

# Capacitors and Inductors

*But in science the credit goes to the man who convinces the world, not to the man to whom the idea first occurs.*

—Francis Darwin

## Enhancing Your Skills and Your Career

**ABET EC 2000 criteria (3.c), “an ability to design a system, component, or process to meet desired needs.”**

The “ability to design a system, component, or process to meet desired needs” is why engineers are hired. That is why this is the most important *technical* skill that an engineer has. Interestingly, your success as an engineer is directly proportional to your ability to communicate but your being able to design is why you will be hired in the first place.

Design takes place when you have what is termed an open-ended problem that eventually is defined by the solution. Within the context of this course or textbook, we can only explore some of the elements of design. Pursuing all of the steps of our problem-solving technique teaches you several of the most important elements of the design process.

Probably the most important part of design is clearly defining what the system, component, process, or, in our case, problem is. Rarely is an engineer given a perfectly clear assignment. Therefore, as a student, you can develop and enhance this skill by asking yourself, your colleagues, or your professors questions designed to clarify the problem statement.

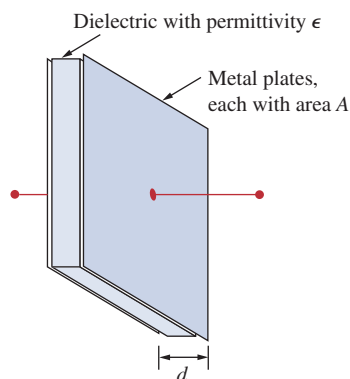
Exploring alternative solutions is another important part of the design process. Again, as a student, you can practice this part of the design process on almost every problem you work.

Evaluating your solutions is critical to any engineering assignment. Again, this is a skill that you as a student can practice on every problem you work.

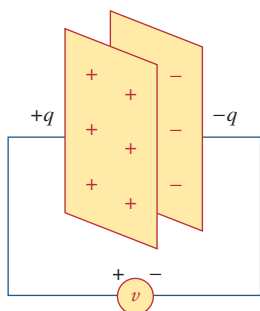


Photo by Charles Alexander

In contrast to a resistor, which spends or dissipates energy irreversibly, an inductor or capacitor stores or releases energy (i.e., has a memory).



**Figure 6.1**  
A typical capacitor.



**Figure 6.2**  
A capacitor with applied voltage  $v$ .

Alternatively, capacitance is the amount of charge stored per plate for a unit voltage difference in a capacitor.

## 6.1 Introduction

So far we have limited our study to resistive circuits. In this chapter, we shall introduce two new and important passive linear circuit elements: the capacitor and the inductor. Unlike resistors, which dissipate energy, capacitors and inductors do not dissipate but store energy, which can be retrieved at a later time. For this reason, capacitors and inductors are called *storage* elements.

The application of resistive circuits is quite limited. With the introduction of capacitors and inductors in this chapter, we will be able to analyze more important and practical circuits. Be assured that the circuit analysis techniques covered in Chapters 3 and 4 are equally applicable to circuits with capacitors and inductors.

We begin by introducing capacitors and describing how to combine them in series or in parallel. Later, we do the same for inductors. As typical applications, we explore how capacitors are combined with op amps to form integrators, differentiators, and analog computers.

## 6.2 Capacitors

A capacitor is a passive element designed to store energy in its electric field. Besides resistors, capacitors are the most common electrical components. Capacitors are used extensively in electronics, communications, computers, and power systems. For example, they are used in the tuning circuits of radio receivers and as dynamic memory elements in computer systems.

A capacitor is typically constructed as depicted in Fig. 6.1.

A **capacitor** consists of two conducting plates separated by an insulator (or dielectric).

In many practical applications, the plates may be aluminum foil while the dielectric may be air, ceramic, paper, or mica.

When a voltage source  $v$  is connected to the capacitor, as in Fig. 6.2, the source deposits a positive charge  $q$  on one plate and a negative charge  $-q$  on the other. The capacitor is said to store the electric charge. The amount of charge stored, represented by  $q$ , is directly proportional to the applied voltage  $v$  so that

$$q = Cv \quad (6.1)$$

where  $C$ , the constant of proportionality, is known as the *capacitance* of the capacitor. The unit of capacitance is the farad (F), in honor of the English physicist Michael Faraday (1791–1867). From Eq. (6.1), we may derive the following definition.

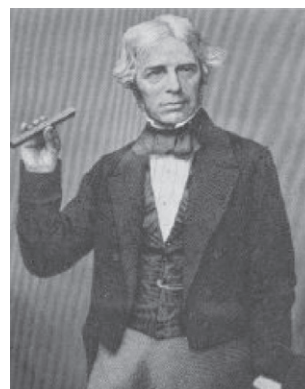
**Capacitance** is the ratio of the charge on one plate of a capacitor to the voltage difference between the two plates, measured in farads (F).

Note from Eq. (6.1) that 1 farad = 1 coulomb/volt.

## Historical

**Michael Faraday** (1791–1867), an English chemist and physicist, was probably the greatest experimentalist who ever lived.

Born near London, Faraday realized his boyhood dream by working with the great chemist Sir Humphry Davy at the Royal Institution, where he worked for 54 years. He made several contributions in all areas of physical science and coined such words as electrolysis, anode, and cathode. His discovery of electromagnetic induction in 1831 was a major breakthrough in engineering because it provided a way of generating electricity. The electric motor and generator operate on this principle. The unit of capacitance, the farad, was named in his honor.



The Burndy Library Collection  
at The Huntington Library,  
San Marino, California.

Although the capacitance  $C$  of a capacitor is the ratio of the charge  $q$  per plate to the applied voltage  $v$ , it does not depend on  $q$  or  $v$ . It depends on the physical dimensions of the capacitor. For example, for the parallel-plate capacitor shown in Fig. 6.1, the capacitance is given by

$$C = \frac{\epsilon A}{d} \quad (6.2)$$

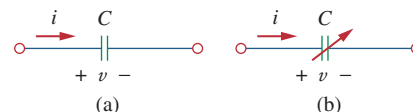
where  $A$  is the surface area of each plate,  $d$  is the distance between the plates, and  $\epsilon$  is the permittivity of the dielectric material between the plates. Although Eq. (6.2) applies to only parallel-plate capacitors, we may infer from it that, in general, three factors determine the value of the capacitance:

1. The surface area of the plates—the larger the area, the greater the capacitance.
2. The spacing between the plates—the smaller the spacing, the greater the capacitance.
3. The permittivity of the material—the higher the permittivity, the greater the capacitance.

Capacitors are commercially available in different values and types. Typically, capacitors have values in the picofarad (pF) to microfarad ( $\mu\text{F}$ ) range. They are described by the dielectric material they are made of and by whether they are of fixed or variable type. Figure 6.3 shows the circuit symbols for fixed and variable capacitors. Note that according to the passive sign convention, if  $v > 0$  and  $i > 0$  or if  $v < 0$  and  $i < 0$ , the capacitor is being charged, and if  $v \cdot i < 0$ , the capacitor is discharging.

