This special case of a scalar product in conjunction with general properties the scalar product is sufficient to derive the general case of the scalar product.

Just as the projection is linear in **A**, we want the scalar product of two vectors to be linear in **A** and **B**, that is, obey the distributive and associative laws

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \tag{1.23a}$$

$$\mathbf{A} \cdot (y\mathbf{B}) = (y\mathbf{A}) \cdot \mathbf{B} = y\mathbf{A} \cdot \mathbf{B}, \tag{1.23b}$$

where y is a number. Now we can use the decomposition of **B** into its Cartesian components according to Eq. (1.5), $\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}$, to construct the general scalar or dot product of the vectors **A** and **B** as

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= \mathbf{A} \cdot (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\ &= B_x\mathbf{A} \cdot \hat{\mathbf{x}} + B_y\mathbf{A} \cdot \hat{\mathbf{y}} + B_z\mathbf{A} \cdot \hat{\mathbf{z}} \qquad \text{upon applying Eqs. (1.23a) and (1.23b)} \\ &= B_xA_x + B_yA_y + B_zA_z \qquad \text{upon substituting Eq. (1.22).}\end{aligned}$$

Hence

$$\boxed{\mathbf{A} \cdot \mathbf{B} \equiv \sum_i B_iA_i = \sum_i A_iB_i = \mathbf{B} \cdot \mathbf{A}.} \tag{1.24}$$

If $\mathbf{A} = \mathbf{B}$ in Eq. (1.24), we recover the magnitude $A = (\sum A_i^2)^{1/2}$ of **A** in Eq. (1.6) from Eq. (1.24).

It is obvious from Eq. (1.24) that the scalar product treats **A** and **B** alike, or is symmetric in **A** and **B**, and is commutative. Thus, alternatively and equivalently, we can first generalize Eqs. (1.22) to the projection A_B of **A** onto the direction of a vector $\mathbf{B} \neq 0$ as $A_B = A\cos\theta \equiv \mathbf{A} \cdot \hat{\mathbf{B}}$, where $\hat{\mathbf{B}} = \mathbf{B}/B$ is the unit vector in the direction of **B** and θ is the angle between **A** and **B**, as shown in Fig. 1.7. Similarly, we project **B** onto **A** as $B_A = B\cos\theta \equiv \mathbf{B} \cdot \hat{\mathbf{A}}$. Second, we make these projections symmetric in **A** and **B**, which leads to the definition

$$\boxed{\mathbf{A} \cdot \mathbf{B} \equiv A_BB = AB_A = AB\cos\theta.} \tag{1.25}$$

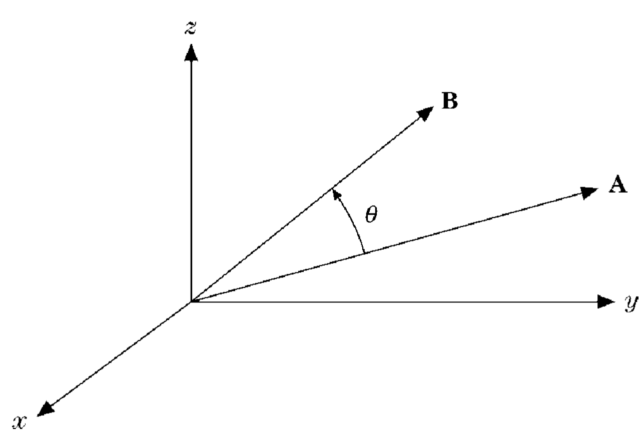

FIGURE 1.7 Scalar product $\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$.

