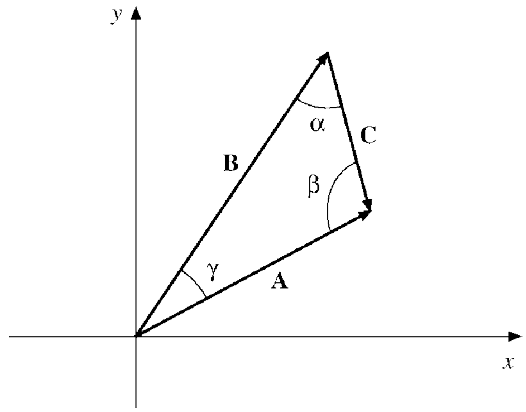

FIGURE 1.15 Law of sines.

and the second rocket path as $\mathbf{r} = \mathbf{r}_2 + t_2\mathbf{v}_2$ with $\mathbf{r}_2 = (5, 2, 1)$ and $\mathbf{v}_2 = (-1, -1, 1)$. Lengths are in kilometers, velocities in kilometers per hour.

1.5 TRIPLE SCALAR PRODUCT, TRIPLE VECTOR PRODUCT

Triple Scalar Product

Sections 1.3 and 1.4 cover the two types of multiplication of interest here. However, there are combinations of three vectors, $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$, that occur with sufficient frequency to deserve further attention. The combination

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$$

is known as the **triple scalar product**. $\mathbf{B} \times \mathbf{C}$ yields a vector that, dotted into $\mathbf{A}$, gives a scalar. We note that $(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$ represents a scalar crossed into a vector, an operation that is not defined. Hence, if we agree to exclude this undefined interpretation, the parentheses may be omitted and the triple scalar product written $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$.

Using Eqs. (1.38) for the cross product and Eq. (1.24) for the dot product, we obtain

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} \times \mathbf{C} &= A_x(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x) \\ &= \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} \\ &= -\mathbf{A} \cdot \mathbf{C} \times \mathbf{B} = -\mathbf{C} \cdot \mathbf{B} \times \mathbf{A} = -\mathbf{B} \cdot \mathbf{A} \times \mathbf{C}, \text{ and so on.} \end{aligned} \tag{1.48}$$

There is a high degree of symmetry in the component expansion. Every term contains the factors A_i, B_j, and C_k. If i, j, and k are in cyclic order (x, y, z), the sign is positive. If the order is anticyclic, the sign is negative. Further, the dot and the cross may be interchanged,

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C}. \tag{1.49}$$

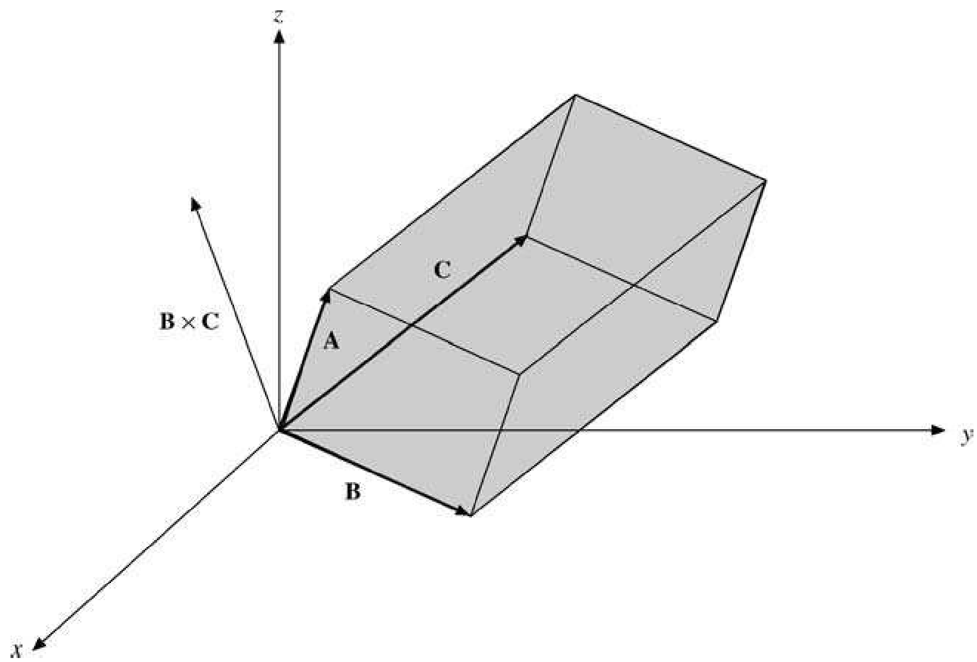

FIGURE 1.16 Parallelepiped representation of triple scalar product.

