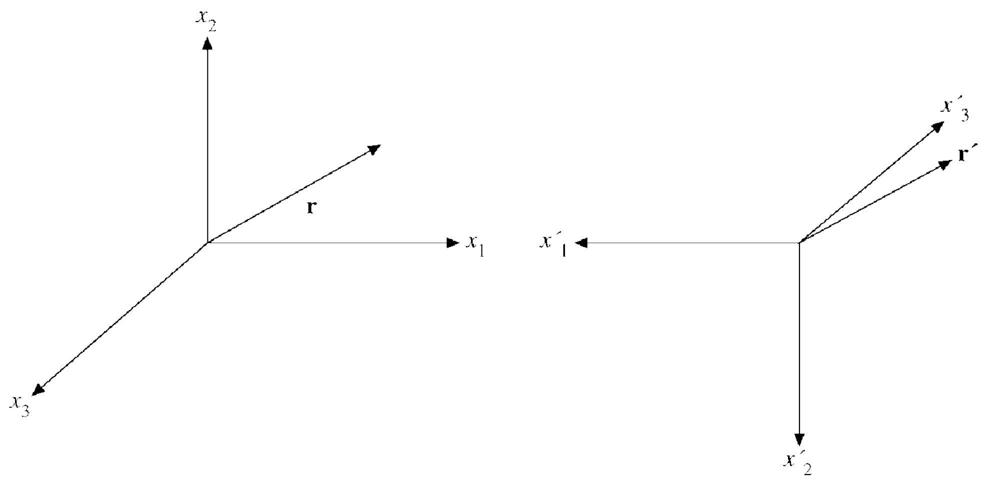

FIGURE 2.9 Inversion of Cartesian coordinates — polar vector.

If we have transformation coefficients $a_{ij} = -\delta_{ij}$, then by Eq. (2.60)

$$x^i = -x'^i, \tag{2.88}$$

which is an inversion or parity transformation. Note that this transformation changes our initial right-handed coordinate system into a left-handed coordinate system.[13] Our prototype vector **r** with components (x^1, x^2, x^3) transforms to

$$\mathbf{r}' = (x'^1, x'^2, x'^3) = (-x^1, -x^2, -x^3).$$

This new vector $\mathbf{r}'$ has negative components, relative to the new transformed set of axes. As shown in Fig. 2.9, reversing the directions of the coordinate axes and changing the signs of the components gives $\mathbf{r}' = \mathbf{r}$. The vector (an arrow in space) stays exactly as it was before the transformation was carried out. The position vector **r** and all other vectors whose components behave this way (reversing sign with a reversal of the coordinate axes) are called **polar vectors** and have odd parity.

A fundamental difference appears when we encounter a vector defined as the cross product of two polar vectors. Let $\mathbf{C} = \mathbf{A} \times \mathbf{B}$, where both **A** and **B** are polar vectors. From Eq. (1.33), the components of **C** are given by

$$C^1 = A^2 B^3 - A^3 B^2 \tag{2.89}$$

and so on. Now, when the coordinate axes are inverted, $A^i \to -A'^i$, $B_j \to -B'_j$, but from its definition $C^k \to +C'^k$; that is, our cross-product vector, vector **C**, does **not** behave like a polar vector under inversion. To distinguish, we label it a pseudovector or axial vector (see Fig. 2.10) that has even parity. The term **axial vector** is frequently used because these cross products often arise from a description of rotation.

[13]This is an inversion of the coordinate system or coordinate axes, objects in the physical world remaining fixed.

