5. **Derivative of a product. (The product Rule)**

If f and g are differentiable at x, then fg is also differentiable at x and

$$[f(x)g(x)]' = f'(x)\, g(x) + f(x)\, g'(x), \text{ that is,}$$

$$\frac{d}{dx}[f(x)g(x)] = \left[\frac{d}{dx}[f(x)]\right] g(x) + f(x)\left[\frac{d}{dx}[g(x)]\right]$$

Proof: Let $\phi(x) = f(x)g(x)$. Then

(i) $\phi(x+\delta x) = f(x+\delta x)\, g(x+\delta x)$ and

(ii) $\phi(x+\delta x) - \phi(x) = f(x+\delta x)\, g(x+\delta x) - f(x)g(x)$

Subtracting and adding $f(x)\, g(x+\delta x)$ in step (ii), gives

$$\phi(x+\delta x) - \phi(x) = f(x+\delta x)\, g(x+\delta x) - f(x)\, g(x+\delta x) + f(x)\, g(x+\delta x) - f(x)g(x)$$

$$= [f(x+\delta x) - f(x)]\, g(x+\delta x) + f(x)\, [g(x+\delta x) - g(x)]$$

(iii) $$\frac{\phi(x+\delta x) - \phi(x)}{\delta x} = \left[\frac{f(x+\delta x) - f(x)}{\delta x}\right] g(x+\delta x) + f(x)\left[\frac{g(x+\delta x) - g(x)}{\delta x}\right]$$

Taking limit when $\delta x \to 0$

(iv) $$\lim_{\delta x \to 0} \frac{\phi(x+\delta x) - \phi(x)}{\delta x}$$

$$= \lim_{\delta x \to 0}\left[\frac{f(x+\delta x) - f(x)}{\delta x} \cdot g(x+\delta x) + f(x) \cdot \frac{g(x+\delta x) - g(x)}{\delta x}\right]$$

$$= \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \cdot \lim_{\delta x \to 0} g(x+\delta x) + \lim_{\delta x \to 0} f(x) \cdot \lim_{\delta x \to 0} \frac{g(x+\delta x) - g(x)}{\delta x}$$

(Using limit theorem

Thus $\phi'(x) = f'(x)\, g(x) + f(x)\, g'(x)$ $\quad \left[\because \lim_{\delta x \to 0} g(x+\delta x) = g(x)\right]$

or $$\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + f(x)\left[\frac{d}{dx} g(x)\right]$$

Example: Find derivative of $y = \left(2\sqrt{x} + 2\right)\left(x - \sqrt{x}\right)$ with respect to x.

Solution: $y = \left(2\sqrt{x} + 2\right)\left(x - \sqrt{x}\right)$

$$= 2\left(\sqrt{x} + 1\right)\left(x - \sqrt{x}\right)$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = 2\frac{d}{dx}\left[(\sqrt{x} + 1)(x - \sqrt{x})\right]$$

$$= 2\left[\left(\frac{d}{dx}(\sqrt{x} + 1)\right)(x - \sqrt{x}) + (\sqrt{x} + 1)\frac{d}{dx}(x - \sqrt{x})\right]$$

$$= 2\left[\left(\frac{1}{2}x^{\frac{1}{2}-1} + 0\right)(x-\sqrt{x}) + (\sqrt{x}+1)\times\left(1-\frac{1}{2}x^{\frac{1}{2}-1}\right)\right]$$

$$= 2\left[\frac{1}{2\sqrt{x}}(x-\sqrt{x}) + (\sqrt{x}+1)\times\left(1-\frac{1}{2\sqrt{x}}\right)\right]$$

$$= 2\left[\frac{x-\sqrt{x}}{2\sqrt{x}} + (\sqrt{x}+1)\left(\frac{2\sqrt{x}-1}{2\sqrt{x}}\right)\right]$$

$$= \frac{1}{\sqrt{x}}\left[x-\sqrt{x}+2x-\sqrt{x}+2\sqrt{x}-1\right]$$

$$= \frac{3x-1}{\sqrt{x}}.$$

6. Derivative of a Quotient (The Quotient Rule)

If f and g are differentiable at x and $g(x) \neq 0$, for any $x \in D(g)$

then $\frac{f}{g}$ is differentiable at x and $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

that is, $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[\frac{d}{dx}[f(x)]\right]g(x) - f(x)\left[\frac{d}{dx}[g(x)]\right]}{[g(x)]^2}$.

