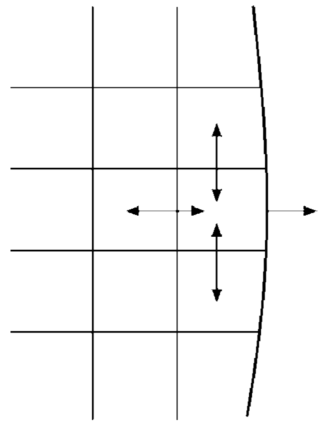

FIGURE 1.28 Exact cancellation of $d\boldsymbol{\sigma}$'s on **interior** surfaces. No cancellation on the **exterior** surface.

of a sum, we take the limit as the number of parallelepipeds approaches infinity ($\to \infty$) and the dimensions of each approach zero ($\to 0$):

$$\begin{array}{ccc} \displaystyle\sum_{\text{exterior surfaces}} \mathbf{V} \cdot d\boldsymbol{\sigma} & = & \displaystyle\sum_{\text{volumes}} \nabla \cdot \mathbf{V}\, d\tau \\ \downarrow & & \downarrow \\ \int_S \mathbf{V} \cdot d\boldsymbol{\sigma} & = & \int_V \nabla \cdot \mathbf{V}\, d\tau. \end{array}$$

The result is Eq. (1.101a), Gauss' theorem.

From a physical point of view Eq. (1.66) has established $\nabla \cdot \mathbf{V}$ as the net outflow of fluid per unit volume. The volume integral then gives the total net outflow. But the surface integral $\int \mathbf{V} \cdot d\boldsymbol{\sigma}$ is just another way of expressing this same quantity, which is the equality, Gauss' theorem.

Green's Theorem

A frequently useful corollary of Gauss' theorem is a relation known as Green's theorem. If u and v are two scalar functions, we have the identities

$$\nabla \cdot (u\, \nabla v) = u \nabla \cdot \nabla v + (\nabla u) \cdot (\nabla v), \tag{1.102}$$

$$\nabla \cdot (v\, \nabla u) = v \nabla \cdot \nabla u + (\nabla v) \cdot (\nabla u). \tag{1.103}$$

Subtracting Eq. (1.103) from Eq. (1.102), integrating over a volume (u, v, and their derivatives, assumed continuous), and applying Eq. (1.101a) (Gauss' theorem), we obtain

$$\boxed{\iiint_V (u \nabla \cdot \nabla v - v \nabla \cdot \nabla u)\, d\tau = \oint\!\!\oint_{\partial V} (u \nabla v - v \nabla u) \cdot d\boldsymbol{\sigma}.} \tag{1.104}$$

This is Green's theorem. We use it for developing Green's functions in Chapter 9. An alternate form of Green's theorem, derived from Eq. (1.102) alone, is

$$\oiint_{\partial V} u\nabla v \cdot d\boldsymbol{\sigma} = \iiint_V u\nabla \cdot \nabla v\, d\tau + \iiint_V \nabla u \cdot \nabla v\, d\tau. \tag{1.105}$$

This is the form of Green's theorem used in Section 1.16.

Alternate Forms of Gauss' Theorem

Although Eq. (1.101a) involving the divergence is by far the most important form of Gauss' theorem, volume integrals involving the gradient and the curl may also appear. Suppose

$$\mathbf{V}(x, y, z) = V(x, y, z)\mathbf{a}, \tag{1.106}$$

in which **a** is a vector with constant magnitude and constant but arbitrary direction. (You pick the direction, but once you have chosen it, hold it fixed.) Equation (1.101a) becomes

$$\mathbf{a} \cdot \oiint_{\partial V} V\, d\boldsymbol{\sigma} = \iiint_V \nabla \cdot \mathbf{a} V\, d\tau = \mathbf{a} \cdot \iiint_V \nabla V\, d\tau \tag{1.107}$$

by Eq. (1.67b). This may be rewritten

$$\mathbf{a} \cdot \left[\oiint_{\partial V} V\, d\boldsymbol{\sigma} - \iiint_V \nabla V\, d\tau \right] = 0. \tag{1.108}$$

Since $|\mathbf{a}| \neq 0$ and its direction is arbitrary, meaning that the cosine of the included angle cannot **always** vanish, the terms in brackets must be zero.[23] The result is

$$\oiint_{\partial V} V\, d\boldsymbol{\sigma} = \iiint_V \nabla V\, d\tau. \tag{1.109}$$

In a similar manner, using $\mathbf{V} = \mathbf{a} \times \mathbf{P}$ in which **a** is a constant vector, we may show

$$\oiint_{\partial V} d\boldsymbol{\sigma} \times \mathbf{P} = \iiint_V \nabla \times \mathbf{P}\, d\tau. \tag{1.110}$$

These last two forms of Gauss' theorem are used in the vector form of Kirchoff diffraction theory. They may also be used to verify Eqs. (1.97) and (1.99). Gauss' theorem may also be extended to tensors (see Section 2.11).

Exercises

1.11.1 Using Gauss' theorem, prove that

$$\oiint_S d\boldsymbol{\sigma} = 0$$

if $S = \partial V$ is a closed surface.

[23] This exploitation of the **arbitrary** nature of a part of a problem is a valuable and widely used technique. The arbitrary vector is used again in Sections 1.12 and 1.13. Other examples appear in Section 1.14 (integrands equated) and in Section 2.8, quotient rule.

