**1.6.2** (a) Find a unit vector perpendicular to the surface

$$x^2 + y^2 + z^2 = 3$$

at the point $(1, 1, 1)$. Lengths are in centimeters.

(b) Derive the equation of the plane tangent to the surface at $(1, 1, 1)$.

ANS. (a) $(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})/\sqrt{3}$, (b) $x + y + z = 3$.

**1.6.3** Given a vector $\mathbf{r}_{12} = \hat{\mathbf{x}}(x_1 - x_2) + \hat{\mathbf{y}}(y_1 - y_2) + \hat{\mathbf{z}}(z_1 - z_2)$, show that $\nabla_1 r_{12}$ (gradient with respect to $x_1$, $y_1$, and $z_1$ of the magnitude $r_{12}$) is a unit vector in the direction of $\mathbf{r}_{12}$.

**1.6.4** If a vector function $\mathbf{F}$ depends on both space coordinates $(x, y, z)$ and time $t$, show that

$$d\mathbf{F} = (d\mathbf{r} \cdot \nabla)\mathbf{F} + \frac{\partial \mathbf{F}}{\partial t} dt.$$

**1.6.5** Show that $\nabla(uv) = v\nabla u + u\nabla v$, where $u$ and $v$ are differentiable scalar functions of $x$, $y$, and $z$.

(a) Show that a necessary and sufficient condition that $u(x, y, z)$ and $v(x, y, z)$ are related by some function $f(u, v) = 0$ is that $(\nabla u) \times (\nabla v) = 0$.

(b) If $u = u(x, y)$ and $v = v(x, y)$, show that the condition $(\nabla u) \times (\nabla v) = 0$ leads to the two-dimensional Jacobian

$$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0.$$

The functions $u$ and $v$ are assumed differentiable.

# 1.7 DIVERGENCE, $\nabla$

Differentiating a vector function is a simple extension of differentiating scalar quantities. Suppose $\mathbf{r}(t)$ describes the position of a satellite at some time $t$. Then, for differentiation with respect to time,

$$\frac{d\mathbf{r}(t)}{dt} = \lim_{\Delta \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \mathbf{v}, \text{ linear velocity.}$$

Graphically, we again have the slope of a curve, orbit, or trajectory, as shown in Fig. 1.22.

If we resolve $\mathbf{r}(t)$ into its Cartesian components, $d\mathbf{r}/dt$ always reduces directly to a vector sum of not more than three (for three-dimensional space) scalar derivatives. In other coordinate systems (Chapter 2) the situation is more complicated, for the unit vectors are no longer constant in direction. Differentiation with respect to the space coordinates is handled in the same way as differentiation with respect to time, as seen in the following paragraphs.

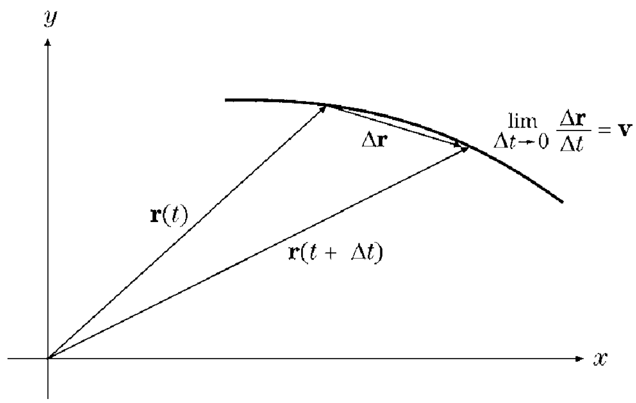


FIGURE 1.22 Differentiation of a vector.

In Section 1.6, $\nabla$ was defined as a vector operator. Now, paying attention to both its vector and its differential properties, we let it operate on a vector. First, as a vector we dot it into a second vector to obtain

$$\boxed{\nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z},} \tag{1.65a}$$

known as the divergence of $\mathbf{V}$. This is a scalar, as discussed in Section 1.3.

