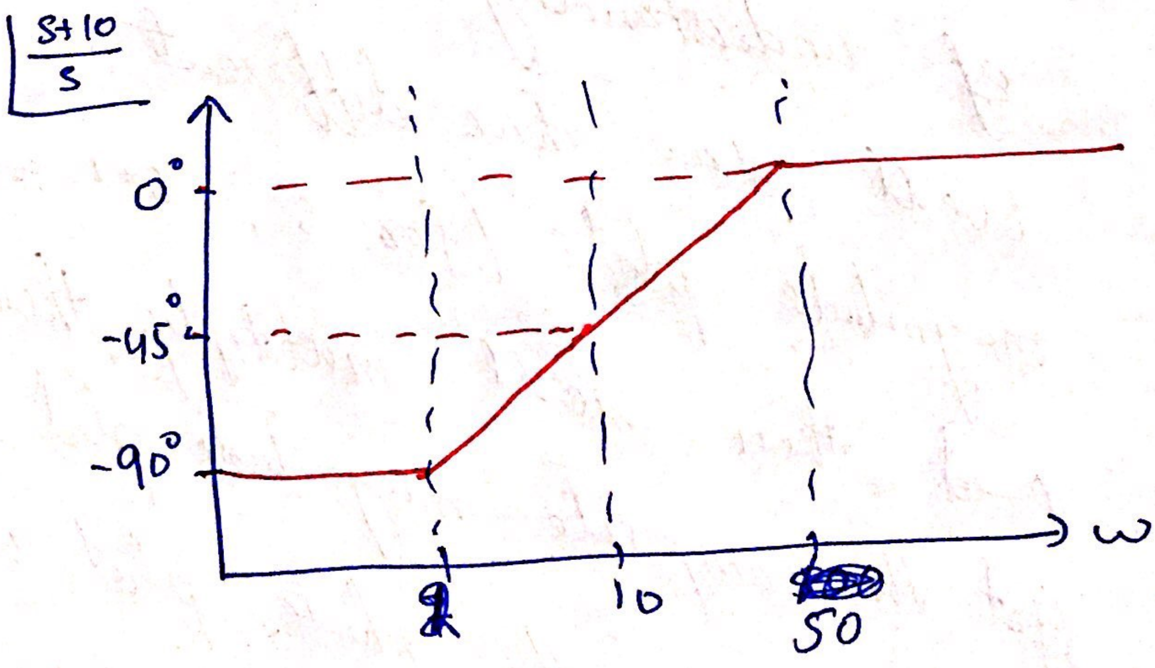
$\angle \frac{10(1+j\omega/10)}{j\omega}$
-1
10
The -1 slope is cancelled by +1 slope.
0.1
1
10
100
ω



$\angle \frac{10(1+s/10)}{s}$
-90°
-45°
0°
2
10
50
ω



10
+1
-1
10



$\angle \frac{s+10}{s}$
0°
-45°
-90°
2
10
50
ω

Example :→

$$L(s) = 50 \frac{S+1}{S+10}$$

Bode plot ?

Solution :→ L(S) in Bode form

$$L(S) = \frac{50}{10(1+ S/10)}(1+S)$$

$$= 5 \frac{S+1}{1+ S/10}$$

⇒ This is easy to plot a composite Bode plot but for the sake of understanding.

We'll draw three different magnitude plots (for 5, S+1 and 1+S/10) and then we'll multiply them to get the composite magnitude plot.

Similarly for the phase plot we will plot 3 different phase plots & then we'll add them to get the composite.