1.11.2 Show that

$$\frac{1}{3}\oiint_S \mathbf{r}\cdot d\boldsymbol{\sigma} = V,$$

where V is the volume enclosed by the closed surface $S = \partial V$.
Note. This is a generalization of Exercise 1.10.5.

1.11.3 If $\mathbf{B} = \nabla \times \mathbf{A}$, show that

$$\oiint_S \mathbf{B}\cdot d\boldsymbol{\sigma} = \mathbf{0}$$

for any closed surface S.

1.11.4 Over some volume V let ψ be a solution of Laplace's equation (with the derivatives appearing there continuous). Prove that the integral over any closed surface in V of the normal derivative of ψ ($\partial\psi/\partial n$, or $\nabla\psi\cdot\mathbf{n}$) will be zero.

1.11.5 In analogy to the integral definition of gradient, divergence, and curl of Section 1.10, show that

$$\nabla^2\varphi = \lim_{\int d\tau\to 0}\frac{\int \nabla\varphi\cdot d\boldsymbol{\sigma}}{\int d\tau}.$$

1.11.6 The electric displacement vector $\mathbf{D}$ satisfies the Maxwell equation $\nabla\cdot\mathbf{D} = \rho$, where ρ is the charge density (per unit volume). At the boundary between two media there is a surface charge density σ (per unit area). Show that a boundary condition for $\mathbf{D}$ is

$$(\mathbf{D}_2 - \mathbf{D}_1)\cdot\mathbf{n} = \sigma.$$

$\mathbf{n}$ is a unit vector normal to the surface and out of medium 1.
Hint. Consider a thin pillbox as shown in Fig. 1.29.

1.11.7 From Eq. (1.67b), with $\mathbf{V}$ the electric field $\mathbf{E}$ and f the electrostatic potential φ, show that, for integration over all space,

$$\int \rho\varphi\, d\tau = \varepsilon_0 \int E^2\, d\tau.$$

This corresponds to a three-dimensional integration by parts.
Hint. $\mathbf{E} = -\nabla\varphi$, $\nabla\cdot\mathbf{E} = \rho/\varepsilon_0$. You may assume that φ vanishes at large r at least as fast as r^{-1}.

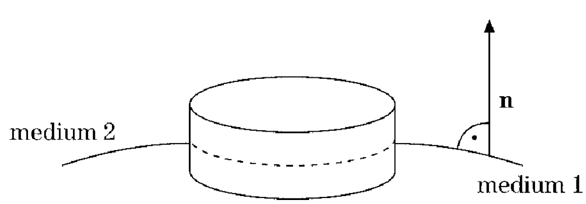

FIGURE 1.29 Pillbox.

1.11.8 A particular steady-state electric current distribution is localized in space. Choosing a bounding surface far enough out so that the current density **J** is zero everywhere on the surface, show that

$$\iiint \mathbf{J}\,d\tau = 0.$$

Hint. Take one component of **J** at a time. With $\nabla \cdot \mathbf{J} = 0$, show that $\mathbf{J}_i = \nabla \cdot (x_i \mathbf{J})$ and apply Gauss' theorem.

1.11.9 The creation of a **localized** system of steady electric currents (current density **J**) and magnetic fields may be shown to require an amount of work

$$W = \frac{1}{2}\iiint \mathbf{H}\cdot\mathbf{B}\,d\tau.$$

Transform this into

$$W = \frac{1}{2}\iiint \mathbf{J}\cdot\mathbf{A}\,d\tau.$$

Here **A** is the magnetic vector potential: $\nabla \times \mathbf{A} = \mathbf{B}$.
Hint. In Maxwell's equations take the displacement current term $\partial\mathbf{D}/\partial t = 0$. If the fields and currents are localized, a bounding surface may be taken far enough out so that the integrals of the fields and currents over the surface yield zero.

1.11.10 Prove the generalization of Green's theorem:

$$\iiint_V (v\mathcal{L}u - u\mathcal{L}v)\,d\tau = \oiint_{\partial V} p(v\nabla u - u\nabla v)\cdot d\boldsymbol{\sigma}.$$

Here $\mathcal{L}$ is the self-adjoint operator (Section 10.1),

$$\mathcal{L} = \nabla\cdot\left[p(\mathbf{r})\nabla\right] + q(\mathbf{r})$$

and p, q, u, and v are functions of position, p and q having continuous first derivatives and u and v having continuous second derivatives.
Note. This generalized Green's theorem appears in Section 9.7.

1.12 STOKES' THEOREM

Gauss' theorem relates the volume integral of a derivative of a function to an integral of the function over the closed surface bounding the volume. Here we consider an analogous relation between the surface integral of a derivative of a function and the line integral of the function, the path of integration being the perimeter bounding the surface.

Let us take the surface and subdivide it into a network of arbitrarily small rectangles. In Section 1.8 we showed that the circulation about such a differential rectangle (in the xy-plane) is $\nabla \times \mathbf{V}|_z\,dx\,dy$. From Eq. (1.76) applied to **one** differential rectangle,

$$\sum_{\text{four sides}} \mathbf{V}\cdot d\boldsymbol{\lambda} = \nabla\times\mathbf{V}\cdot d\boldsymbol{\sigma}. \tag{1.111}$$