Example 1.8.1 VECTOR POTENTIAL OF A CONSTANT **B** FIELD

From electrodynamics we know that $\nabla \cdot \mathbf{B} = 0$, which has the general solution $\mathbf{B} = \nabla \times \mathbf{A}$, where $\mathbf{A}(\mathbf{r})$ is called the vector potential (of the magnetic induction), because $\nabla \cdot (\nabla \times \mathbf{A}) = (\nabla \times \nabla) \cdot \mathbf{A} \equiv 0$, as a triple scalar product with two identical vectors. This last identity will not change if we add the gradient of some scalar function to the vector potential, which, therefore, is not unique.

In our case, we want to show that a vector potential is $\mathbf{A} = \frac{1}{2}(\mathbf{B} \times \mathbf{r})$.

Using the *BAC–BAC* rule in conjunction with Example 1.7.1, we find that

$$2\nabla \times \mathbf{A} = \nabla \times (\mathbf{B} \times \mathbf{r}) = (\nabla \cdot \mathbf{r})\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{r} = 3\mathbf{B} - \mathbf{B} = 2\mathbf{B},$$

where we indicate by the ordering of the scalar product of the second term that the gradient still acts on the coordinate vector. ■

Example 1.8.2 CURL OF A CENTRAL FORCE FIELD

Calculate $\nabla \times (\mathbf{r} f(r))$.

By Eq. (1.71),

$$\nabla \times \big(\mathbf{r} f(r)\big) = f(r)\nabla \times \mathbf{r} + \big[\nabla f(r)\big] \times \mathbf{r}. \tag{1.72}$$

First,

$$\nabla \times \mathbf{r} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0. \tag{1.73}$$

Second, using $\nabla f(r) = \hat{\mathbf{r}}(df/dr)$ (Example 1.6.1), we obtain

$$\nabla \times \mathbf{r} f(r) = \frac{df}{dr}\hat{\mathbf{r}} \times \mathbf{r} = 0. \tag{1.74}$$

This vector product vanishes, since $\mathbf{r} = \hat{\mathbf{r}}r$ and $\hat{\mathbf{r}} \times \hat{\mathbf{r}} = 0$. ■

To develop a better feeling for the physical significance of the curl, we consider the circulation of fluid around a differential loop in the xy-plane, Fig. 1.24.

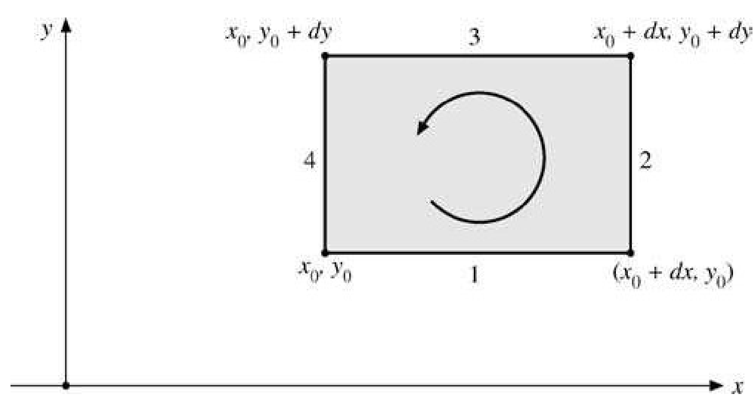

FIGURE 1.24 Circulation around a differential loop.

Although the circulation is technically given by a vector line integral $\int \mathbf{V}\cdot d\boldsymbol{\lambda}$ (Section 1.10), we can set up the equivalent scalar integrals here. Let us take the circulation to be

$$\begin{aligned}\text{circulation}_{1234} &= \int_1 V_x(x,y)\,d\lambda_x + \int_2 V_y(x,y)\,d\lambda_y \\ &\quad + \int_3 V_x(x,y)\,d\lambda_x + \int_4 V_y(x,y)\,d\lambda_y. \end{aligned}\tag{1.75}$$

The numbers 1, 2, 3, and 4 refer to the numbered line segments in Fig. 1.24. In the first integral, $d\lambda_x = +dx$; but in the third integral, $d\lambda_x = -dx$ because the third line segment is traversed in the negative x-direction. Similarly, $d\lambda_y = +dy$ for the second integral, $-dy$ for the fourth. Next, the integrands are referred to the point (x_0, y_0) with a Taylor expansion[18] taking into account the displacement of line segment 3 from 1 and that of 2 from 4. For our differential line segments this leads to

$$\begin{aligned}\text{circulation}_{1234} &= V_x(x_0,y_0)\,dx + \left[V_y(x_0,y_0) + \frac{\partial V_y}{\partial x}dx\right]dy \\ &\quad + \left[V_x(x_0,y_0) + \frac{\partial V_x}{\partial y}dy\right](-dx) + V_y(x_0,y_0)(-dy) \\ &= \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)dx\,dy. \end{aligned}\tag{1.76}$$

Dividing by $dx\,dy$, we have

$$\text{circulation per unit area} = \nabla\times\mathbf{V}|_z. \tag{1.77}$$

The circulation[19] about our differential area in the xy-plane is given by the z-component of $\nabla\times\mathbf{V}$. In principle, the curl $\nabla\times\mathbf{V}$ at (x_0, y_0) could be determined by inserting a (differential) paddle wheel into the moving fluid at point (x_0, y_0). The rotation of the little paddle wheel would be a measure of the curl, and its axis would be along the direction of $\nabla\times\mathbf{V}$, which is perpendicular to the plane of circulation.