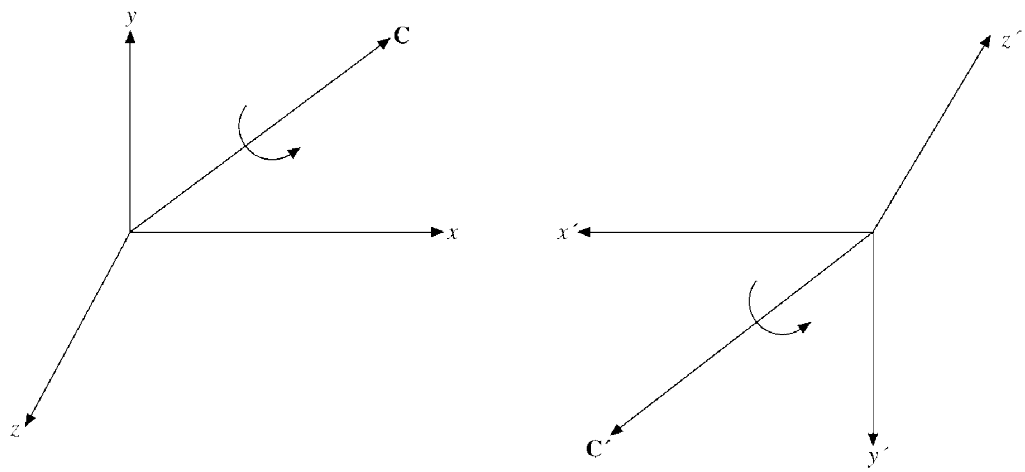

FIGURE 2.10 Inversion of Cartesian coordinates—axial vector.

Examples are

angular velocity,	$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$,
orbital angular momentum,	$\mathbf{L} = \mathbf{r} \times \mathbf{p}$,
torque, force $= \mathbf{F}$,	$\mathbf{N} = \mathbf{r} \times \mathbf{F}$,
magnetic induction field $\mathbf{B}$,	$\dfrac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$.

In $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, the axial vector is the angular velocity $\boldsymbol{\omega}$, and $\mathbf{r}$ and $\mathbf{v} = d\mathbf{r}/dt$ are polar vectors. Clearly, axial vectors occur frequently in physics, although this fact is usually not pointed out. In a right-handed coordinate system an axial vector $\mathbf{C}$ has a sense of rotation associated with it given by a right-hand rule (compare Section 1.4). In the inverted left-handed system the sense of rotation is a left-handed rotation. This is indicated by the curved arrows in Fig. 2.10.

The distinction between polar and axial vectors may also be illustrated by a reflection. A polar vector reflects in a mirror like a real physical arrow, Fig. 2.11a. In Figs. 2.9 and 2.10 the coordinates are inverted; the physical world remains fixed. Here the coordinate axes remain fixed; the world is reflected—as in a mirror in the xz-plane. Specifically, in this representation we keep the axes fixed and associate a change of sign with the component of the vector. For a mirror in the xz-plane, $P_y \to -P_y$. We have

$$\mathbf{P} = (P_x, P_y, P_z)$$
$$\mathbf{P}' = (P_x, -P_y, P_z) \qquad \text{polar vector.}$$

An axial vector such as a magnetic field $\mathbf{H}$ or a magnetic moment $\boldsymbol{\mu}$ (= current $\times$ area of current loop) behaves quite differently under reflection. Consider the magnetic field $\mathbf{H}$ and magnetic moment $\boldsymbol{\mu}$ to be produced by an electric charge moving in a circular path (Exercise 5.8.4 and Example 12.5.3). Reflection reverses the sense of rotation of the charge.