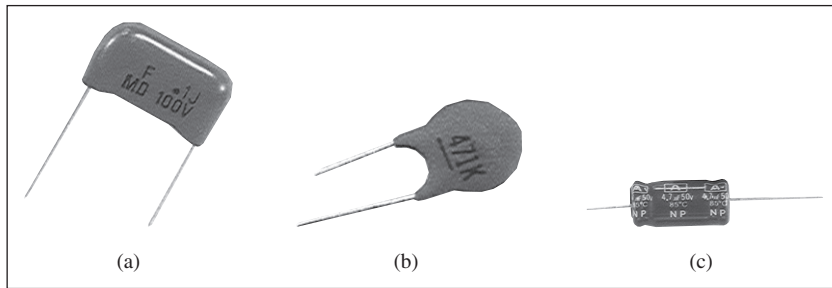
Figure 6.4 shows common types of fixed-value capacitors. Polyester capacitors are light in weight, stable, and their change with temperature is predictable. Instead of polyester, other dielectric materials such as mica and polystyrene may be used. Film capacitors are rolled and housed in metal or plastic films. Electrolytic capacitors produce very high capacitance. Figure 6.5 shows the most common types of variable capacitors. The capacitance of a trimmer (or padder) capacitor

Capacitor voltage rating and capacitance are typically inversely related due to the relationships in Eqs. (6.1) and (6.2). Arcing occurs if  $d$  is small and  $V$  is high.



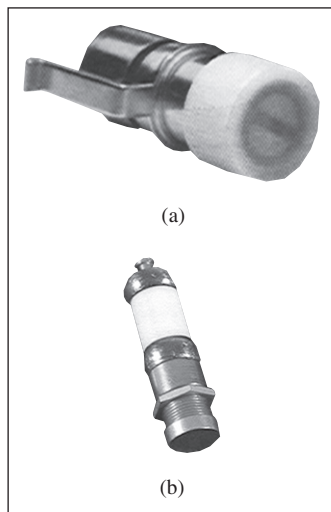
**Figure 6.3**

Circuit symbols for capacitors: (a) fixed capacitor, (b) variable capacitor.



**Figure 6.4**

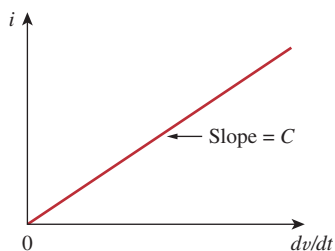
Fixed capacitors: (a) polyester capacitor, (b) ceramic capacitor, (c) electrolytic capacitor. Courtesy of Tech America.



**Figure 6.5**

Variable capacitors: (a) trimmer capacitor, (b) filmtrim capacitor. Courtesy of Johanson.

According to Eq. (6.4), for a capacitor to carry current, its voltage must vary with time. Hence, for constant voltage,  $i = 0$ .



**Figure 6.6**

Current-voltage relationship of a capacitor.

is often placed in parallel with another capacitor so that the equivalent capacitance can be varied slightly. The capacitance of the variable air capacitor (meshed plates) is varied by turning the shaft. Variable capacitors are used in radio receivers allowing one to tune to various stations. In addition, capacitors are used to block dc, pass ac, shift phase, store energy, start motors, and suppress noise.

To obtain the current-voltage relationship of the capacitor, we take the derivative of both sides of Eq. (6.1). Since

$$i = \frac{dq}{dt} \quad (6.3)$$

differentiating both sides of Eq. (6.1) gives

$$i = C \frac{dv}{dt} \quad (6.4)$$

This is the current-voltage relationship for a capacitor, assuming the passive sign convention. The relationship is illustrated in Fig. 6.6 for a capacitor whose capacitance is independent of voltage. Capacitors that satisfy Eq. (6.4) are said to be *linear*. For a *nonlinear capacitor*, the plot of the current-voltage relationship is not a straight line. Although some capacitors are nonlinear, most are linear. We will assume linear capacitors in this book.

The voltage-current relation of the capacitor can be obtained by integrating both sides of Eq. (6.4). We get

$$v(t) = \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau \quad (6.5)$$

or

$$v(t) = \frac{1}{C} \int_{t_0}^t i(\tau) d\tau + v(t_0) \quad (6.6)$$

where  $v(t_0) = q(t_0)/C$  is the voltage across the capacitor at time  $t_0$ . Equation (6.6) shows that capacitor voltage depends on the past history

of the capacitor current. Hence, the capacitor has memory—a property that is often exploited.

The instantaneous power delivered to the capacitor is

$$p = vi = Cv \frac{dv}{dt} \quad (6.7)$$

The energy stored in the capacitor is therefore

$$w = \int_{-\infty}^t p(\tau) d\tau = C \int_{-\infty}^t v \frac{dv}{d\tau} d\tau = C \int_{v(-\infty)}^{v(t)} v dv = \frac{1}{2} Cv^2 \Big|_{v(-\infty)}^{v(t)} \quad (6.8)$$

We note that  $v(-\infty) = 0$ , because the capacitor was uncharged at  $t = -\infty$ . Thus,

$$w = \frac{1}{2} Cv^2 \quad (6.9)$$

Using Eq. (6.1), we may rewrite Eq. (6.9) as

$$w = \frac{q^2}{2C} \quad (6.10)$$

Equation (6.9) or (6.10) represents the energy stored in the electric field that exists between the plates of the capacitor. This energy can be retrieved, since an ideal capacitor cannot dissipate energy. In fact, the word *capacitor* is derived from this element's capacity to store energy in an electric field.

We should note the following important properties of a capacitor:

1. Note from Eq. (6.4) that when the voltage across a capacitor is not changing with time (i.e., dc voltage), the current through the capacitor is zero. Thus,

A capacitor is an open circuit to dc.

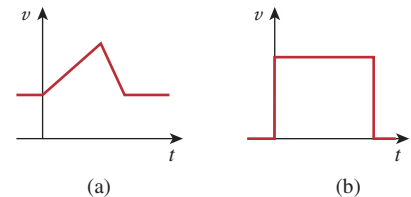
However, if a battery (dc voltage) is connected across a capacitor, the capacitor charges.

2. The voltage on the capacitor must be continuous.

The voltage on a capacitor cannot change abruptly.

The capacitor resists an abrupt change in the voltage across it. According to Eq. (6.4), a discontinuous change in voltage requires an infinite current, which is physically impossible. For example, the voltage across a capacitor may take the form shown in Fig. 6.7(a), whereas it is not physically possible for the capacitor voltage to take the form shown in Fig. 6.7(b) because of the abrupt changes. Conversely, the current through a capacitor can change instantaneously.

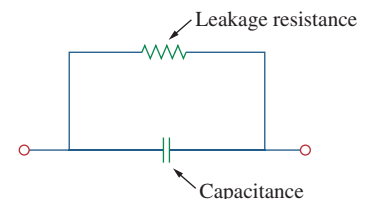
3. The ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy in its field and returns previously stored energy when delivering power to the circuit.
4. A real, nonideal capacitor has a parallel-model leakage resistance, as shown in Fig. 6.8. The leakage resistance may be as high as



**Figure 6.7**

Voltage across a capacitor: (a) allowed, (b) not allowable; an abrupt change is not possible.