We shall use the result, Eq. (1.76), in Section 1.12 to derive Stokes' theorem. Whenever the curl of a vector $\mathbf{V}$ vanishes,

$$\nabla\times\mathbf{V} = 0, \tag{1.78}$$

$\mathbf{V}$ is labeled **irrotational**. The most important physical examples of irrotational vectors are the gravitational and electrostatic forces. In each case

$$\mathbf{V} = C\frac{\hat{\mathbf{r}}}{r^2} = C\frac{\mathbf{r}}{r^3}, \tag{1.79}$$

where C is a constant and $\hat{\mathbf{r}}$ is the unit vector in the outward radial direction. For the gravitational case we have $C = -Gm_1m_2$, given by Newton's law of universal gravitation. If $C = q_1q_2/4\pi\varepsilon_0$, we have Coulomb's law of electrostatics (mks units). The force $\mathbf{V}$

[18] Here, $V_y(x_0+dx, y_0) = V_y(x_0,y_0) + (\frac{\partial V_y}{\partial x})_{x_0y_0}\,dx + \cdots$. The higher-order terms will drop out in the limit as $dx \to 0$. A correction term for the variation of V_y with y is canceled by the corresponding term in the fourth integral.

[19] In fluid dynamics $\nabla\times\mathbf{V}$ is called the "vorticity."

given in Eq. (1.79) may be shown to be irrotational by direct expansion into Cartesian components, as we did in Example 1.8.1. Another approach is developed in Chapter 2, in which we express $\nabla\times$, the curl, in terms of spherical polar coordinates. In Section 1.13 we shall see that whenever a vector is irrotational, the vector may be written as the (negative) gradient of a scalar potential. In Section 1.16 we shall prove that a vector field may be resolved into an irrotational part and a solenoidal part (subject to conditions at infinity). In terms of the electromagnetic field this corresponds to the resolution into an irrotational electric field and a solenoidal magnetic field.

For waves in an elastic medium, if the displacement $\mathbf{u}$ is irrotational, $\nabla \times \mathbf{u} = 0$, plane waves (or spherical waves at large distances) become longitudinal. If $\mathbf{u}$ is solenoidal, $\nabla \cdot \mathbf{u} = 0$, then the waves become transverse. A seismic disturbance will produce a displacement that may be resolved into a solenoidal part and an irrotational part (compare Section 1.16). The irrotational part yields the longitudinal P (primary) earthquake waves. The solenoidal part gives rise to the slower transverse S (secondary) waves.

Using the gradient, divergence, and curl, and of course the *BAC–CAB* rule, we may construct or verify a large number of useful vector identities. For verification, complete expansion into Cartesian components is always a possibility. Sometimes if we use insight instead of routine shuffling of Cartesian components, the verification process can be shortened drastically.

Remember that ∇ is a vector operator, a hybrid creature satisfying two sets of rules:

1. vector rules, and
2. partial differentiation rules — including differentiation of a product.

Example 1.8.3 Gradient of a Dot Product

Verify that

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B}). \tag{1.80}$$

This particular example hinges on the recognition that $\nabla(\mathbf{A}\cdot\mathbf{B})$ is the type of term that appears in the *BAC–CAB* expansion of a triple vector product, Eq. (1.55). For instance,

$$\mathbf{A}\times(\nabla\times\mathbf{B}) = \nabla(\mathbf{A}\cdot\mathbf{B}) - (\mathbf{A}\cdot\nabla)\mathbf{B},$$

with the ∇ differentiating only $\mathbf{B}$, not $\mathbf{A}$. From the commutativity of factors in a scalar product we may interchange $\mathbf{A}$ and $\mathbf{B}$ and write

$$\mathbf{B}\times(\nabla\times\mathbf{A}) = \nabla(\mathbf{A}\cdot\mathbf{B}) - (\mathbf{B}\cdot\nabla)\mathbf{A},$$

now with ∇ differentiating only $\mathbf{A}$, not $\mathbf{B}$. Adding these two equations, we obtain ∇ differentiating the product $\mathbf{A}\cdot\mathbf{B}$ and the identity, Eq. (1.80). This identity is used frequently in electromagnetic theory. Exercise 1.8.13 is a simple illustration. ■