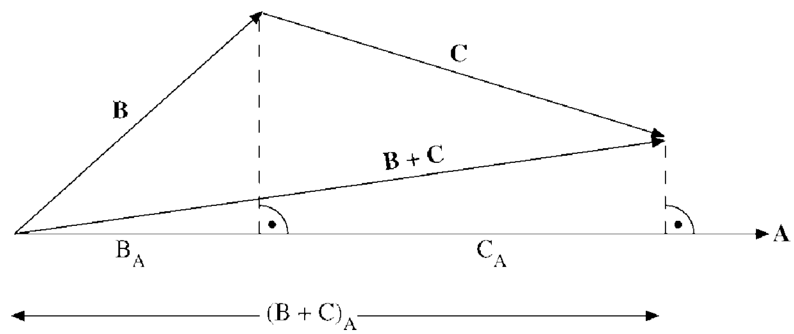

FIGURE 1.8 The distributive law
$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = AB_A + AC_A = A(\mathbf{B} + \mathbf{C})_A$, Eq. (1.23a).

The distributive law in Eq. (1.23a) is illustrated in Fig. 1.8, which shows that the sum of the projections of **B** and **C** onto **A**, $B_A + C_A$ is equal to the projection of $\mathbf{B} + \mathbf{C}$ onto **A**, $(\mathbf{B} + \mathbf{C})_A$.

It follows from Eqs. (1.22), (1.24), and (1.25) that the coordinate unit vectors satisfy the relations

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1, \tag{1.26a}$$

whereas

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = 0. \tag{1.26b}$$

If the component definition, Eq. (1.24), is labeled an algebraic definition, then Eq. (1.25) is a geometric definition. One of the most common applications of the scalar product in physics is in the calculation of **work** = **force**·**displacement**· $\cos\theta$, which is interpreted as displacement times the projection of the force along the displacement direction, i.e., the scalar product of force and displacement, $W = \mathbf{F} \cdot \mathbf{S}$.

If $\mathbf{A} \cdot \mathbf{B} = 0$ and we know that $\mathbf{A} \neq 0$ and $\mathbf{B} \neq 0$, then, from Eq. (1.25), $\cos\theta = 0$, or $\theta = 90^\circ, 270^\circ$, and so on. The vectors **A** and **B** must be perpendicular. Alternately, we may say **A** and **B** are orthogonal. The unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are mutually orthogonal. To develop this notion of orthogonality one more step, suppose that **n** is a unit vector and **r** is a nonzero vector in the xy-plane; that is, $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$ (Fig. 1.9). If

$$\mathbf{n} \cdot \mathbf{r} = 0$$

for **all** choices of **r**, then **n** must be perpendicular (orthogonal) to the xy-plane.

Often it is convenient to replace $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ by subscripted unit vectors $\mathbf{e}_m$, $m = 1, 2, 3$, with $\hat{\mathbf{x}} = \mathbf{e}_1$, and so on. Then Eqs. (1.26a) and (1.26b) become

$$\mathbf{e}_m \cdot \mathbf{e}_n = \delta_{mn}. \tag{1.26c}$$

For $m \neq n$ the unit vectors $\mathbf{e}_m$ and $\mathbf{e}_n$ are orthogonal. For $m = n$ each vector is normalized to unity, that is, has unit magnitude. The set $\mathbf{e}_m$ is said to be **orthonormal**. A major advantage of Eq. (1.26c) over Eqs. (1.26a) and (1.26b) is that Eq. (1.26c) may readily be generalized to N-dimensional space: $m, n = 1, 2, \ldots, N$. Finally, we are picking sets of unit vectors $\mathbf{e}_m$ that are orthonormal for convenience – a very great convenience.

