1.9.10   In a (nonrotating) isolated mass such as a star, the condition for equilibrium is

$$\nabla P + \rho \nabla \varphi = 0.$$

Here P is the total pressure, ρ is the density, and φ is the gravitational potential. Show that at any given point the normals to the surfaces of constant pressure and constant gravitational potential are parallel.

1.9.11   In the Pauli theory of the electron, one encounters the expression

$$(\mathbf{p} - e\mathbf{A}) \times (\mathbf{p} - e\mathbf{A})\psi,$$

where ψ is a scalar (wave) function. $\mathbf{A}$ is the magnetic vector potential related to the magnetic induction $\mathbf{B}$ by $\mathbf{B} = \nabla \times \mathbf{A}$. Given that $\mathbf{p} = -i\nabla$, show that this expression reduces to $ie\mathbf{B}\psi$. Show that this leads to the orbital g-factor $g_L = 1$ upon writing the magnetic moment as $\boldsymbol{\mu} = g_L \mathbf{L}$ in units of Bohr magnetons and $\mathbf{L} = -i\mathbf{r} \times \nabla$. See also Exercise 1.13.7.

1.9.12   Show that any solution of the equation

$$\nabla \times (\nabla \times \mathbf{A}) - k^2 \mathbf{A} = 0$$

automatically satisfies the vector Helmholtz equation

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = 0$$

and the solenoidal condition

$$\nabla \cdot \mathbf{A} = 0.$$

Hint. Let $\nabla \cdot$ operate on the first equation.

1.9.13   The theory of heat conduction leads to an equation

$$\nabla^2 \Psi = k|\nabla \Phi|^2,$$

where Φ is a potential satisfying Laplace's equation: $\nabla^2 \Phi = 0$. Show that a solution of this equation is

$$\Psi = \frac{1}{2}k\Phi^2.$$

1.10   VECTOR INTEGRATION

The next step after differentiating vectors is to integrate them. Let us start with line integrals and then proceed to surface and volume integrals. In each case the method of attack will be to reduce the vector integral to scalar integrals with which the reader is assumed familiar.

Line Integrals

Using an increment of length $d\mathbf{r} = \hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$, we may encounter the line integrals

$$\int_C \varphi\,d\mathbf{r}, \tag{1.92a}$$

$$\int_C \mathbf{V} \cdot d\mathbf{r}, \tag{1.92b}$$

$$\int_C \mathbf{V} \times d\mathbf{r}, \tag{1.92c}$$

in each of which the integral is over some contour C that may be open (with starting point and ending point separated) or closed (forming a loop). Because of its physical interpretation that follows, the second form, Eq. (1.92b) is by far the most important of the three.

With φ, a scalar, the first integral reduces immediately to

$$\int_C \varphi\,d\mathbf{r} = \hat{\mathbf{x}}\int_C \varphi(x,y,z)\,dx + \hat{\mathbf{y}}\int_C \varphi(x,y,z)\,dy + \hat{\mathbf{z}}\int_C \varphi(x,y,z)\,dz. \tag{1.93}$$

This separation has employed the relation

$$\int \hat{\mathbf{x}}\varphi\,dx = \hat{\mathbf{x}}\int \varphi\,dx, \tag{1.94}$$

which is permissible because the Cartesian unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are constant in both magnitude and direction. Perhaps this relation is obvious here, but it will not be true in the non-Cartesian systems encountered in Chapter 2.

The three integrals on the right side of Eq. (1.93) are ordinary scalar integrals and, to avoid complications, we assume that they are Riemann integrals. Note, however, that the integral with respect to x cannot be evaluated unless y and z are known in terms of x and similarly for the integrals with respect to y and z. This simply means that the path of integration C must be specified. Unless the integrand has special properties so that the integral depends only on the value of the end points, the value will depend on the particular choice of contour C. For instance, if we choose the very special case $\varphi = 1$, Eq. (1.92a) is just the vector distance from the start of contour C to the endpoint, in this case independent of the choice of path connecting fixed endpoints. With $d\mathbf{r} = \hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$, the second and third forms also reduce to scalar integrals and, like Eq. (1.92a), are dependent, in general, on the choice of path. The form (Eq. (1.92b)) is exactly the same as that encountered when we calculate the work done by a force that varies along the path,

$$W = \int \mathbf{F} \cdot d\mathbf{r} = \int F_x(x,y,z)\,dx + \int F_y(x,y,z)\,dy + \int F_z(x,y,z)\,dz. \tag{1.95a}$$

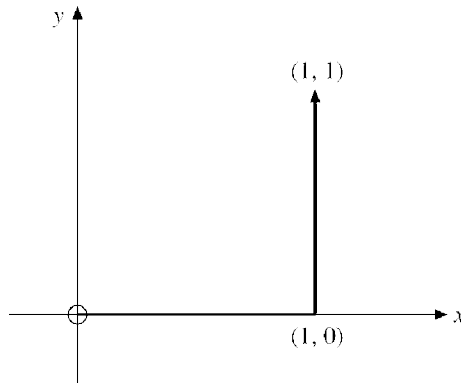In this expression $\mathbf{F}$ is the force exerted on a particle.