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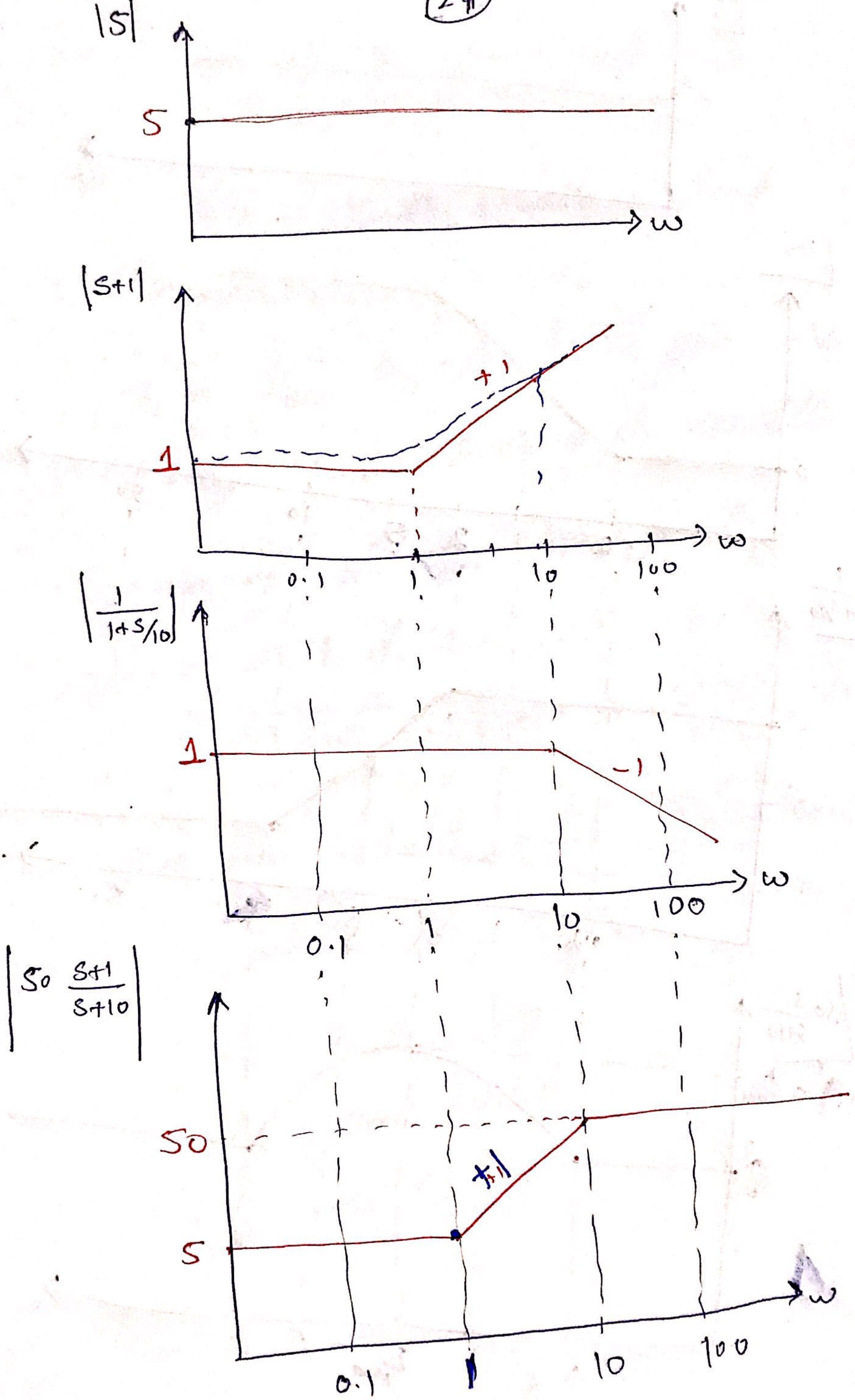
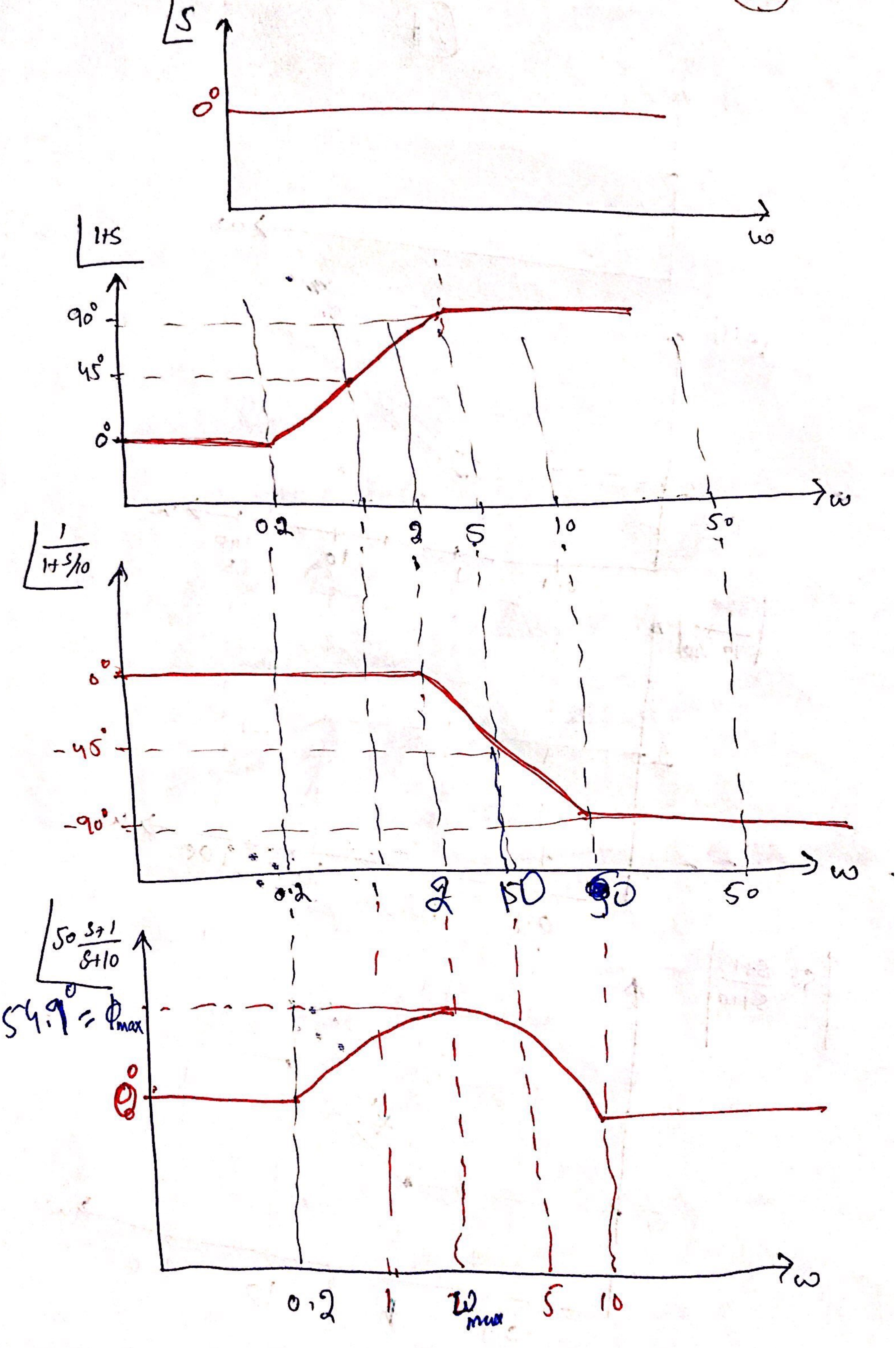


