2.5.24 The magnetic vector potential for a uniformly charged rotating spherical shell is

$$\mathbf{A} = \begin{cases} \hat{\boldsymbol{\varphi}} \dfrac{\mu_0 a^4 \sigma \omega}{3} \cdot \dfrac{\sin\theta}{r^2}, & r > a \\ \hat{\boldsymbol{\varphi}} \dfrac{\mu_0 a \sigma \omega}{3} \cdot r\cos\theta, & r < a. \end{cases}$$

(a = radius of spherical shell, σ = surface charge density, and ω = angular velocity.) Find the magnetic induction $\mathbf{B} = \nabla \times \mathbf{A}$.

$$\text{ANS. } B_r(r,\theta) = \frac{2\mu_0 a^4 \sigma\omega}{3} \cdot \frac{\cos\theta}{r^3}, \qquad r > a,$$

$$B_\theta(r,\theta) = \frac{\mu_0 a^4 \sigma\omega}{3} \cdot \frac{\sin\theta}{r^3}, \qquad r > a,$$

$$\mathbf{B} = \hat{\mathbf{z}} \frac{2\mu_0 a\sigma\omega}{3}, \qquad r < a.$$

2.5.25 (a) Explain why ∇^2 in plane polar coordinates follows from ∇^2 in circular cylindrical coordinates with z = constant.

(b) Explain why taking ∇^2 in spherical polar coordinates and restricting θ to $\pi/2$ does **not** lead to the plane polar form of ∇.

Note.

$$\nabla^2(\rho,\varphi) = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}.$$

2.6 Tensor Analysis

Introduction, Definitions

Tensors are important in many areas of physics, including general relativity and electrodynamics. Scalars and vectors are special cases of tensors. In Chapter 1, a quantity that did not change under rotations of the coordinate system in three-dimensional space, an invariant, was labeled a scalar. A **scalar** is specified by one real number and is a **tensor of rank 0**. A quantity whose components transformed under rotations like those of the distance of a point from a chosen origin (Eq. (1.9), Section 1.2) was called a vector. The transformation of the components of the vector under a rotation of the coordinates preserves the vector as a geometric entity (such as an arrow in space), independent of the orientation of the reference frame. In three-dimensional space, a **vector** is specified by $3 = 3^1$ real numbers, for example, its Cartesian components, and is a **tensor of rank 1**. A **tensor of rank** n has 3^n components that transform in a definite way.[5] This transformation philosophy is of central importance for tensor analysis and conforms with the mathematician's concept of vector and vector (or linear) space and the physicist's notion that physical observables must not depend on the choice of coordinate frames. There is a physical basis for such a philosophy: We describe the physical world by mathematics, but any physical predictions we make

[5] In N-dimensional space a tensor of rank n has N^n components.

must be independent of our mathematical conventions, such as a coordinate system with its arbitrary origin and orientation of its axes.

There is a possible ambiguity in the transformation law of a vector

$$A'_i = \sum_j a_{ij} A_j, \tag{2.59}$$

in which a_{ij} is the cosine of the angle between the x'_i-axis and the x_j-axis.

If we start with a differential distance vector $d\mathbf{r}$, then, taking dx'_i to be a function of the unprimed variables,

$$dx'_i = \sum_j \frac{\partial x'_i}{\partial x_j} dx_j \tag{2.60}$$

by partial differentiation. If we set

$$a_{ij} = \frac{\partial x'_i}{\partial x_j}, \tag{2.61}$$

Eqs. (2.59) and (2.60) are consistent. Any set of quantities A^j transforming according to

$$A'^i = \sum_j \frac{\partial x'_i}{\partial x_j} A^j \tag{2.62a}$$

is defined as a **contravariant** vector, whose indices we write as **superscript**; this includes the Cartesian coordinate vector $x^i = x_i$ from now on.