An alternative way of looking at this is using Eq. (6.9), which indicates that energy is proportional to voltage squared. Since injecting or extracting energy can only be done over some finite time, voltage cannot change instantaneously across a capacitor.



**Figure 6.8**

Circuit model of a nonideal capacitor.

100 M $\Omega$  and can be neglected for most practical applications. For this reason, we will assume ideal capacitors in this book.

### Example 6.1

- (a) Calculate the charge stored on a 3-pF capacitor with 20 V across it.  
 (b) Find the energy stored in the capacitor.

#### Solution:

- (a) Since  $q = Cv$ ,

$$q = 3 \times 10^{-12} \times 20 = 60 \text{ pC}$$

- (b) The energy stored is

$$w = \frac{1}{2}Cv^2 = \frac{1}{2} \times 3 \times 10^{-12} \times 400 = 600 \text{ pJ}$$

### Practice Problem 6.1

What is the voltage across a 4.5- $\mu\text{F}$  capacitor if the charge on one plate is 0.12 mC? How much energy is stored?

**Answer:** 26.67 V, 1.6 mJ.

### Example 6.2

The voltage across a 5- $\mu\text{F}$  capacitor is

$$v(t) = 10 \cos 6000t \text{ V}$$

Calculate the current through it.

#### Solution:

By definition, the current is

$$\begin{aligned} i(t) &= C \frac{dv}{dt} = 5 \times 10^{-6} \frac{d}{dt}(10 \cos 6000t) \\ &= -5 \times 10^{-6} \times 6000 \times 10 \sin 6000t = -0.3 \sin 6000t \text{ A} \end{aligned}$$

### Practice Problem 6.2

If a 10- $\mu\text{F}$  capacitor is connected to a voltage source with

$$v(t) = 75 \sin 2000t \text{ V}$$

determine the current through the capacitor.

**Answer:** 1.5 cos 2000t A.

### Example 6.3

Determine the voltage across a 2- $\mu\text{F}$  capacitor if the current through it is

$$i(t) = 6e^{-3000t} \text{ mA}$$

Assume that the initial capacitor voltage is zero.

**Solution:**

Since  $v = \frac{1}{C} \int_0^t i dt + v(0)$  and  $v(0) = 0$ ,

$$\begin{aligned} v &= \frac{1}{2 \times 10^{-6}} \int_0^t 6e^{-3000t} dt \cdot 10^{-3} \\ &= \frac{3 \times 10^3}{-3000} e^{-3000t} \Big|_0^t = (1 - e^{-3000t}) \text{ V} \end{aligned}$$

The current through a  $100\text{-}\mu\text{F}$  capacitor is  $i(t) = 50 \sin 120\pi t$  mA. Calculate the voltage across it at  $t = 1$  ms and  $t = 5$  ms. Take  $v(0) = 0$ .

**Practice Problem 6.3**

**Answer:** 93.14 mV, 1.736 V.

Determine the current through a  $200\text{-}\mu\text{F}$  capacitor whose voltage is shown in Fig. 6.9.

**Example 6.4****Solution:**

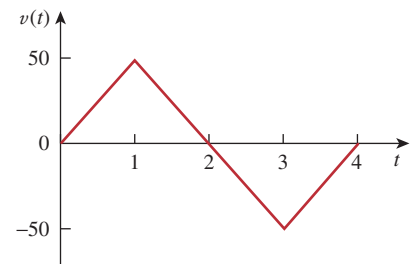
The voltage waveform can be described mathematically as

$$v(t) = \begin{cases} 50t \text{ V} & 0 < t < 1 \\ 100 - 50t \text{ V} & 1 < t < 3 \\ -200 + 50t \text{ V} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

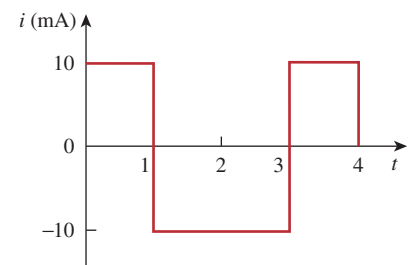
Since  $i = C dv/dt$  and  $C = 200 \mu\text{F}$ , we take the derivative of  $v$  to obtain

$$\begin{aligned} i(t) &= 200 \times 10^{-6} \times \begin{cases} 50 & 0 < t < 1 \\ -50 & 1 < t < 3 \\ 50 & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 10 \text{ mA} & 0 < t < 1 \\ -10 \text{ mA} & 1 < t < 3 \\ 10 \text{ mA} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Thus the current waveform is as shown in Fig. 6.10.



**Figure 6.9**  
For Example 6.4.

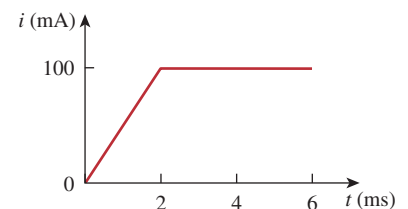


**Figure 6.10**  
For Example 6.4.

An initially uncharged  $1\text{-mF}$  capacitor has the current shown in Fig. 6.11 across it. Calculate the voltage across it at  $t = 2$  ms and  $t = 5$  ms.

**Practice Problem 6.4**

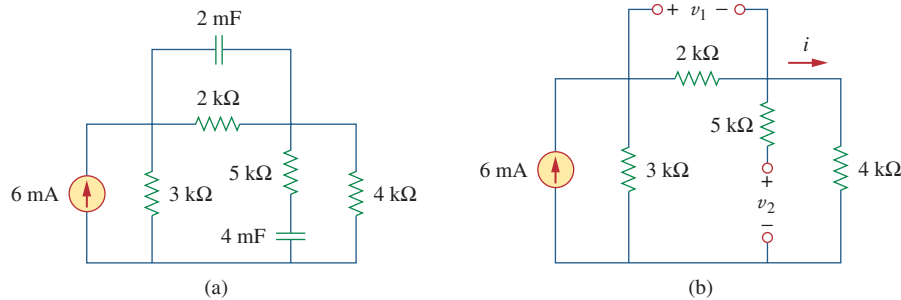
**Answer:** 100 mV, 400 mV.



**Figure 6.11**  
For Practice Prob. 6.4.

**Example 6.5**

Obtain the energy stored in each capacitor in Fig. 6.12(a) under dc conditions.



**Figure 6.12**  
For Example 6.5.

**Solution:**

Under dc conditions, we replace each capacitor with an open circuit, as shown in Fig. 6.12(b). The current through the series combination of the 2-k $\Omega$  and 4-k $\Omega$  resistors is obtained by current division as

$$i = \frac{3}{3 + 2 + 4}(6 \text{ mA}) = 2 \text{ mA}$$

Hence, the voltages  $v_1$  and  $v_2$  across the capacitors are

$$v_1 = 2000i = 4 \text{ V} \quad v_2 = 4000i = 8 \text{ V}$$

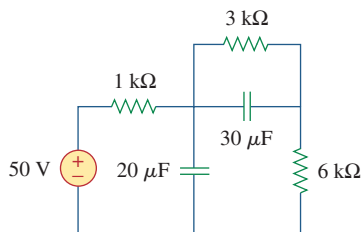
and the energies stored in them are

$$w_1 = \frac{1}{2}C_1v_1^2 = \frac{1}{2}(2 \times 10^{-3})(4)^2 = 16 \text{ mJ}$$

$$w_2 = \frac{1}{2}C_2v_2^2 = \frac{1}{2}(4 \times 10^{-3})(8)^2 = 128 \text{ mJ}$$

**Practice Problem 6.5**

Under dc conditions, find the energy stored in the capacitors in Fig. 6.13.