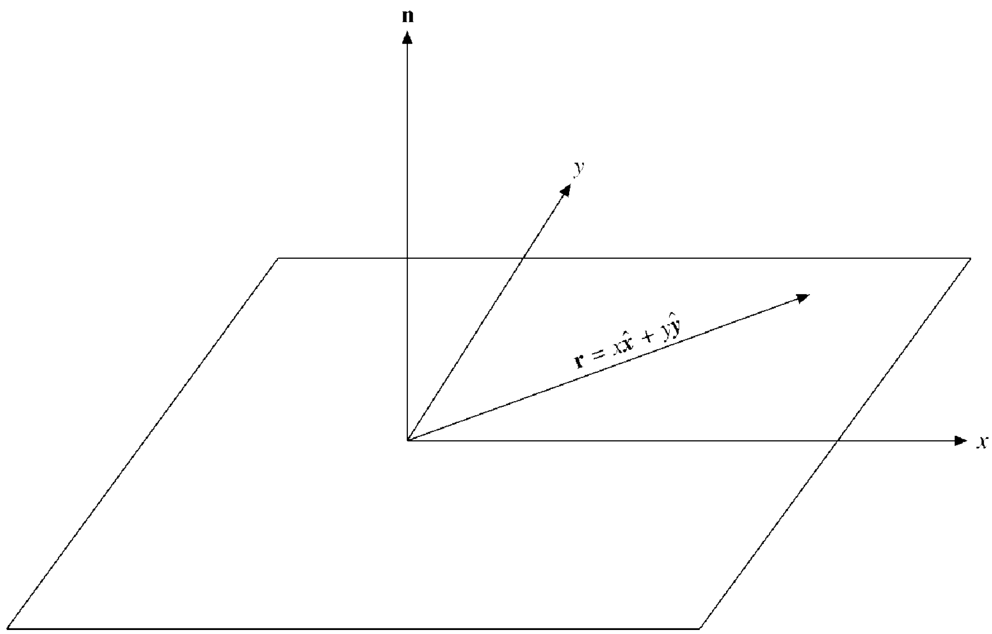

FIGURE 1.9 A normal vector.

Invariance of the Scalar Product Under Rotations

We have not yet shown that the word *scalar* is justified or that the scalar product is indeed a scalar quantity. To do this, we investigate the behavior of $\mathbf{A} \cdot \mathbf{B}$ under a rotation of the coordinate system. By use of Eq. (1.15),

$$A'_x B'_x + A'_y B'_y + A'_z B'_z = \sum_i a_{xi} A_i \sum_j a_{xj} B_j + \sum_i a_{yi} A_i \sum_j a_{yj} B_j + \sum_i a_{zi} A_i \sum_j a_{zj} B_j. \tag{1.27}$$

Using the indices k and l to sum over x, y, and z, we obtain

$$\sum_k A'_k B'_k = \sum_l \sum_i \sum_j a_{li} A_i a_{lj} B_j, \tag{1.28}$$

and, by rearranging the terms on the right-hand side, we have

$$\sum_k A'_k B'_k = \sum_l \sum_i \sum_j (a_{li} a_{lj}) A_i B_j = \sum_i \sum_j \delta_{ij} A_i B_j = \sum_i A_i B_i. \tag{1.29}$$

The last two steps follow by using Eq. (1.18), the orthogonality condition of the direction cosines, and Eqs. (1.20), which define the Kronecker delta. The effect of the Kronecker delta is to cancel all terms in a summation over either index except the term for which the indices are equal. In Eq. (1.29) its effect is to set $j = i$ and to eliminate the summation over j. Of course, we could equally well set $i = j$ and eliminate the summation over i.

Equation (1.29) gives us

$$\boxed{\sum_k A'_k B'_k = \sum_i A_i B_i,} \tag{1.30}$$

which is just our definition of a scalar quantity, one that remains invariant under the rotation of the coordinate system.

In a similar approach that exploits this concept of invariance, we take $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and dot it into itself:

$$\begin{aligned} \mathbf{C} \cdot \mathbf{C} &= (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B}) \\ &= \mathbf{A} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B} + 2\mathbf{A} \cdot \mathbf{B}. \end{aligned} \tag{1.31}$$

Since

$$\mathbf{C} \cdot \mathbf{C} = C^2, \tag{1.32}$$

the square of the magnitude of vector $\mathbf{C}$ and thus an invariant quantity, we see that

$$\mathbf{A} \cdot \mathbf{B} = \frac{1}{2}\left(C^2 - A^2 - B^2\right), \qquad \text{invariant.} \tag{1.33}$$

Since the right-hand side of Eq. (1.33) is invariant—that is, a scalar quantity—the left-hand side, $\mathbf{A} \cdot \mathbf{B}$, must also be invariant under rotation of the coordinate system. Hence $\mathbf{A} \cdot \mathbf{B}$ is a scalar.

