using $x_4 = ict$. This is the four-dimensional Laplacian, sometimes called the d'Alembertian and denoted by $\Box^2$. Show that it is a **scalar** operator, that is, is invariant under Lorentz transformations.

2.8 Quotient Rule

If A_i and B_j are vectors, as seen in Section 2.7, we can easily show that $A_i B_j$ is a second-rank tensor. Here we are concerned with a variety of inverse relations. Consider such equations as

$$K_i A_i = B \tag{2.82a}$$

$$K_{ij} A_j = B_i \tag{2.82b}$$

$$K_{ij} A_{jk} = B_{ik} \tag{2.82c}$$

$$K_{ijkl} A_{ij} = B_{kl} \tag{2.82d}$$

$$K_{ij} A_k = B_{ijk}. \tag{2.82e}$$

Inline with our restriction to Cartesian systems, we write all indices as subscripts and, unless specified otherwise, sum repeated indices.

In each of these expressions **A** and **B** are known tensors of rank indicated by the number of indices and **A** is arbitrary. In each case K is an unknown quantity. We wish to establish the transformation properties of K. The quotient rule asserts that if the equation of interest holds in all (rotated) Cartesian coordinate systems, K is a tensor of the indicated rank. The importance in physical theory is that the quotient rule can establish the tensor nature of quantities. Exercise 2.8.1 is a simple illustration of this. The quotient rule (Eq. (2.82b)) shows that the inertia matrix appearing in the angular momentum equation $\mathbf{L} = I\boldsymbol{\omega}$, Section 3.5, is a tensor.

In proving the quotient rule, we consider Eq. (2.82b) as a typical case. In our primed coordinate system

$$K'_{ij} A'_j = B'_i = a_{ik} B_k, \tag{2.83}$$

using the vector transformation properties of **B**. Since the equation holds in all rotated Cartesian coordinate systems,

$$a_{ik} B_k = a_{ik}(K_{kl} A_l). \tag{2.84}$$

Now, transforming **A** back into the primed coordinate system[11] (compare Eq. (2.62)), we have

$$K'_{ij} A'_j = a_{ik} K_{kl} a_{jl} A'_j. \tag{2.85}$$

Rearranging, we obtain

$$(K'_{ij} - a_{ik} a_{jl} K_{kl}) A'_j = 0. \tag{2.86}$$

[11] Note the order of the indices of the direction cosine a_{jl} in this **inverse** transformation. We have

$$A_l = \sum_j \frac{\partial x_l}{\partial x'_j} A'_j = \sum_j a_{jl} A'_j.$$

This must hold for each value of the index i and for every primed coordinate system. Since the A'_j is arbitrary,[12] we conclude

$$K'_{ij} = a_{ik}a_{jl}K_{kl}, \tag{2.87}$$

which is our definition of second-rank tensor.

The other equations may be treated similarly, giving rise to other forms of the quotient rule. One minor pitfall should be noted: The quotient rule does not necessarily apply if B is zero. The transformation properties of zero are indeterminate.

Example 2.8.1 Equations of Motion and Field Equations

In classical mechanics, Newton's equations of motion $m\dot{\mathbf{v}} = \mathbf{F}$ tell us on the basis of the quotient rule that, if the mass is a scalar and the force a vector, then the acceleration $\mathbf{a} \equiv \dot{\mathbf{v}}$ is a vector. In other words, the vector character of the force as the driving term imposes its vector character on the acceleration, provided the scale factor m is scalar.

The wave equation of electrodynamics $\partial^2 A^\mu = J^\mu$ involves the four-dimensional version of the Laplacian $\partial^2 = \frac{\partial^2}{c^2\partial t^2} - \nabla^2$, a Lorentz scalar, and the external four-vector current J^μ as its driving term. From the quotient rule, we infer that the vector potential A^μ is a four-vector as well. If the driving current is a four-vector, the vector potential must be of rank 1 by the quotient rule. ■

The quotient rule is a substitute for the illegal division of tensors.

Exercises

2.8.1 The double summation $K_{ij}A_iB_j$ is invariant for any two vectors A_i and B_j. Prove that K_{ij} is a second-rank tensor.
Note. In the form ds^2 (invariant) $= g_{ij}\,dx^i\,dx^j$, this result shows that the matrix g_{ij} is a tensor.

2.8.2 The equation $K_{ij}A_{jk} = B_{ik}$ holds for all orientations of the coordinate system. If $\mathbf{A}$ and $\mathbf{B}$ are arbitrary second-rank tensors, show that $\mathbf{K}$ is a second-rank tensor also.

2.8.3 The exponential in a plane wave is $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$. We recognize $x^\mu = (ct, x_1, x_2, x_3)$ as a prototype vector in Minkowski space. If $\mathbf{k}\cdot\mathbf{r} - \omega t$ is a scalar under Lorentz transformations (Section 4.5), show that $k^\mu = (\omega/c, k_1, k_2, k_3)$ is a vector in Minkowski space.
Note. Multiplication by $\hbar$ yields $(E/c, \mathbf{p})$ as a vector in Minkowski space.

2.9 Pseudotensors, Dual Tensors

So far our coordinate transformations have been restricted to pure passive rotations. We now consider the effect of reflections or inversions.

[12] We might, for instance, take $A'_1 = 1$ and $A'_m = 0$ for $m \neq 1$. Then the equation $K'_{i1} = a_{ik}a_{1l}K_{kl}$ follows immediately. The rest of Eq. (2.87) comes from other special choices of the arbitrary A'_j.