However, we have already encountered a slightly different type of vector transformation. The gradient of a scalar $\nabla\varphi$, defined by

$$\nabla\varphi = \hat{\mathbf{x}}\frac{\partial\varphi}{\partial x^1} + \hat{\mathbf{y}}\frac{\partial\varphi}{\partial x^2} + \hat{\mathbf{z}}\frac{\partial\varphi}{\partial x^3} \tag{2.63}$$

(using x^1, x^2, x^3 for x, y, z), transforms as

$$\frac{\partial\varphi'}{\partial x'^i} = \sum_j \frac{\partial\varphi}{\partial x^j}\frac{\partial x^j}{\partial x'^i}, \tag{2.64}$$

using $\varphi = \varphi(x, y, z) = \varphi(x', y', z') = \varphi'$, φ defined as a scalar quantity. Notice that this differs from Eq. (2.62) in that we have $\partial x^j/\partial x'^i$ instead of $\partial x'^i/\partial x^j$. Equation (2.64) is taken as the definition of a **covariant** vector, with the gradient as the prototype. The covariant analog of Eq. (2.62a) is

$$A'_i = \sum_j \frac{\partial x^j}{\partial x'^i} A_j. \tag{2.62b}$$

Only in Cartesian coordinates is

$$\frac{\partial x^j}{\partial x'^i} = \frac{\partial x'^i}{\partial x^j} = a_{ij} \tag{2.65}$$

so that there no difference between contravariant and covariant transformations. In other systems, Eq. (2.65) in general does not apply, and the distinction between contravariant and covariant is real and must be observed. This is of prime importance in the curved Riemannian space of general relativity.

In the remainder of this section the components of any **contravariant** vector are denoted by a **superscript**, A^i, whereas a **subscript** is used for the components of a **covariant** vector A_i.[6]

Definition of Tensors of Rank 2

Now we proceed to define **contravariant, mixed, and covariant tensors of rank 2** by the following equations for their components under coordinate transformations:

$$\begin{aligned} A'^{ij} &= \sum_{kl} \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l} A^{kl}, \\ B'^i{}_j &= \sum_{kl} \frac{\partial x'^i}{\partial x^k}\frac{\partial x^l}{\partial x'^j} B^k{}_l, \\ C'_{ij} &= \sum_{kl} \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j} C_{kl}. \end{aligned} \tag{2.66}$$

Clearly, the rank goes as the number of partial derivatives (or direction cosines) in the definition: 0 for a scalar, 1 for a vector, 2 for a second-rank tensor, and so on. Each index (subscript or superscript) ranges over the number of dimensions of the space. The number of indices (equal to the rank of tensor) is independent of the dimensions of the space. We see that A^{kl} is contravariant with respect to both indices, C_{kl} is covariant with respect to both indices, and $B^k{}_l$ transforms contravariantly with respect to the first index k but covariantly with respect to the second index l. Once again, if we are using Cartesian coordinates, all three forms of the tensors of second rank contravariant, mixed, and covariant are — the same.

As with the components of a vector, the transformation laws for the components of a tensor, Eq. (2.66), yield entities (and properties) that are independent of the choice of reference frame. This is what makes tensor analysis important in physics. The independence of reference frame (invariance) is ideal for expressing and investigating universal physical laws.

The second-rank tensor A (components A^{kl}) may be conveniently represented by writing out its components in a square array (3×3 if we are in three-dimensional space):

$$\mathsf{A} = \begin{pmatrix} A^{11} & A^{12} & A^{13} \\ A^{21} & A^{22} & A^{23} \\ A^{31} & A^{32} & A^{33} \end{pmatrix}. \tag{2.67}$$

This does not mean that any square array of numbers or functions forms a tensor. The essential condition is that the components transform according to Eq. (2.66).

[6]This means that the coordinates (x, y, z) are written (x^1, x^2, x^3) since $\mathbf{r}$ transforms as a contravariant vector. The ambiguity of x^2 representing both x squared and y is the price we pay.

In the context of matrix analysis the preceding transformation equations become (for Cartesian coordinates) an orthogonal similarity transformation; see Section 3.3. A geometrical interpretation of a second-rank tensor (the inertia tensor) is developed in Section 3.5.