## *Example 1.7.1* DIVERGENCE OF COORDINATE VECTOR

Calculate $\nabla \cdot \mathbf{r}$:

$$\begin{aligned}\nabla \cdot \mathbf{r} &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot (\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z)\\ &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z},\end{aligned}$$

or $\nabla \cdot \mathbf{r} = 3$. ■

## *Example 1.7.2* DIVERGENCE OF CENTRAL FORCE FIELD

Generalizing Example 1.7.1,

$$\begin{aligned}\nabla \cdot \big(\mathbf{r} f(r)\big) &= \frac{\partial}{\partial x}\big[x\, f(r)\big] + \frac{\partial}{\partial y}\big[y\, f(r)\big] + \frac{\partial}{\partial z}\big[z\, f(r)\big]\\ &= 3 f(r) + \frac{x^2}{r}\frac{df}{dr} + \frac{y^2}{r}\frac{df}{dr} + \frac{z^2}{r}\frac{df}{dr}\\ &= 3 f(r) + r\frac{df}{dr}.\end{aligned}$$

The manipulation of the partial derivatives leading to the second equation in Example 1.7.2 is discussed in Example 1.6.1. In particular, if $f(r) = r^{n-1}$,

$$\begin{aligned}\nabla \cdot \left(\mathbf{r}r^{n-1}\right) &= \nabla \cdot \hat{\mathbf{r}}r^{n} \\ &= 3r^{n-1} + (n-1)r^{n-1} \\ &= (n+2)r^{n-1}. \end{aligned} \tag{1.65b}$$

This divergence vanishes for $n = -2$, except at $r = 0$, an important fact in Section 1.14. ■

## *Example 1.7.3* Integration by Parts of Divergence

Let us prove the formula $\int f(\mathbf{r})\nabla \cdot \mathbf{A}(\mathbf{r})\,d^3r = -\int \mathbf{A} \cdot \nabla f\,d^3r$, where $\mathbf{A}$ or $f$ or both vanish at infinity.

To show this, we proceed, as in Example 1.6.3, by integration by parts after writing the inner product in Cartesian coordinates. Because the integrated terms are evaluated at infinity, where they vanish, we obtain

$$\begin{aligned}\int f(\mathbf{r})\nabla \cdot \mathbf{A}(\mathbf{r})\,d^3r &= \int f\left(\frac{\partial A_x}{\partial x}dx\,dy\,dz + \frac{\partial A_y}{\partial y}dy\,dx\,dz + \frac{\partial A_z}{\partial z}dz\,dx\,dy\right) \\ &= -\int \left(A_x\frac{\partial f}{\partial x}dx\,dy\,dz + A_y\frac{\partial f}{\partial y}dy\,dx\,dz + A_z\frac{\partial f}{\partial z}dz\,dx\,dy\right) \\ &= -\int \mathbf{A} \cdot \nabla f\,d^3r. \end{aligned}$$

■

# A Physical Interpretation

To develop a feeling for the physical significance of the divergence, consider $\nabla \cdot (\rho\mathbf{v})$ with $\mathbf{v}(x, y, z)$, the velocity of a compressible fluid, and $\rho(x, y, z)$, its density at point $(x, y, z)$. If we consider a small volume $dx\,dy\,dz$ (Fig. 1.23) at $x = y = z = 0$, the fluid flowing into this volume per unit time (positive $x$-direction) through the face $EFGH$ is (rate of flow in)$_{EFGH} = \rho v_x|_{x=0} = dy\,dz$. The components of the flow $\rho v_y$ and $\rho v_z$ tangential to this face contribute nothing to the flow through this face. The rate of flow out (still positive $x$-direction) through face $ABCD$ is $\rho v_x|_{x=dx}\,dy\,dz$. To compare these flows and to find the net flow out, we expand this last result, like the total variation in Section 1.6.[15] This yields

$$\begin{aligned}(\text{rate of flow out})_{ABCD} &= \rho v_x|_{x=dx}\,dy\,dz \\ &= \left[\rho v_x + \frac{\partial}{\partial x}(\rho v_x)\,dx\right]_{x=0} dy\,dz. \end{aligned}$$

Here the derivative term is a first correction term, allowing for the possibility of nonuniform density or velocity or both.[16] The zero-order term $\rho v_x|_{x=0}$ (corresponding to uniform flow)

[15] Here we have the increment $dx$ and we show a partial derivative with respect to $x$ since $\rho v_x$ may also depend on $y$ and $z$.