**Figure 6.13**  
For Practice Prob. 6.5.

**Answer:** 20.25 mJ, 3.375 mJ.

**6.3 Series and Parallel Capacitors**

We know from resistive circuits that the series-parallel combination is a powerful tool for reducing circuits. This technique can be extended to series-parallel connections of capacitors, which are sometimes encountered. We desire to replace these capacitors by a single equivalent capacitor  $C_{\text{eq}}$ .

In order to obtain the equivalent capacitor  $C_{\text{eq}}$  of  $N$  capacitors in parallel, consider the circuit in Fig. 6.14(a). The equivalent circuit is



in Fig. 6.14(b). Note that the capacitors have the same voltage  $v$  across them. Applying KCL to Fig. 6.14(a),

$$i = i_1 + i_2 + i_3 + \cdots + i_N \quad (6.11)$$

But  $i_k = C_k dv/dt$ . Hence,

$$\begin{aligned} i &= C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt} + \cdots + C_N \frac{dv}{dt} \\ &= \left( \sum_{k=1}^N C_k \right) \frac{dv}{dt} = C_{\text{eq}} \frac{dv}{dt} \end{aligned} \quad (6.12)$$

where

$$C_{\text{eq}} = C_1 + C_2 + C_3 + \cdots + C_N \quad (6.13)$$

The **equivalent capacitance** of  $N$  parallel-connected capacitors is the sum of the individual capacitances.

We observe that capacitors in parallel combine in the same manner as resistors in series.

We now obtain  $C_{\text{eq}}$  of  $N$  capacitors connected in series by comparing the circuit in Fig. 6.15(a) with the equivalent circuit in Fig. 6.15(b). Note that the same current  $i$  flows (and consequently the same charge) through the capacitors. Applying KVL to the loop in Fig. 6.15(a),

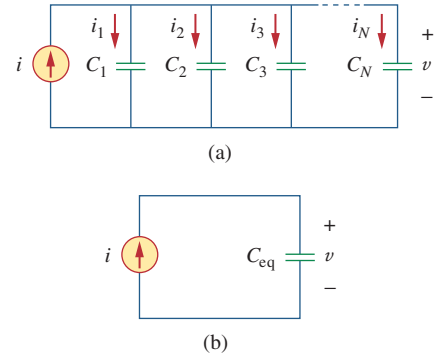
$$v = v_1 + v_2 + v_3 + \cdots + v_N \quad (6.14)$$

But  $v_k = \frac{1}{C_k} \int_{t_0}^t i(\tau) d\tau + v_k(t_0)$ . Therefore,

$$\begin{aligned} v &= \frac{1}{C_1} \int_{t_0}^t i(\tau) d\tau + v_1(t_0) + \frac{1}{C_2} \int_{t_0}^t i(\tau) d\tau + v_2(t_0) \\ &\quad + \cdots + \frac{1}{C_N} \int_{t_0}^t i(\tau) d\tau + v_N(t_0) \\ &= \left( \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N} \right) \int_{t_0}^t i(\tau) d\tau + v_1(t_0) + v_2(t_0) \\ &\quad + \cdots + v_N(t_0) \\ &= \frac{1}{C_{\text{eq}}} \int_{t_0}^t i(\tau) d\tau + v(t_0) \end{aligned} \quad (6.15)$$

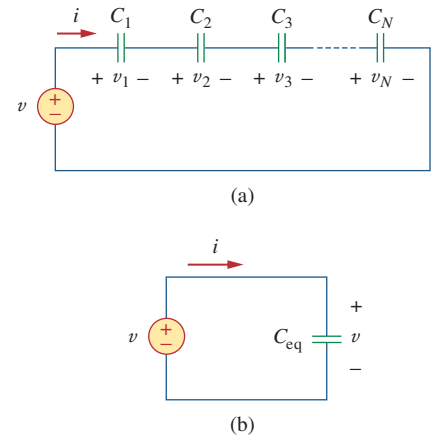
where

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_N} \quad (6.16)$$



**Figure 6.14**

(a) Parallel-connected  $N$  capacitors, (b) equivalent circuit for the parallel capacitors.



**Figure 6.15**

(a) Series-connected  $N$  capacitors, (b) equivalent circuit for the series capacitor.

The initial voltage  $v(t_0)$  across  $C_{\text{eq}}$  is required by KVL to be the sum of the capacitor voltages at  $t_0$ . Or according to Eq. (6.15),

$$v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0)$$

Thus, according to Eq. (6.16),

The **equivalent capacitance** of series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.

Note that capacitors in series combine in the same manner as resistors in parallel. For  $N = 2$  (i.e., two capacitors in series), Eq. (6.16) becomes

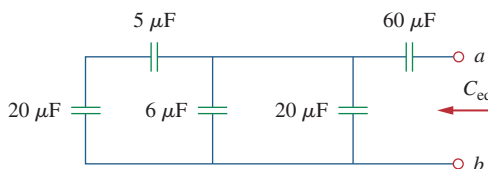
$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2}$$

or

$$C_{\text{eq}} = \frac{C_1 C_2}{C_1 + C_2} \quad (6.17)$$

### Example 6.6

Find the equivalent capacitance seen between terminals  $a$  and  $b$  of the circuit in Fig. 6.16.



**Figure 6.16**  
For Example 6.6.

#### Solution:

The  $20\text{-}\mu\text{F}$  and  $5\text{-}\mu\text{F}$  capacitors are in series; their equivalent capacitance is

$$\frac{20 \times 5}{20 + 5} = 4 \mu\text{F}$$

This  $4\text{-}\mu\text{F}$  capacitor is in parallel with the  $6\text{-}\mu\text{F}$  and  $20\text{-}\mu\text{F}$  capacitors; their combined capacitance is

$$4 + 6 + 20 = 30 \mu\text{F}$$

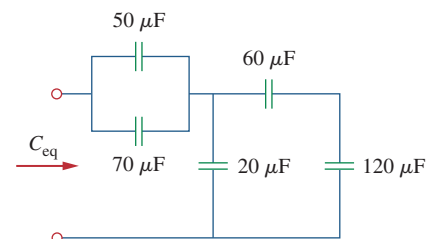
This  $30\text{-}\mu\text{F}$  capacitor is in series with the  $60\text{-}\mu\text{F}$  capacitor. Hence, the equivalent capacitance for the entire circuit is

$$C_{\text{eq}} = \frac{30 \times 60}{30 + 60} = 20 \mu\text{F}$$

Find the equivalent capacitance seen at the terminals of the circuit in Fig. 6.17.