Proof: Let $\phi(x) = \frac{f(x)}{g(x)}$. Then

(i) $\phi(x+\delta x) = \frac{f(x+\delta x)}{g(x+\delta x)}$ and

(ii) $\phi(x+\delta x) - \phi(x) = \frac{f(x+\delta x)}{g(x+\delta x)} - \frac{f(x)}{g(x)} = \frac{f(x+\delta x)\,g(x) - f(x)g(x+\delta x)}{g(x)\,g(x+\delta x)}$

Subtracting and adding $f(x)\,g(x)$ in the numerator of step (ii), gives

$$\phi(x+\delta x) - \phi(x) = \frac{f(x+\delta x)g(x) - f(x)g(x) - f(x)g(x+\delta x) + f(x)g(x)}{g(x)g(x+\delta x)}$$

$$= \frac{1}{g(x)g(x+\delta x)}\left[(f(x+\delta x) - f(x))\,g(x) - f(x)\,(g(x+\delta x) - g(x))\right]$$

(iii) $\dfrac{\phi(x+\delta x)-\phi(x)}{\delta x}=\dfrac{1}{g(x)\,g(x+\delta x)}\left[\dfrac{f(x+\delta x)-f(x)}{\delta x}\cdot g(x)-f(x)\cdot\dfrac{g(x+\delta x)-g(x)}{\delta x}\right]$

Taking limit when $\delta x\to 0$

(iv) $\lim\limits_{\delta x\to 0}\dfrac{\phi(x+\delta x)-\phi(x)}{\delta x}$

$$=\lim_{x\to 0}\left[\frac{1}{g(x)g(x+\delta x)}\left(\frac{f(x+\delta x)-f(x)}{\delta x}\cdot g(x)-f(x)\cdot\frac{g(x+\delta x)-g(x)}{\delta x}\right)\right]$$

Using limit theorems, we have

$$\phi'(x)=\frac{1}{g(x)\,.\,g(x)}\,[f'(x)\,g(x)-f(x)\,g'(x)]\quad\left(\because\ \lim_{\delta x\to 0}g(x+\delta x)=g(x)\right)$$

Thus $\left(\dfrac{f(x)}{g(x)}\right)'=\dfrac{f'(x)\,g(x)-f(x)\,g'(x)}{[g(x)]^2}$ or $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)=\dfrac{\left[\dfrac{d}{dx}[f(x)]\right]g(x)-f(x)\left[\dfrac{d}{dx}[g(x)]\right]}{[g(x)]^2}$

First Alternative Proof:

$\phi(x)=\dfrac{f(x)}{g(x)}$ can be written as $f(x)=\phi(x)\,g(x)$

Using the procedure used to prove product rule, quotient rule can be proved.

Second Alternative Proof: We first prove the reciprocal rule and then use product rule to prove the quotient rule.

The reciprocal rule. If g is differentiable at x and $g(x)\neq 0$, then $\dfrac{1}{g}$ is differentiable at

and

$$\frac{d}{dx}\left[\frac{1}{g(x)}\right]=\frac{-\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$$ (Proof of the reciprocal rule is left as an exercise)

Using the product rule to $f(x)\cdot\dfrac{1}{g(x)}$, we have

$$\frac{d}{dx}\left[f(x)\cdot\frac{1}{g(x)}\right]=\left(\frac{d}{dx}[f(x)]\right)\cdot\frac{1}{g(x)}+f(x)\cdot\frac{d}{dx}\left[\frac{1}{g(x)}\right]$$

$$=\frac{\dfrac{d}{dx}[f(x)]}{g(x)}+f(x)\,\frac{-\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$$

i.e., $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{\left[\dfrac{d}{dx}[f(x)]\right]g(x)-f(x)\left[\dfrac{d}{dx}[g(x)]\right]}{[g(x)]^2}$$

Example 2. Find $\frac{dy}{dx}$ if $y = \frac{(\sqrt{x}+1)(x^{3/2}-1)}{x^{1/2}-1}$, $(x \neq 1)$