In summary, tensors are systems of components organized by one or more indices that transform according to specific rules under a set of transformations. The number of indices is called the rank of the tensor. If the transformations are coordinate rotations in three-dimensional space, then tensor analysis amounts to what we did in the sections on curvilinear coordinates and in Cartesian coordinates in Chapter 1. *In four dimensions of Minkowski space–time, the transformations are Lorentz transformations, and tensors of rank* 1 *are called four-vectors.*

Addition and Subtraction of Tensors

The addition and subtraction of tensors is defined in terms of the individual elements, just as for vectors. If

$$\mathsf{A}+\mathsf{B}=\mathsf{C}, \tag{2.68}$$

then

$$A^{ij}+B^{ij}=C^{ij}.$$

Of course, A and B must be tensors of the same rank and both expressed in a space of the same number of dimensions.

Summation Convention

In tensor analysis it is customary to adopt a summation convention to put Eq. (2.66) and subsequent tensor equations in a more compact form. As long as we are distinguishing between contravariance and covariance, let us agree that when an index appears on one side of an equation, once as a superscript and once as a subscript (except for the coordinates where both are subscripts), we automatically sum over that index. Then we may write the second expression in Eq. (2.66) as

$$B'^{i}{}_{j}=\frac{\partial x'^{i}}{\partial x^{k}}\frac{\partial x^{l}}{\partial x'^{j}}B^{k}{}_{l}, \tag{2.69}$$

with the summation of the right-hand side over k and l implied. This is Einstein's summation convention.[7] The index i is superscript because it is associated with the contravariant x'^{i}; likewise j is subscript because it is related to the covariant gradient.

To illustrate the use of the summation convention and some of the techniques of tensor analysis, let us show that the now-familiar Kronecker delta, δ_{kl}, is really a mixed tensor

[7] In this context $\partial x'^{i}/\partial x^{k}$ might better be written as a^{i}_{k} and $\partial x^{l}/\partial x'^{j}$ as b^{l}_{j}.

of rank 2, $\delta^k{}_l$.[8] The question is: Does $\delta^k{}_l$ transform according to Eq. (2.66)? This is our criterion for calling it a tensor. We have, using the summation convention,

$$\delta^k{}_l \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x^k}{\partial x'^j} \tag{2.70}$$

by definition of the Kronecker delta. Now,

$$\frac{\partial x'^i}{\partial x^k} \frac{\partial x^k}{\partial x'^j} = \frac{\partial x'^i}{\partial x'^j} \tag{2.71}$$

by direct partial differentiation of the right-hand side (chain rule). However, x'^i and x'^j are independent coordinates, and therefore the variation of one with respect to the other must be zero if they are different, unity if they coincide; that is,

$$\frac{\partial x'^i}{\partial x'^j} = \delta'^i{}_j. \tag{2.72}$$

Hence

$$\delta'^i{}_j = \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j} \delta^k{}_l,$$

showing that the $\delta^k{}_l$ are indeed the components of a mixed second-rank tensor. Notice that this result is independent of the number of dimensions of our space. The reason for the upper index i and lower index j is the same as in Eq. (2.69).

The Kronecker delta has one further interesting property. It has the same components in all of our rotated coordinate systems and is therefore called **isotropic**. In Section 2.9 we shall meet a third-rank isotropic tensor and three fourth-rank isotropic tensors. No isotropic first-rank tensor (vector) exists.

Symmetry–Antisymmetry

The order in which the indices appear in our description of a tensor is important. In general, A^{mn} is independent of A^{nm}, but there are some cases of special interest. If, for all m and n,

$$A^{mn} = A^{nm}, \tag{2.73}$$

we call the tensor **symmetric**. If, on the other hand,

$$A^{mn} = -A^{nm}, \tag{2.74}$$

the tensor is **antisymmetric**. Clearly, every (second-rank) tensor can be resolved into symmetric and antisymmetric parts by the identity

$$A^{mn} = \tfrac{1}{2}\left(A^{mn} + A^{nm}\right) + \tfrac{1}{2}\left(A^{mn} - A^{nm}\right), \tag{2.75}$$

the first term on the right being a symmetric tensor, the second, an antisymmetric tensor. A similar resolution of functions into symmetric and antisymmetric parts is of extreme importance to quantum mechanics.

[8] It is common practice to refer to a tensor $\mathbf{A}$ by specifying a typical component, A_{ij}. As long as the reader refrains from writing nonsense such as $\mathbf{A} = A_{ij}$, no harm is done.