|s|
5
ω
|s+1|
1
+1
0.1
1
10
100
ω
|1/(1+s/10)|
1
-1
0.1
1
10
100
ω
|50 (s+1)/(s+10)|
50
5
+1
0.1
1
10
100
ω



∠s
0°
ω
∠1+s
90°
45°
0°
0.2
2
5
10
50
ω
∠1/(1+s/10)
0°
-45°
-90°
0.2
2
20
50
ω
∠50 (s+1)/(s+10)
54.9° = φmax
0
0.2
1
ωmax
5
10
ω

Look at the phase plot and

$50 \frac{S+1}{S+10}$

This is ~~a~~ actually a lead compensator.

⇒ The ϕ_{max} occurs at the ~~geo~~ freq. ω_{max} which is the geometric mean of of two breakpoints

i.e., 1 & 10.

$\sqrt{\omega_{b_1} \times \omega_{b_2}} = \sqrt{1 \times 10} = 3.16$ rad/sec

Max. phase lead $= \phi_{max} = \sin^{-1}\left(\frac{\alpha - 1}{\alpha + 1}\right)$

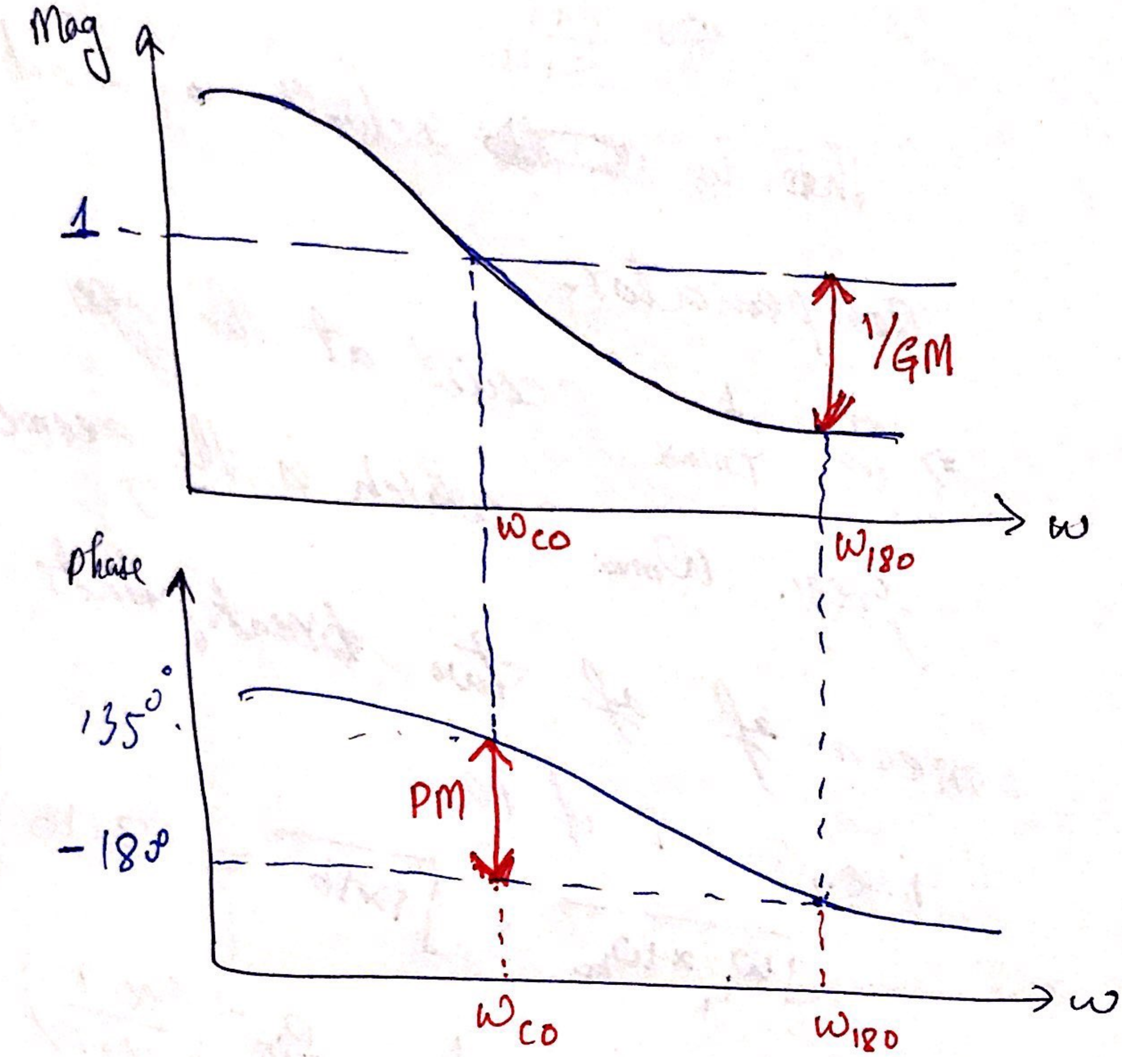
where $\alpha = \frac{\omega_{b_2}}{\omega_{b_1}}$

$\omega_{b_2} \rightarrow$ Break point 2

$\omega_{b_1} \rightarrow$ Break point 1

$\phi_{max} = \sin^{-1}\left(\frac{10-1}{10+1}\right) = 54.9^0$

Phase Margin & Gain Margin



$\omega_{co} \rightarrow \omega$ cross over or Cross over freq.

ω_{co} is related to the Bandwidth

$\Rightarrow$ Bandwidth is the frequency upto which the control sys. performs well.

$\Rightarrow$ For stability we need $GM > 1$ and $PM > 0^\circ$