**Answer:**  $40 \mu\text{F}$ .

### Practice Problem 6.6



**Figure 6.17**

For Practice Prob. 6.6.

For the circuit in Fig. 6.18, find the voltage across each capacitor.

### Example 6.7

#### Solution:

We first find the equivalent capacitance  $C_{\text{eq}}$ , shown in Fig. 6.19. The two parallel capacitors in Fig. 6.18 can be combined to get  $40 + 20 = 60 \text{ mF}$ . This 60-mF capacitor is in series with the 20-mF and 30-mF capacitors. Thus,

$$C_{\text{eq}} = \frac{1}{\frac{1}{60} + \frac{1}{30} + \frac{1}{20}} \text{ mF} = 10 \text{ mF}$$

The total charge is

$$q = C_{\text{eq}}v = 10 \times 10^{-3} \times 30 = 0.3 \text{ C}$$

This is the charge on the 20-mF and 30-mF capacitors, because they are in series with the 30-V source. (A crude way to see this is to imagine that charge acts like current, since  $i = dq/dt$ .) Therefore,

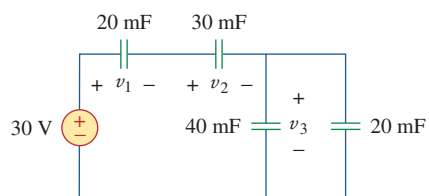
$$v_1 = \frac{q}{C_1} = \frac{0.3}{20 \times 10^{-3}} = 15 \text{ V} \quad v_2 = \frac{q}{C_2} = \frac{0.3}{30 \times 10^{-3}} = 10 \text{ V}$$

Having determined  $v_1$  and  $v_2$ , we now use KVL to determine  $v_3$  by

$$v_3 = 30 - v_1 - v_2 = 5 \text{ V}$$

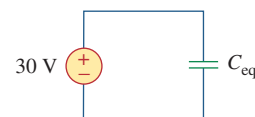
Alternatively, since the 40-mF and 20-mF capacitors are in parallel, they have the same voltage  $v_3$  and their combined capacitance is  $40 + 20 = 60 \text{ mF}$ . This combined capacitance is in series with the 20-mF and 30-mF capacitors and consequently has the same charge on it. Hence,

$$v_3 = \frac{q}{60 \text{ mF}} = \frac{0.3}{60 \times 10^{-3}} = 5 \text{ V}$$



**Figure 6.18**

For Example 6.7.



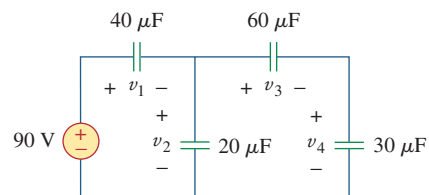
**Figure 6.19**

Equivalent circuit for Fig. 6.18.

Find the voltage across each of the capacitors in Fig. 6.20.

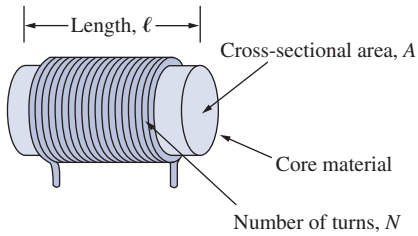
### Practice Problem 6.7

**Answer:**  $v_1 = 45 \text{ V}$ ,  $v_2 = 45 \text{ V}$ ,  $v_3 = 15 \text{ V}$ ,  $v_4 = 30 \text{ V}$ .



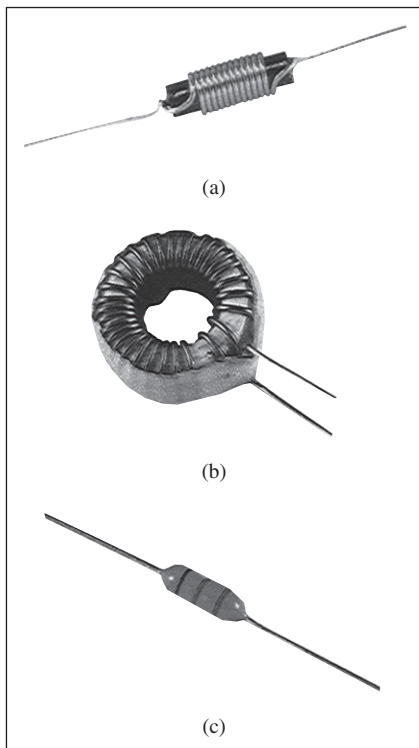
**Figure 6.20**

For Practice Prob. 6.7.



**Figure 6.21**  
Typical form of an inductor.

In view of Eq. (6.18), for an inductor to have voltage across its terminals, its current must vary with time. Hence,  $v = 0$  for constant current through the inductor.



**Figure 6.22**  
Various types of inductors: (a) solenoidal wound inductor, (b) toroidal inductor, (c) chip inductor.  
Courtesy of Tech America.

## 6.4 Inductors

An inductor is a passive element designed to store energy in its magnetic field. Inductors find numerous applications in electronic and power systems. They are used in power supplies, transformers, radios, TVs, radars, and electric motors.

Any conductor of electric current has inductive properties and may be regarded as an inductor. But in order to enhance the inductive effect, a practical inductor is usually formed into a cylindrical coil with many turns of conducting wire, as shown in Fig. 6.21.

An **inductor** consists of a coil of conducting wire.

If current is allowed to pass through an inductor, it is found that the voltage across the inductor is directly proportional to the time rate of change of the current. Using the passive sign convention,

$$v = L \frac{di}{dt} \quad (6.18)$$

where  $L$  is the constant of proportionality called the *inductance* of the inductor. The unit of inductance is the henry (H), named in honor of the American inventor Joseph Henry (1797–1878). It is clear from Eq. (6.18) that 1 henry equals 1 volt-second per ampere.

**Inductance** is the property whereby an inductor exhibits opposition to the change of current flowing through it, measured in henrys (H).

The inductance of an inductor depends on its physical dimension and construction. Formulas for calculating the inductance of inductors of different shapes are derived from electromagnetic theory and can be found in standard electrical engineering handbooks. For example, for the inductor, (solenoid) shown in Fig. 6.21,

$$L = \frac{N^2 \mu A}{\ell} \quad (6.19)$$

where  $N$  is the number of turns,  $\ell$  is the length,  $A$  is the cross-sectional area, and  $\mu$  is the permeability of the core. We can see from Eq. (6.19) that inductance can be increased by increasing the number of turns of coil, using material with higher permeability as the core, increasing the cross-sectional area, or reducing the length of the coil.

Like capacitors, commercially available inductors come in different values and types. Typical practical inductors have inductance values ranging from a few microhenrys ( $\mu\text{H}$ ), as in communication systems, to tens of henrys (H) as in power systems. Inductors may be fixed or variable. **The core may be made of iron, steel, plastic, or air. The terms *coil* and *choke* are also used for inductors.** Common inductors are shown in Fig. 6.22. The circuit symbols for inductors are shown in Fig. 6.23, following the passive sign convention.