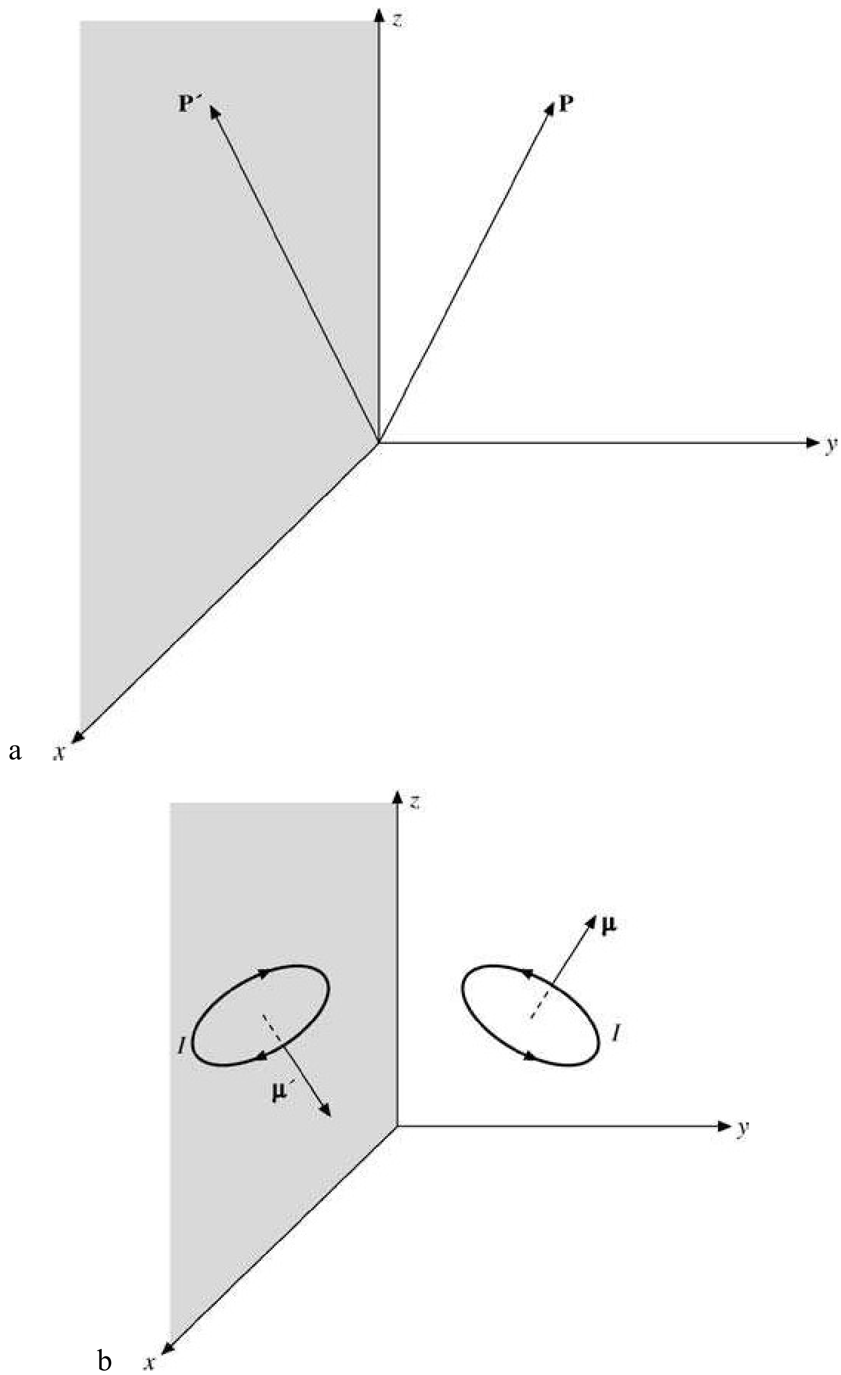

FIGURE 2.11 (a) Mirror in xz-plane; (b) mirror in xz-plane.

The two current loops and the resulting magnetic moments are shown in Fig. 2.11b. We have

$$\boldsymbol{\mu} = (\mu_x, \mu_y, \mu_z)$$

$$\boldsymbol{\mu}' = (-\mu_x, \mu_y, -\mu_z) \quad \text{reflected axial vector.}$$

If we agree that the universe does not care whether we use a right- or left-handed coordinate system, then it does not make sense to add an axial vector to a polar vector. In the vector equation $\mathbf{A} = \mathbf{B}$, both $\mathbf{A}$ and $\mathbf{B}$ are either polar vectors or axial vectors.[14] Similar restrictions apply to scalars and pseudoscalars and, in general, to the tensors and pseudotensors considered subsequently.