FIGURE 1.25   A path of integration.

Example 1.10.1   PATH-DEPENDENT WORK

The force exerted on a body is $\mathbf{F} = -\hat{\mathbf{x}}y + \hat{\mathbf{y}}x$. The problem is to calculate the work done going from the origin to the point $(1, 1)$:

$$W = \int_{0,0}^{1,1} \mathbf{F} \cdot d\mathbf{r} = \int_{0,0}^{1,1} (-y\,dx + x\,dy). \tag{1.95b}$$

Separating the two integrals, we obtain

$$W = -\int_0^1 y\,dx + \int_0^1 x\,dy. \tag{1.95c}$$

The first integral cannot be evaluated until we specify the values of y as x ranges from 0 to 1. Likewise, the second integral requires x as a function of y. Consider first the path shown in Fig. 1.25. Then

$$W = -\int_0^1 0\,dx + \int_0^1 1\,dy = 1, \tag{1.95d}$$

since $y = 0$ along the first segment of the path and $x = 1$ along the second. If we select the path $[x = 0, 0 \leqslant y \leqslant 1]$ and $[0 \leqslant x \leqslant 1, y = 1]$, then Eq. (1.95c) gives $W = -1$. For this force the work done depends on the choice of path.   ∎

Surface Integrals

Surface integrals appear in the same forms as line integrals, the element of area also being a vector, $d\boldsymbol{\sigma}$.[20] Often this area element is written $\mathbf{n}\,dA$, in which $\mathbf{n}$ is a unit (normal) vector to indicate the positive direction.[21] There are two conventions for choosing the positive direction. First, if the surface is a closed surface, we agree to take the outward normal as positive. Second, if the surface is an open surface, the positive normal depends on the direction in which the perimeter of the open surface is traversed. If the right-hand fingers

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[20]Recall that in Section 1.4 the area (of a parallelogram) is represented by a cross-product **vector**.
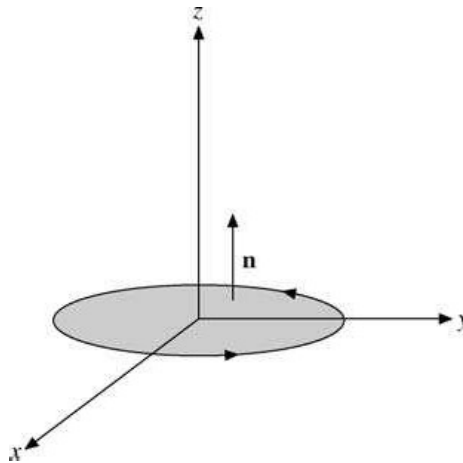[21]Although $\mathbf{n}$ always has unit length, its direction may well be a function of position.

FIGURE 1.26   Right-hand rule for
the positive normal.

are placed in the direction of travel around the perimeter, the positive normal is indicated by the thumb of the right hand. As an illustration, a circle in the xy-plane (Fig. 1.26) mapped out from x to y to $-x$ to $-y$ and back to x will have its positive normal parallel to the positive z-axis (for the right-handed coordinate system).

Analogous to the line integrals, Eqs. (1.92a) to (1.92c), surface integrals may appear in the forms

$$\int \varphi \, d\boldsymbol{\sigma}, \qquad \int \mathbf{V} \cdot d\boldsymbol{\sigma}, \qquad \int \mathbf{V} \times d\boldsymbol{\sigma}.$$

Again, the dot product is by far the most commonly encountered form. The surface integral $\int \mathbf{V} \cdot d\boldsymbol{\sigma}$ may be interpreted as a flow or flux through the given surface. This is really what we did in Section 1.7 to obtain the significance of the term divergence. This identification reappears in Section 1.11 as Gauss' theorem. Note that both physically and from the dot product the tangential components of the velocity contribute nothing to the flow through the surface.

Volume Integrals

Volume integrals are somewhat simpler, for the volume element $d\tau$ is a scalar quantity.[22] We have

$$\int_V \mathbf{V} \, d\tau = \hat{\mathbf{x}} \int_V V_x \, d\tau + \hat{\mathbf{y}} \int_V V_y \, d\tau + \hat{\mathbf{z}} \int_V V_z \, d\tau, \tag{1.96}$$

again reducing the vector integral to a vector sum of scalar integrals.

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[22]Frequently the symbols d^3r and d^3x are used to denote a volume element in coordinate (xyz or $x_1 x_2 x_3$) space.