Equation (6.18) is the voltage-current relationship for an inductor. Figure 6.24 shows this relationship graphically for an inductor whose

## Historical

**Joseph Henry** (1797–1878), an American physicist, discovered inductance and constructed an electric motor.

Born in Albany, New York, Henry graduated from Albany Academy and taught philosophy at Princeton University from 1832 to 1846. He was the first secretary of the Smithsonian Institution. He conducted several experiments on electromagnetism and developed powerful electromagnets that could lift objects weighing thousands of pounds. Interestingly, Joseph Henry discovered electromagnetic induction before Faraday but failed to publish his findings. The unit of inductance, the henry, was named after him.



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inductance is independent of current. Such an inductor is known as a *linear inductor*. For a *nonlinear inductor*, the plot of Eq. (6.18) will not be a straight line because its inductance varies with current. We will assume linear inductors in this textbook unless stated otherwise.

The current-voltage relationship is obtained from Eq. (6.18) as

$$di = \frac{1}{L} v dt$$

Integrating gives

$$i = \frac{1}{L} \int_{-\infty}^t v(\tau) d\tau \quad (6.20)$$

or

$$i = \frac{1}{L} \int_{t_0}^t v(\tau) d\tau + i(t_0) \quad (6.21)$$

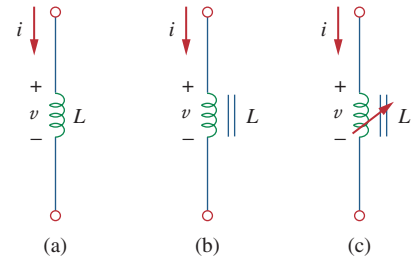
where  $i(t_0)$  is the total current for  $-\infty < t < t_0$  and  $i(-\infty) = 0$ . The idea of making  $i(-\infty) = 0$  is practical and reasonable, because there must be a time in the past when there was no current in the inductor.

The inductor is designed to store energy in its magnetic field. The energy stored can be obtained from Eq. (6.18). The power delivered to the inductor is

$$p = vi = \left( L \frac{di}{dt} \right) i \quad (6.22)$$

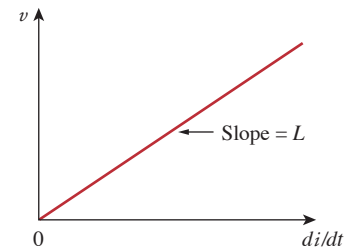
The energy stored is

$$\begin{aligned} w &= \int_{-\infty}^t p(\tau) d\tau = L \int_{-\infty}^t \frac{di}{d\tau} i d\tau \\ &= L \int_{-\infty}^t i di = \frac{1}{2} Li^2(t) - \frac{1}{2} Li^2(-\infty) \end{aligned} \quad (6.23)$$



**Figure 6.23**

Circuit symbols for inductors: (a) air-core, (b) iron-core, (c) variable iron-core.



**Figure 6.24**

Voltage-current relationship of an inductor.

Since  $i(-\infty) = 0$ ,

$$w = \frac{1}{2}Li^2 \quad (6.24)$$

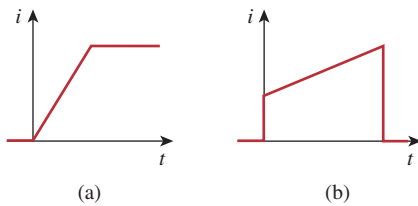
We should note the following important properties of an inductor.

1. Note from Eq. (6.18) that the voltage across an inductor is zero when the current is constant. Thus,

An inductor acts like a short circuit to dc.

2. An important property of the inductor is its opposition to the change in current flowing through it.

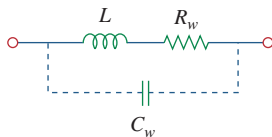
The current through an inductor cannot change instantaneously.



**Figure 6.25**

Current through an inductor: (a) allowed, (b) not allowable; an abrupt change is not possible.

Since an inductor is often made of a highly conducting wire, it has a very small resistance.



**Figure 6.26**

Circuit model for a practical inductor.

According to Eq. (6.18), a discontinuous change in the current through an inductor requires an infinite voltage, which is not physically possible. Thus, an inductor opposes an abrupt change in the current through it. For example, the current through an inductor may take the form shown in Fig. 6.25(a), whereas the inductor current cannot take the form shown in Fig. 6.25(b) in real-life situations due to the discontinuities. However, the voltage across an inductor can change abruptly.

3. Like the ideal capacitor, the ideal inductor does not dissipate energy. The energy stored in it can be retrieved at a later time. The inductor takes power from the circuit when storing energy and delivers power to the circuit when returning previously stored energy.
4. A practical, nonideal inductor has a significant resistive component, as shown in Fig. 6.26. This is due to the fact that the inductor is made of a conducting material such as copper, which has some resistance. This resistance is called the *winding resistance*  $R_w$ , and it appears in series with the inductance of the inductor. The presence of  $R_w$  makes it both an energy storage device and an energy dissipation device. Since  $R_w$  is usually very small, it is ignored in most cases. The nonideal inductor also has a *winding capacitance*  $C_w$  due to the capacitive coupling between the conducting coils.  $C_w$  is very small and can be ignored in most cases, except at high frequencies. We will assume ideal inductors in this book.

### Example 6.8

The current through a 0.1-H inductor is  $i(t) = 10te^{-5t}$  A. Find the voltage across the inductor and the energy stored in it.

#### Solution:

Since  $v = L di/dt$  and  $L = 0.1$  H,

$$v = 0.1 \frac{d}{dt}(10te^{-5t}) = e^{-5t} + t(-5)e^{-5t} = e^{-5t}(1 - 5t) \text{ V}$$

The energy stored is

$$w = \frac{1}{2}Li^2 = \frac{1}{2}(0.1)100t^2e^{-10t} = 5t^2e^{-10t} \text{ J}$$

If the current through a 1-mH inductor is  $i(t) = 60 \cos 100t$  mA, find the terminal voltage and the energy stored.

### Practice Problem 6.8

**Answer:**  $-6 \sin 100t$  mV,  $1.8 \cos^2(100t) \mu\text{J}$ .

Find the current through a 5-H inductor if the voltage across it is

### Example 6.9

$$v(t) = \begin{cases} 30t^2, & t > 0 \\ 0, & t < 0 \end{cases}$$

Also, find the energy stored at  $t = 5$  s. Assume  $i(v) > 0$ .