A convenient representation of the component expansion of Eq. (1.48) is provided by the determinant

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \tag{1.50}$$

The rules for interchanging rows and columns of a determinant[12] provide an immediate verification of the permutations listed in Eq. (1.48), whereas the symmetry of **A**, **B**, and **C** in the determinant form suggests the relation given in Eq. (1.49). The triple products encountered in Section 1.4, which showed that $\mathbf{A}\times\mathbf{B}$ was perpendicular to both **A** and **B**, were special cases of the general result (Eq. (1.48)).

The triple scalar product has a direct geometrical interpretation. The three vectors **A**, **B**, and **C** may be interpreted as defining a parallelepiped (Fig. 1.16):

$$\begin{aligned}|\mathbf{B}\times\mathbf{C}| &= BC\sin\theta \\ &= \text{area of parallelogram base.}\end{aligned} \tag{1.51}$$

The direction, of course, is normal to the base. Dotting **A** into this means multiplying the base area by the projection of **A** onto the normal, or base times height. Therefore

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \text{volume of parallelepiped defined by } \mathbf{A}, \mathbf{B}, \text{ and } \mathbf{C}.$$

The triple scalar product finds an interesting and important application in the construction of a reciprocal crystal lattice. Let **a**, **b**, and **c** (not necessarily mutually perpendicular)

[12]See Section 3.1 for a summary of the properties of determinants.

represent the vectors that define a crystal lattice. The displacement from one lattice point to another may then be written

$$\mathbf{r} = n_a\mathbf{a} + n_b\mathbf{b} + n_c\mathbf{c}, \tag{1.52}$$

with n_a, n_b, and n_c taking on integral values. With these vectors we may form

$$\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}, \qquad \mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}, \qquad \mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}. \tag{1.53a}$$

We see that $\mathbf{a}'$ is perpendicular to the plane containing $\mathbf{b}$ and $\mathbf{c}$, and we can readily show that

$$\mathbf{a}'\cdot\mathbf{a} = \mathbf{b}'\cdot\mathbf{b} = \mathbf{c}'\cdot\mathbf{c} = 1, \tag{1.53b}$$

whereas

$$\mathbf{a}'\cdot\mathbf{b} = \mathbf{a}'\cdot\mathbf{c} = \mathbf{b}'\cdot\mathbf{a} = \mathbf{b}'\cdot\mathbf{c} = \mathbf{c}'\cdot\mathbf{a} = \mathbf{c}'\cdot\mathbf{b} = 0. \tag{1.53c}$$

It is from Eqs. (1.53b) and (1.53c) that the name **reciprocal lattice** is associated with the points $\mathbf{r}' = n'_a\mathbf{a}' + n'_b\mathbf{b}' + n'_c\mathbf{c}'$. The mathematical space in which this reciprocal lattice exists is sometimes called a *Fourier space*, on the basis of relations to the Fourier analysis of Chapters 14 and 15. This reciprocal lattice is useful in problems involving the scattering of waves from the various planes in a crystal. Further details may be found in R. B. Leighton's *Principles of Modern Physics*, pp. 440–448 [New York: McGraw-Hill (1959)].

Triple Vector Product

The second triple product of interest is $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$, which is a vector. Here the parentheses must be retained, as may be seen from a special case $(\hat{\mathbf{x}}\times\hat{\mathbf{x}})\times\hat{\mathbf{y}} = 0$, while $\hat{\mathbf{x}}\times(\hat{\mathbf{x}}\times\hat{\mathbf{y}}) = \hat{\mathbf{x}}\times\hat{\mathbf{z}} = -\hat{\mathbf{y}}$.

