further reduces the number of independent components to 21. Finally, if the components satisfy an identity $R_{iklm} + R_{ilmk} + R_{imkl} = 0$, show that the number of independent components is reduced to 20.
Note. The final three-term identity furnishes new information only if all four indices are different. Then it reduces the number of independent components by one-third.

2.6.6 T_{iklm} is antisymmetric with respect to all pairs of indices. How many independent components has it (in three-dimensional space)?

2.7 CONTRACTION, DIRECT PRODUCT

Contraction

When dealing with vectors, we formed a scalar product (Section 1.3) by summing products of corresponding components:

$$\mathbf{A} \cdot \mathbf{B} = A_i B_i \qquad (\textbf{summation convention}). \tag{2.76}$$

The generalization of this expression in tensor analysis is a process known as contraction. Two indices, one covariant and the other contravariant, are set equal to each other, and then (as implied by the summation convention) we sum over this repeated index. For example, let us contract the second-rank mixed tensor $B'^{i}{}_{j}$,

$$B'^{i}{}_{i} = \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x^{l}}{\partial x'^{i}} B^{k}{}_{l} = \frac{\partial x^{l}}{\partial x^{k}} B^{k}{}_{l} \tag{2.77}$$

using Eq. (2.71), and then by Eq. (2.72)

$$B'^{i}{}_{i} = \delta^{l}{}_{k} B^{k}{}_{l} = B^{k}{}_{k}. \tag{2.78}$$

Our contracted second-rank mixed tensor is invariant and therefore a scalar.[10] This is exactly what we obtained in Section 1.3 for the dot product of two vectors and in Section 1.7 for the divergence of a vector. In general, the operation of contraction reduces the rank of a tensor by 2. An example of the use of contraction appears in Chapter 4.

Direct Product

The components of a covariant vector (first-rank tensor) a_i and those of a contravariant vector (first-rank tensor) b^j may be multiplied component by component to give the general term $a_i b^j$. This, by Eq. (2.66) is actually a second-rank tensor, for

$$a'_i b'^{j} = \frac{\partial x^{k}}{\partial x'^{i}} a_k \frac{\partial x'^{j}}{\partial x^{l}} b^{l} = \frac{\partial x^{k}}{\partial x'^{i}} \frac{\partial x'^{j}}{\partial x^{l}} \left(a_k b^{l}\right). \tag{2.79}$$

Contracting, we obtain

$$a'_i b'^{i} = a_k b^{k}, \tag{2.80}$$

[10]In matrix analysis this scalar is the **trace** of the matrix, Section 3.2.

as in Eqs. (2.77) and (2.78), to give the regular scalar product.

The operation of adjoining two vectors a_i and b^j as in the last paragraph is known as forming the **direct product**. For the case of two vectors, the direct product is a tensor of second rank. In this sense we may attach meaning to $\nabla\mathbf{E}$, which was not defined within the framework of vector analysis. In general, the direct product of two tensors is a tensor of rank equal to the sum of the two initial ranks; that is,

$$A^i{}_j B^{kl} = C^i{}_j{}^{kl}, \tag{2.81a}$$

where $C^i{}_j{}^{kl}$ is a tensor of fourth rank. From Eqs. (2.66),

$$C'^i{}_j{}^{kl} = \frac{\partial x'^i}{\partial x^m}\frac{\partial x^n}{\partial x'^j}\frac{\partial x'^k}{\partial x^p}\frac{\partial x'^l}{\partial x^q} C^m{}_n{}^{pq}. \tag{2.81b}$$

The direct product is a technique for creating new, higher-rank tensors. Exercise 2.7.1 is a form of the direct product in which the first factor is ∇. Applications appear in Section 4.6.

When $\mathbf{T}$ is an nth-rank Cartesian tensor, $(\partial/\partial x^i)T_{jkl}\ldots$, a component of $\nabla\mathbf{T}$, is a **Cartesian** tensor of rank $n+1$ (Exercise 2.7.1). However, $(\partial/\partial x^i)T_{jkl}\ldots$ is not a tensor in more general spaces. In non-Cartesian systems $\partial/\partial x'^i$ will act on the partial derivatives $\partial x^p/\partial x'^q$ and destroy the simple tensor transformation relation (see Eq. (2.129)).

So far the distinction between a covariant transformation and a contravariant transformation has been maintained because it does exist in non-Euclidean space and because it is of great importance in general relativity. In Sections 2.10 and 2.11 we shall develop differential relations for general tensors. Often, however, because of the simplification achieved, we restrict ourselves to Cartesian tensors. As noted in Section 2.6, the distinction between contravariance and covariance disappears.

Exercises

2.7.1 If $T_{\cdots i}$ is a tensor of rank n, show that $\partial T_{\cdots i}/\partial x^j$ is a tensor of rank $n+1$ (Cartesian coordinates).

Note. In non-Cartesian coordinate systems the coefficients a_{ij} are, in general, functions of the coordinates, and the simple derivative of a tensor of rank n is not a tensor except in the special case of $n=0$. In this case the derivative does yield a covariant vector (tensor of rank 1) by Eq. (2.64).

2.7.2 If $T_{ijk\cdots}$ is a tensor of rank n, show that $\sum_j \partial T_{ijk\cdots}/\partial x^j$ is a tensor of rank $n-1$ (Cartesian coordinates).

2.7.3 The operator

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

may be written as

$$\sum_{i=1}^{4}\frac{\partial^2}{\partial x_i^2},$$