Usually, pseudoscalars, pseudovectors, and pseudotensors will transform as

$$S' = JS, \qquad C'_i = J a_{ij} C_j, \qquad A'_{ij} = J a_{ik} a_{jl} A_{kl}, \tag{2.90}$$

where J is the determinant[15] of the array of coefficients a_{mn}, the Jacobian of the parity transformation. In our inversion the Jacobian is

$$J = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1. \tag{2.91}$$

For a reflection of one axis, the x-axis,

$$J = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1, \tag{2.92}$$

and again the Jacobian $J = -1$. On the other hand, for all pure rotations, the Jacobian J is always $+1$. Rotation matrices discussed further in Section 3.3.

In Chapter 1 the triple scalar product $S = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$ was shown to be a scalar (under rotations). Now by considering the parity transformation given by Eq. (2.88), we see that $S \to -S$, proving that the triple scalar product is actually a pseudoscalar: This behavior was foreshadowed by the geometrical analogy of a volume. If all three parameters of the volume — length, depth, and height — change from positive distances to negative distances, the product of the three will be negative.

Levi-Civita Symbol

For future use it is convenient to introduce the three-dimensional Levi-Civita symbol ε_{ijk}, defined by

$$\begin{gathered} \varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \\ \varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1, \\ \text{all other } \varepsilon_{ijk} = 0. \end{gathered} \tag{2.93}$$

Note that ε_{ijk} is antisymmetric with respect to all pairs of indices. Suppose now that we have a third-rank pseudotensor δ_{ijk}, which in one particular coordinate system is equal to ε_{ijk}. Then

$$\delta'_{ijk} = |a| a_{ip} a_{jq} a_{kr} \varepsilon_{pqr} \tag{2.94}$$

[14] The big exception to this is in beta decay, weak interactions. Here the universe distinguishes between right- and left-handed systems, and we add polar and axial vector interactions.

[15] Determinants are described in Section 3.1.

by definition of pseudotensor. Now,

$$a_{1p}a_{2q}a_{3r}\varepsilon_{pqr} = |a| \tag{2.95}$$

by direct expansion of the determinant, showing that $\delta'_{123} = |a|^2 = 1 = \varepsilon_{123}$. Considering the other possibilities one by one, we find

$$\delta'_{ijk} = \varepsilon_{ijk} \tag{2.96}$$

for rotations and reflections. Hence ε_{ijk} is a pseudotensor.[16,17] Furthermore, it is seen to be an isotropic pseudotensor with the same components in all rotated Cartesian coordinate systems.

Dual Tensors

With any **antisymmetric** second-rank tensor **C** (in three-dimensional space) we may associate a dual pseudovector C_i defined by

$$C_i = \frac{1}{2}\varepsilon_{ijk}C^{jk}. \tag{2.97}$$

Here the antisymmetric **C** may be written

$$\mathbf{C} = \begin{pmatrix} 0 & C^{12} & -C^{31} \\ -C^{12} & 0 & C^{23} \\ C^{31} & -C^{23} & 0 \end{pmatrix}. \tag{2.98}$$

We know that C_i must transform as a vector under rotations from the double contraction of the fifth-rank (pseudo) tensor $\varepsilon_{ijk}C_{mn}$ but that it is really a pseudovector from the pseudo nature of ε_{ijk}. Specifically, the components of **C** are given by

$$(C_1, C_2, C_3) = \left(C^{23}, C^{31}, C^{12}\right). \tag{2.99}$$

Notice the cyclic order of the indices that comes from the cyclic order of the components of ε_{ijk}. Eq. (2.99) means that our three-dimensional vector product may literally be taken to be either a pseudovector or an antisymmetric second-rank tensor, depending on how we choose to write it out.