[16] Strictly speaking, $\rho v_x$ is averaged over face $EFGH$ and the expression $\rho v_x + (\partial/\partial x)(\rho v_x)\,dx$ is similarly averaged over face $ABCD$. Using an arbitrarily small differential volume, we find that the averages reduce to the values employed here.

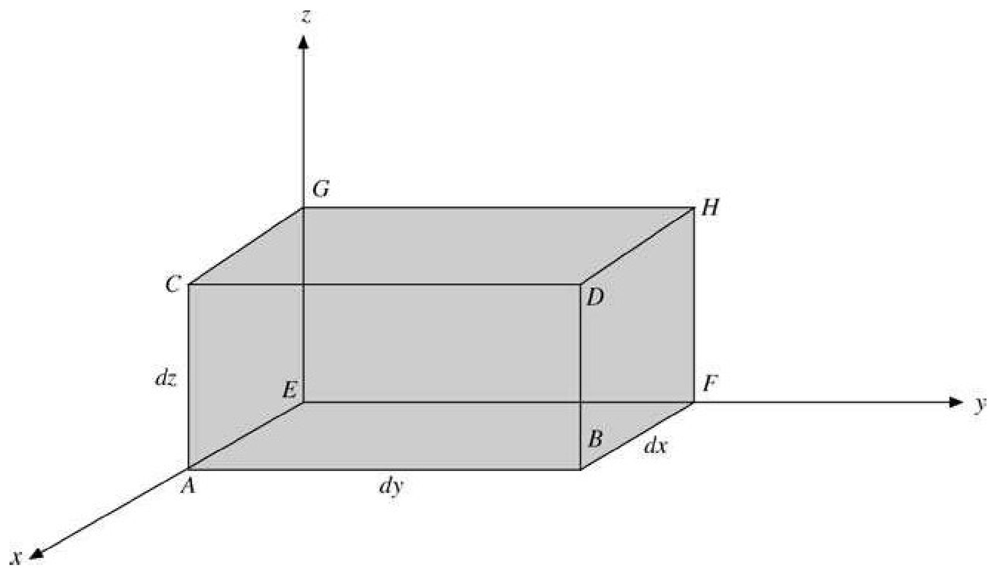


FIGURE 1.23 Differential rectangular parallelepiped (in first octant).

cancels out:

$$\text{Net rate of flow out}|_x = \frac{\partial}{\partial x}(\rho v_x)\,dx\,dy\,dz.$$

Equivalently, we can arrive at this result by

$$\lim_{\Delta x \to 0} \frac{\rho v_x(\Delta x, 0, 0) - \rho v_x(0,0,0)}{\Delta x} \equiv \left.\frac{\partial[\rho v_x(x, y, z)]}{\partial x}\right|_{0,0,0}.$$

Now, the $x$-axis is not entitled to any preferred treatment. The preceding result for the two faces perpendicular to the $x$-axis must hold for the two faces perpendicular to the $y$-axis, with $x$ replaced by $y$ and the corresponding changes for $y$ and $z$: $y \to z$, $z \to x$. This is a cyclic permutation of the coordinates. A further cyclic permutation yields the result for the remaining two faces of our parallelepiped. Adding the net rate of flow out for all three pairs of surfaces of our volume element, we have

$$\begin{aligned} \begin{array}{l}\text{net flow out}\\ \text{(per unit time)}\end{array} &= \left[\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right] dx\,dy\,dz \\ &= \nabla \cdot (\rho \mathbf{v})\,dx\,dy\,dz. \end{aligned} \tag{1.66}$$