Example 1.5.1 A Triple Vector Product

For the vectors

$$\mathbf{A} = \hat{\mathbf{x}} + 2\hat{\mathbf{y}} - \hat{\mathbf{z}} = (1, 2, -1), \quad \mathbf{B} = \hat{\mathbf{y}} + \hat{\mathbf{z}} = (0, 1, 1), \quad \mathbf{C} = \hat{\mathbf{x}} - \hat{\mathbf{y}} = (0, 1, 1),$$

$$\mathbf{B}\times\mathbf{C} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}},$$

and

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 2 & -1 \\ 1 & 1 & -1 \end{vmatrix} = -\hat{\mathbf{x}} - \hat{\mathbf{z}} = -(\hat{\mathbf{y}} + \hat{\mathbf{z}}) - (\hat{\mathbf{x}} - \hat{\mathbf{y}})$$

$$= -\mathbf{B} - \mathbf{C}.$$

■

By rewriting the result in the last line of Example 1.5.1 as a linear combination of **B** and **C**, we notice that, taking a geometric approach, the triple vector product is perpendicular

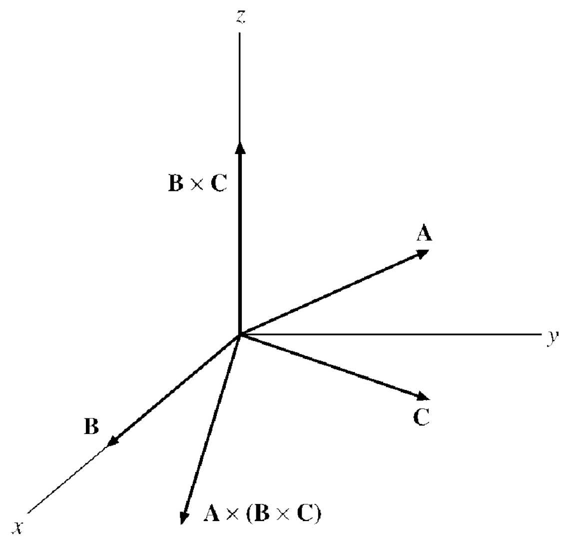

FIGURE 1.17 **B** and **C** are in the xy-plane. $\mathbf{B} \times \mathbf{C}$ is perpendicular to the xy-plane and is shown here along the z-axis. Then $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is perpendicular to the z-axis and therefore is back in the xy-plane.

to **A** and to $\mathbf{B} \times \mathbf{C}$. The plane defined by **B** and **C** is perpendicular to $\mathbf{B} \times \mathbf{C}$, and so the triple product lies in this plane (see Fig. 1.17):

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = u\mathbf{B} + v\mathbf{C}. \tag{1.54}$$

Taking the scalar product of Eq. (1.54) with **A** gives zero for the left-hand side, so $u\mathbf{A} \cdot \mathbf{B} + v\mathbf{A} \cdot \mathbf{C} = 0$. Hence $u = w\mathbf{A} \cdot \mathbf{C}$ and $v = -w\mathbf{A} \cdot \mathbf{B}$ for a suitable w. Substituting these values into Eq. (1.54) gives

$$\boxed{\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = w\big[\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})\big];} \tag{1.55}$$

we want to show that

$$w = 1$$

in Eq. (1.55), an important relation sometimes known as the **BAC–CAB** rule. Since Eq. (1.55) is linear in A, B, and C, w is independent of these magnitudes. That is, we only need to show that $w = 1$ for unit vectors $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$. Let us denote $\hat{\mathbf{B}} \cdot \hat{\mathbf{C}} = \cos\alpha$, $\hat{\mathbf{C}} \cdot \hat{\mathbf{A}} = \cos\beta$, $\hat{\mathbf{A}} \cdot \hat{\mathbf{B}} = \cos\gamma$, and square Eq. (1.55) to obtain

$$\begin{aligned}
\big[\hat{\mathbf{A}} \times (\hat{\mathbf{B}} \times \hat{\mathbf{C}})\big]^2 &= \hat{\mathbf{A}}^2(\hat{\mathbf{B}} \times \hat{\mathbf{C}})^2 - \big[\hat{\mathbf{A}} \cdot (\hat{\mathbf{B}} \times \hat{\mathbf{C}})\big]^2 \\
&= 1 - \cos^2\alpha - \big[\hat{\mathbf{A}} \cdot (\hat{\mathbf{B}} \times \hat{\mathbf{C}})\big]^2 \\
&= w^2\big[(\hat{\mathbf{A}} \cdot \hat{\mathbf{C}})^2 + (\hat{\mathbf{A}} \cdot \hat{\mathbf{B}})^2 - 2(\hat{\mathbf{A}} \cdot \hat{\mathbf{B}})(\hat{\mathbf{A}} \cdot \hat{\mathbf{C}})(\hat{\mathbf{B}} \cdot \hat{\mathbf{C}})\big] \\
&= w^2\big(\cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma\big),
\end{aligned} \tag{1.56}$$