**Solution:**

Since  $i = \frac{1}{L} \int_{t_0}^t v(t) dt + i(t_0)$  and  $L = 5$  H,

$$i = \frac{1}{5} \int_0^t 30t^2 dt + 0 = 6 \times \frac{t^3}{3} = 2t^3 \text{ A}$$

The power  $p = vi = 60t^5$ , and the energy stored is then

$$w = \int p dt = \int_0^5 60t^5 dt = 60 \left. \frac{t^6}{6} \right|_0^5 = 156.25 \text{ kJ}$$

Alternatively, we can obtain the energy stored using Eq. (6.24), by writing

$$w|_0^5 = \frac{1}{2}Li^2(5) - \frac{1}{2}Li^2(0) = \frac{1}{2}(5)(2 \times 5^3)^2 - 0 = 156.25 \text{ kJ}$$

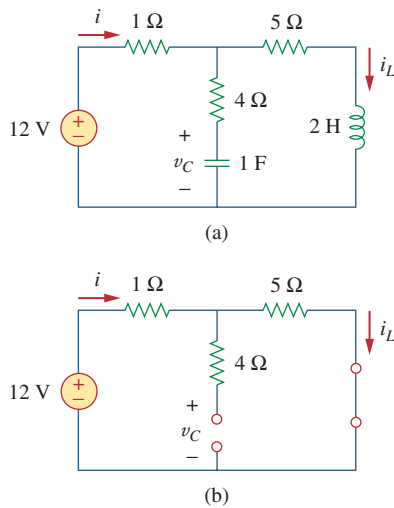
as obtained before.

The terminal voltage of a 2-H inductor is  $v = 10(1 - t)$  V. Find the current flowing through it at  $t = 4$  s and the energy stored in it at  $t = 4$  s. Assume  $i(0) = 2$  A.

### Practice Problem 6.9

**Answer:**  $-18$  A,  $320$  J.

### Example 6.10



**Figure 6.27**

For Example 6.10.

Consider the circuit in Fig. 6.27(a). Under dc conditions, find: (a)  $i$ ,  $v_C$ , and  $i_L$ , (b) the energy stored in the capacitor and inductor.

#### Solution:

(a) Under dc conditions, we replace the capacitor with an open circuit and the inductor with a short circuit, as in Fig. 6.27(b). It is evident from Fig. 6.27(b) that

$$i = i_L = \frac{12}{1 + 5} = 2 \text{ A}$$

The voltage  $v_C$  is the same as the voltage across the 5- $\Omega$  resistor. Hence,

$$v_C = 5i = 10 \text{ V}$$

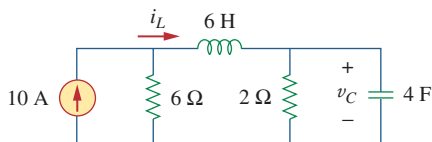
(b) The energy in the capacitor is

$$w_C = \frac{1}{2} C v_C^2 = \frac{1}{2} (1)(10^2) = 50 \text{ J}$$

and that in the inductor is

$$w_L = \frac{1}{2} L i_L^2 = \frac{1}{2} (2)(2^2) = 4 \text{ J}$$

### Practice Problem 6.10



**Figure 6.28**

For Practice Prob. 6.10.

Determine  $v_C$ ,  $i_L$ , and the energy stored in the capacitor and inductor in the circuit of Fig. 6.28 under dc conditions.

**Answer:** 15 V, 7.5 A, 450 J, 168.75 J.

## 6.5 Series and Parallel Inductors

Now that the inductor has been added to our list of passive elements, it is necessary to extend the powerful tool of series-parallel combination. We need to know how to find the equivalent inductance of a series-connected or parallel-connected set of inductors found in practical circuits.

Consider a series connection of  $N$  inductors, as shown in Fig. 6.29(a), with the equivalent circuit shown in Fig. 6.29(b). The inductors have the same current through them. Applying KVL to the loop,

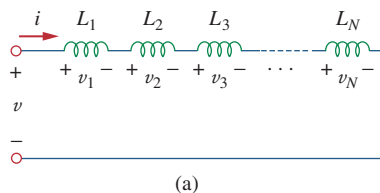
$$v = v_1 + v_2 + v_3 + \cdots + v_N \quad (6.25)$$

Substituting  $v_k = L_k di/dt$  results in

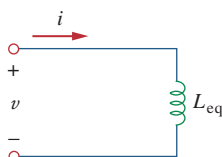
$$\begin{aligned} v &= L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt} + \cdots + L_N \frac{di}{dt} \\ &= (L_1 + L_2 + L_3 + \cdots + L_N) \frac{di}{dt} \\ &= \left( \sum_{k=1}^N L_k \right) \frac{di}{dt} = L_{\text{eq}} \frac{di}{dt} \end{aligned} \quad (6.26)$$

where

$$L_{\text{eq}} = L_1 + L_2 + L_3 + \cdots + L_N \quad (6.27)$$



(a)



(b)

**Figure 6.29**

(a) A series connection of  $N$  inductors, (b) equivalent circuit for the series inductors.



Thus,

The **equivalent inductance** of series-connected inductors is the sum of the individual inductances.

Inductors in series are combined in exactly the same way as resistors in series.

We now consider a parallel connection of  $N$  inductors, as shown in Fig. 6.30(a), with the equivalent circuit in Fig. 6.30(b). The inductors have the same voltage across them. Using KCL,

$$i = i_1 + i_2 + i_3 + \cdots + i_N \quad (6.28)$$

But  $i_k = \frac{1}{L_k} \int_{t_0}^t v dt + i_k(t_0)$ ; hence,

$$\begin{aligned} i &= \frac{1}{L_1} \int_{t_0}^t v dt + i_1(t_0) + \frac{1}{L_2} \int_{t_0}^t v dt + i_2(t_0) \\ &\quad + \cdots + \frac{1}{L_N} \int_{t_0}^t v dt + i_N(t_0) \\ &= \left( \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N} \right) \int_{t_0}^t v dt + i_1(t_0) + i_2(t_0) \\ &\quad + \cdots + i_N(t_0) \\ &= \left( \sum_{k=1}^N \frac{1}{L_k} \right) \int_{t_0}^t v dt + \sum_{k=1}^N i_k(t_0) = \frac{1}{L_{\text{eq}}} \int_{t_0}^t v dt + i(t_0) \end{aligned} \quad (6.29)$$

where

$$\frac{1}{L_{\text{eq}}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_N} \quad (6.30)$$

The initial current  $i(t_0)$  through  $L_{\text{eq}}$  at  $t = t_0$  is expected by KCL to be the sum of the inductor currents at  $t_0$ . Thus, according to Eq. (6.29),

$$i(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$

According to Eq. (6.30),

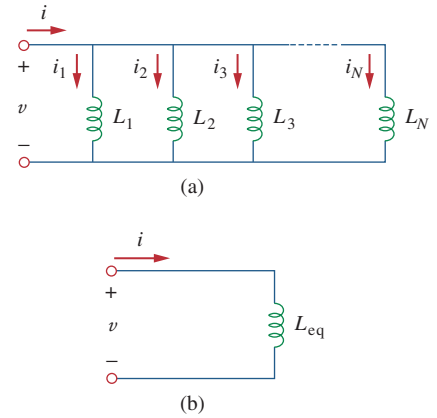
The **equivalent inductance** of parallel inductors is the reciprocal of the sum of the reciprocals of the individual inductances.

Note that the inductors in parallel are combined in the same way as resistors in parallel.

For two inductors in parallel ( $N = 2$ ), Eq. (6.30) becomes

$$\frac{1}{L_{\text{eq}}} = \frac{1}{L_1} + \frac{1}{L_2} \quad \text{or} \quad L_{\text{eq}} = \frac{L_1 L_2}{L_1 + L_2} \quad (6.31)$$

As long as all the elements are of the same type, the  $\Delta$ -Y transformations for resistors discussed in Section 2.7 can be extended to capacitors and inductors.