Solution. Given that

$$y = \frac{(\sqrt{x}+1)(x^{3/2}-1)}{x^{1/2}-1} = \frac{(\sqrt{x}+1)[(\sqrt{x})^3-(1)^3]}{\sqrt{x}-1}$$

$$= \frac{(\sqrt{x}+1)(\sqrt{x}-1)(x+1+\sqrt{x})}{\sqrt{x}-1} = (\sqrt{x}+1)(x+1+\sqrt{x})$$

$$= (\sqrt{x}+1)(\sqrt{x}+1+x) = (\sqrt{x}+1)^2 + (\sqrt{x}+1)\,x$$

$$= x+1+2\sqrt{x}+x\sqrt{x}+x = x^{3/2}+2x+2x^{1/2}+1$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^{3/2}+2x+2x^{1/2}+1) = \frac{d}{dx}(x^{3/2}) + \frac{d}{dx}(2x) + \frac{d}{dx}(2x^{1/2}) + \frac{d}{dx}(1)$$

$$= \frac{3}{2}x^{1/2} + 2(1) + 2\cdot\frac{1}{2\sqrt{x}} + 0 = \frac{3}{2}\sqrt{x} + 2 + \frac{1}{\sqrt{x}}$$

Example 3: Differentiate $\frac{(\sqrt{x}+1)(x^{3/2}-1)}{x^{3/2}-x^{1/2}}$ with respect to x.

Solution: Let $y = \frac{(\sqrt{x}+1)(x^{3/2}-1)}{x^{3/2}-x^{1/2}}$

$$= \frac{(\sqrt{x}+1)\left[(\sqrt{x})^3-1\right]}{\sqrt{x}(x-1)}$$

$$= \frac{(\sqrt{x}+1)(\sqrt{x}-1)(x+\sqrt{x}+1)}{\sqrt{x}(x-1)} = \frac{(x-1)(x+\sqrt{x}+1)}{\sqrt{x}(x-1)}$$

$$= \frac{x+\sqrt{x}+1}{\sqrt{x}}$$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x+\sqrt{x}+1}{\sqrt{x}}\right]$$

$$= \frac{\sqrt{x}\frac{d}{dx}(x+\sqrt{x}+1) - (x+\sqrt{x}+1)\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}$$

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$$= \frac{\sqrt{x}\left(1+\frac{1}{2}x^{-\frac{1}{2}}+0\right)-(x+\sqrt{x}+1).\left(\frac{1}{2}x^{-\frac{1}{2}}\right)}{x}$$

$$= \frac{\sqrt{x}\left(1+\frac{1}{2\sqrt{x}}\right)-(x+\sqrt{x}+1)\frac{1}{2\sqrt{x}}}{x}$$

$$= \frac{\sqrt{x}\left(\frac{2\sqrt{x}+1}{2\sqrt{x}}\right)-\frac{x+\sqrt{x}+1}{2\sqrt{x}}}{x} = \frac{2x+\sqrt{x}-x-\sqrt{x}-1}{x.2\sqrt{x}} = \frac{x-1}{2x^{3/2}}$$

Example 4: Differentiate $\frac{2x^3-3x^2+5}{x^2+1}$ with respect to x.

Solution: Let $\phi(x) = \frac{2x^3-3x^2+5}{x^2+1}$. Then we take

$$f(x) = 2x^3-3x^2+5 \quad \text{and} \quad g(x) = x^2+1$$

Now $f'(x) = \frac{d}{dx}[2x^3-3x^2+5] = 2(3x^2)-3(2x)+0 = 6x^2-6x$

and $g'(x) = \frac{d}{dx}[x^2+1] = 2x+0 = 2x$

Using the quotient formula: $\phi'(x) = \frac{f'(x)\,g(x)-f(x)\,g'(x)}{[g(x)]^2}$, we obt

$$\frac{d}{dx}\left[\frac{2x^3-3x^2+5}{x^2+1}\right] = \frac{(6x^2-6x)(x^2+1)-(2x^3-3x^2+5)(2x)}{(x^2+1)^2}$$

$$= \frac{6x^4-6x^3+6x^2-6x-(4x^4-6x^3+10x)}{(x^2+1)^2}$$

$$= \frac{6x^4-6x^3+6x^2-6x-4x^4+6x^3-10x}{(x^2+1)^2}$$

$$= \frac{2x^4+6x^2-16x}{(x^2+1)^2}$$

EXER

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