If we take three (polar) vectors **A**, **B**, and **C**, we may define the direct product

$$V^{ijk} = A^i B^j C^k. \tag{2.100}$$

By an extension of the analysis of Section 2.6, V^{ijk} is a tensor of third rank. The dual quantity

$$V = \frac{1}{3!}\varepsilon_{ijk}V^{ijk} \tag{2.101}$$

[16]The usefulness of ε_{pqr} extends far beyond this section. For instance, the matrices M_k of Exercise 3.2.16 are derived from $(M_r)_{pq} = -i\varepsilon_{pqr}$. Much of elementary vector analysis can be written in a very compact form by using ε_{ijk} and the identity of Exercise 2.9.4 See A. A. Evett, Permutation symbol approach to elementary vector analysis. *Am. J. Phys.* **34**: 503 (1966).

[17]The numerical value of ε_{pqr} is given by the triple scalar product of coordinate unit vectors:

$$\hat{\mathbf{x}}_p \cdot \hat{\mathbf{x}}_q \times \hat{\mathbf{x}}_r.$$

From this point of view each element of ε_{pqr} is a pseudoscalar, but the ε_{pqr} collectively form a third-rank pseudotensor.

is clearly a pseudoscalar. By expansion it is seen that

$$V = \begin{vmatrix} A^1 & B^1 & C^1 \\ A^2 & B^2 & C^2 \\ A^3 & B^3 & C^3 \end{vmatrix} \tag{2.102}$$

is our familiar triple scalar product.

For use in writing Maxwell's equations in covariant form, Section 4.6, we want to extend this dual vector analysis to four-dimensional space and, in particular, to indicate that the four-dimensional volume element $dx^0\,dx^1\,dx^2\,dx^3$ is a pseudoscalar.

We introduce the Levi-Civita symbol ε_{ijkl}, the four-dimensional analog of ε_{ijk}. This quantity ε_{ijkl} is defined as totally antisymmetric in all four indices. If $(ijkl)$ is an even permutation[18] of (0, 1, 2, 3), then ε_{ijkl} is defined as $+1$; if it is an odd permutation, then ε_{ijkl} is -1, and 0 if any two indices are equal. The Levi-Civita ε_{ijkl} may be proved a pseudotensor of rank 4 by analysis similar to that used for establishing the tensor nature of ε_{ijk}. Introducing the direct product of four vectors as fourth-rank tensor with components

$$H^{ijkl} = A^i B^j C^k D^l, \tag{2.103}$$

built from the polar vectors **A**, **B**, **C**, and **D**, we may define the dual quantity

$$H = \frac{1}{4!}\varepsilon_{ijkl} H^{ijkl}, \tag{2.104}$$

a pseudoscalar due to the quadruple contraction with the pseudotensor ε_{ijkl}. Now we let **A**, **B**, **C**, and **D** be infinitesimal displacements along the four coordinate axes (Minkowski space),

$$\begin{aligned} \mathbf{A} &= \left(dx^0, 0, 0, 0\right) \\ \mathbf{B} &= \left(0, dx^1, 0, 0\right), \qquad \text{and so on,} \end{aligned} \tag{2.105}$$

and

$$H = dx^0\,dx^1\,dx^2\,dx^3. \tag{2.106}$$

The four-dimensional volume element is now identified as a pseudoscalar. We use this result in Section 4.6. This result could have been expected from the results of the special theory of relativity. The Lorentz–Fitzgerald contraction of $dx^1\,dx^2\,dx^3$ just balances the time dilation of dx^0.

We slipped into this four-dimensional space as a simple mathematical extension of the three-dimensional space and, indeed, we could just as easily have discussed 5-, 6-, or N-dimensional space. This is typical of the power of the component analysis. Physically, this four-dimensional space may be taken as Minkowski space,

$$\left(x^0, x^1, x^2, x^3\right) = (ct, x, y, z), \tag{2.107}$$

where t is time. This is the merger of space and time achieved in special relativity. The transformations that describe the rotations in four-dimensional space are the Lorentz transformations of special relativity. We encounter these Lorentz transformations in Section 4.6.

[18] A permutation is odd if it involves an odd number of interchanges of adjacent indices, such as (0 1 2 3) $\rightarrow$ (0 2 1 3). Even permutations arise from an even number of transpositions of adjacent indices. (Actually the word *adjacent* is unnecessary.) $\varepsilon_{0123} = +1$.