**Figure 6.30**

(a) A parallel connection of  $N$  inductors, (b) equivalent circuit for the parallel inductors.

TABLE 6.1

Important characteristics of the basic elements.<sup>†</sup>

Relation	Resistor ( $R$ )	Capacitor ( $C$ )	Inductor ( $L$ )
$v$ - $i$ :	$v = iR$	$v = \frac{1}{C} \int_{t_0}^t i(\tau) d\tau + v(t_0)$	$v = L \frac{di}{dt}$
$i$ - $v$ :	$i = v/R$	$i = C \frac{dv}{dt}$	$i = \frac{1}{L} \int_{t_0}^t v(\tau) d\tau + i(t_0)$
$p$ or $w$ :	$p = i^2 R = \frac{v^2}{R}$	$w = \frac{1}{2} C v^2$	$w = \frac{1}{2} L i^2$
Series:	$R_{\text{eq}} = R_1 + R_2$	$C_{\text{eq}} = \frac{C_1 C_2}{C_1 + C_2}$	$L_{\text{eq}} = L_1 + L_2$
Parallel:	$R_{\text{eq}} = \frac{R_1 R_2}{R_1 + R_2}$	$C_{\text{eq}} = C_1 + C_2$	$L_{\text{eq}} = \frac{L_1 L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	$v$	$i$

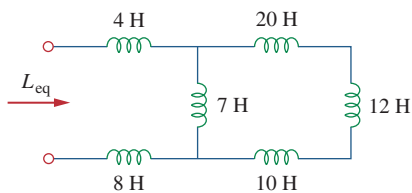
<sup>†</sup> Passive sign convention is assumed.

It is appropriate at this point to summarize the most important characteristics of the three basic circuit elements we have studied. The summary is given in Table 6.1.

The wye-delta transformation discussed in Section 2.7 for resistors can be extended to capacitors and inductors.

**Example 6.11**

Find the equivalent inductance of the circuit shown in Fig. 6.31.

**Figure 6.31**

For Example 6.11.

**Solution:**

The 10-H, 12-H, and 20-H inductors are in series; thus, combining them gives a 42-H inductance. This 42-H inductor is in parallel with the 7-H inductor so that they are combined, to give

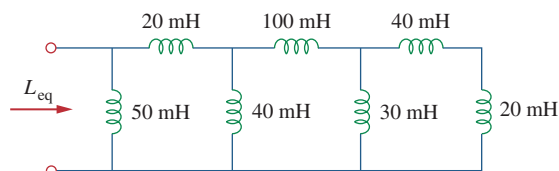
$$\frac{7 \times 42}{7 + 42} = 6 \text{ H}$$

This 6-H inductor is in series with the 4-H and 8-H inductors. Hence,

$$L_{\text{eq}} = 4 + 6 + 8 = 18 \text{ H}$$

**Practice Problem 6.11**

Calculate the equivalent inductance for the inductive ladder network in Fig. 6.32.

**Figure 6.32**

For Practice Prob. 6.11.

**Answer:** 25 mH.

For the circuit in Fig. 6.33,  $i(t) = 4(2 - e^{-10t})$  mA. If  $i_2(0) = -1$  mA, find: (a)  $i_1(0)$ ; (b)  $v(t)$ ,  $v_1(t)$ , and  $v_2(t)$ ; (c)  $i_1(t)$  and  $i_2(t)$ .

**Solution:**

(a) From  $i(t) = 4(2 - e^{-10t})$  mA,  $i(0) = 4(2 - 1) = 4$  mA. Since  $i = i_1 + i_2$ ,

$$i_1(0) = i(0) - i_2(0) = 4 - (-1) = 5 \text{ mA}$$

(b) The equivalent inductance is

$$L_{\text{eq}} = 2 + 4 \parallel 12 = 2 + 3 = 5 \text{ H}$$

Thus,

$$v(t) = L_{\text{eq}} \frac{di}{dt} = 5(4)(-1)(-10)e^{-10t} \text{ mV} = 200e^{-10t} \text{ mV}$$

and

$$v_1(t) = 2 \frac{di}{dt} = 2(-4)(-10)e^{-10t} \text{ mV} = 80e^{-10t} \text{ mV}$$

Since  $v = v_1 + v_2$ ,

$$v_2(t) = v(t) - v_1(t) = 120e^{-10t} \text{ mV}$$

(c) The current  $i_1$  is obtained as

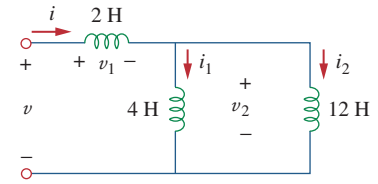
$$\begin{aligned} i_1(t) &= \frac{1}{4} \int_0^t v_2 dt + i_1(0) = \frac{120}{4} \int_0^t e^{-10t} dt + 5 \text{ mA} \\ &= -3e^{-10t} \Big|_0^t + 5 \text{ mA} = -3e^{-10t} + 3 + 5 = 8 - 3e^{-10t} \text{ mA} \end{aligned}$$

Similarly,

$$\begin{aligned} i_2(t) &= \frac{1}{12} \int_0^t v_2 dt + i_2(0) = \frac{120}{12} \int_0^t e^{-10t} dt - 1 \text{ mA} \\ &= -e^{-10t} \Big|_0^t - 1 \text{ mA} = -e^{-10t} + 1 - 1 = -e^{-10t} \text{ mA} \end{aligned}$$

Note that  $i_1(t) + i_2(t) = i(t)$ .

### Example 6.12



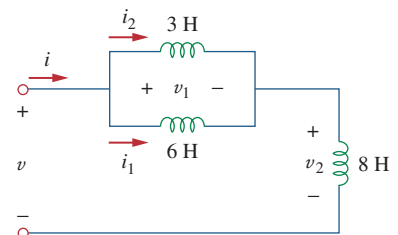
**Figure 6.33**

For Example 6.12.

In the circuit of Fig. 6.34,  $i_1(t) = 0.6e^{-2t}$  A. If  $i(0) = 1.4$  A, find: (a)  $i_2(0)$ ; (b)  $i_2(t)$  and  $i(t)$ ; (c)  $v_1(t)$ ,  $v_2(t)$ , and  $v(t)$ .

**Answer:** (a) 0.8 A, (b)  $(-0.4 + 1.2e^{-2t})$  A,  $(-0.4 + 1.8e^{-2t})$  A, (c)  $-36e^{-2t}$  V,  $-7.2e^{-2t}$  V,  $-28.8e^{-2t}$  V.

### Practice Problem 6.12



**Figure 6.34**

For Practice Prob. 6.12.

## 6.6 Applications

Circuit elements such as resistors and capacitors are commercially available in either discrete form or integrated-circuit (IC) form. Unlike capacitors and resistors, inductors with appreciable inductance are difficult to produce on IC substrates. Therefore, inductors (coils) usually