Therefore the net flow of our compressible fluid out of the volume element $dx\,dy\,dz$ per unit volume per unit time is $\nabla \cdot (\rho \mathbf{v})$. Hence the name **divergence**. A direct application is in the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1.67a}$$

which states that a net flow out of the volume results in a decreased density inside the volume. Note that in Eq. (1.67a), $\rho$ is considered to be a possible function of time as well as of space: $\rho(x, y, z, t)$. The divergence appears in a wide variety of physical problems, ranging from a probability current density in quantum mechanics to neutron leakage in a nuclear reactor.

The combination $\nabla \cdot (f\mathbf{V})$, in which $f$ is a scalar function and $\mathbf{V}$ is a vector function, may be written

$$\begin{aligned}\nabla \cdot (f\mathbf{V}) &= \frac{\partial}{\partial x}(fV_x) + \frac{\partial}{\partial y}(fV_y) + \frac{\partial}{\partial z}(fV_z) \\ &= \frac{\partial f}{\partial x}V_x + f\frac{\partial V_x}{\partial x} + \frac{\partial f}{\partial y}V_y + f\frac{\partial V_y}{\partial y} + \frac{\partial f}{\partial z}V_z + f\frac{\partial V_z}{\partial z} \\ &= (\nabla f)\cdot \mathbf{V} + f\nabla \cdot \mathbf{V}, \end{aligned} \tag{1.67b}$$

which is just what we would expect for the derivative of a product. Notice that $\nabla$ as a differential operator differentiates both $f$ and $\mathbf{V}$; as a vector it is dotted into $\mathbf{V}$ (in each term).

If we have the special case of the divergence of a vector vanishing,

$$\nabla \cdot \mathbf{B} = 0, \tag{1.68}$$

the vector $\mathbf{B}$ is said to be **solenoidal**, the term coming from the example in which $\mathbf{B}$ is the magnetic induction and Eq. (1.68) appears as one of Maxwell's equations. When a vector is solenoidal, it may be written as the curl of another vector known as the vector potential. (In Section 1.13 we shall calculate such a vector potential.)

## *Exercises*

**1.7.1** For a particle moving in a circular orbit $\mathbf{r} = \hat{\mathbf{x}}r\cos\omega t + \hat{\mathbf{y}}r\sin\omega t$,

(a) evaluate $\mathbf{r}\times\dot{\mathbf{r}}$, with $\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \mathbf{v}$.
(b) Show that $\ddot{\mathbf{r}} + \omega^2\mathbf{r} = 0$ with $\ddot{\mathbf{r}} = \frac{d\mathbf{v}}{dt}$.
The radius $r$ and the angular velocity $\omega$ are constant.

ANS. (a) $\hat{\mathbf{z}}\omega r^2$.

**1.7.2** Vector $\mathbf{A}$ satisfies the vector transformation law, Eq. (1.15). Show directly that its time derivative $d\mathbf{A}/dt$ also satisfies Eq. (1.15) and is therefore a vector.

**1.7.3** Show, by differentiating components, that

(a) $\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \frac{d\mathbf{A}}{dt}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{dt}$,
(b) $\frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \frac{d\mathbf{A}}{dt}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{dt}$,
just like the derivative of the product of two algebraic functions.

**1.7.4** In Chapter 2 it will be seen that the unit vectors in non-Cartesian coordinate systems are usually functions of the coordinate variables, $\mathbf{e}_i = \mathbf{e}_i(q_1, q_2, q_3)$ but $|\mathbf{e}_i| = 1$. Show that either $\partial\mathbf{e}_i/\partial q_j = 0$ or $\partial\mathbf{e}_i/\partial q_j$ is orthogonal to $\mathbf{e}_i$.
*Hint*. $\partial\mathbf{e}_i^2/\partial q_j = 0$.