Equation (1.31) is really another form of the law of cosines, which is

$$C^2 = A^2 + B^2 + 2AB\cos\theta. \tag{1.34}$$

Comparing Eqs. (1.31) and (1.34), we have another verification of Eq. (1.25), or, if preferred, a vector derivation of the law of cosines (Fig. 1.10).

The dot product, given by Eq. (1.24), may be generalized in two ways. The space need not be restricted to three dimensions. In n-dimensional space, Eq. (1.24) applies with the sum running from 1 to n. Moreover, n may be infinity, with the sum then a convergent infinite series (Section 5.2). The other generalization extends the concept of vector to embrace functions. The function analog of a dot, or inner, product appears in Section 10.4.

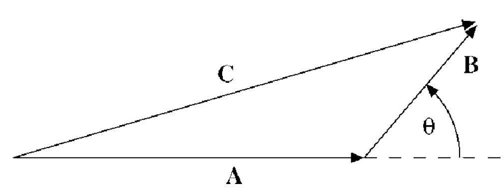

FIGURE 1.10 The law of cosines.

Exercises

1.3.1 Two unit magnitude vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ are required to be either parallel or perpendicular to each other. Show that $\mathbf{e}_i \cdot \mathbf{e}_j$ provides an interpretation of Eq. (1.18), the direction cosine orthogonality relation.

1.3.2 Given that (1) the dot product of a unit vector with itself is unity and (2) this relation is valid in all (rotated) coordinate systems, show that $\hat{\mathbf{x}}' \cdot \hat{\mathbf{x}}' = 1$ (with the primed system rotated 45° about the z-axis relative to the unprimed) implies that $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0$.

1.3.3 The vector $\mathbf{r}$, starting at the origin, terminates at and specifies the point in space (x, y, z). Find the surface swept out by the tip of $\mathbf{r}$ if

(a) $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{a} = 0$. Characterize $\mathbf{a}$ geometrically.
(b) $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{r} = 0$. Describe the geometric role of $\mathbf{a}$.
The vector $\mathbf{a}$ is constant (in magnitude and direction).

1.3.4 The interaction energy between two dipoles of moments $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ may be written in the vector form

$$V = -\frac{\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2}{r^3} + \frac{3(\boldsymbol{\mu}_1 \cdot \mathbf{r})(\boldsymbol{\mu}_2 \cdot \mathbf{r})}{r^5}$$

and in the scalar form

$$V = \frac{\mu_1 \mu_2}{r^3}(2\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos\varphi).$$

Here θ_1 and θ_2 are the angles of $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ relative to $\mathbf{r}$, while φ is the azimuth of $\boldsymbol{\mu}_2$ relative to the $\boldsymbol{\mu}_1$–$\mathbf{r}$ plane (Fig. 1.11). Show that these two forms are equivalent.
Hint: Equation (12.178) will be helpful.

1.3.5 A pipe comes diagonally down the south wall of a building, making an angle of 45° with the horizontal. Coming into a corner, the pipe turns and continues diagonally down a west-facing wall, still making an angle of 45° with the horizontal. What is the angle between the south-wall and west-wall sections of the pipe?

ANS. 120°.

1.3.6 Find the shortest distance of an observer at the point $(2, 1, 3)$ from a rocket in free flight with velocity $(1, 2, 3)$ m/s. The rocket was launched at time $t = 0$ from $(1, 1, 1)$. Lengths are in kilometers.

1.3.7 Prove the law of cosines from the triangle with corners at the point of $\mathbf{C}$ and $\mathbf{A}$ in Fig. 1.10 and the projection of vector $\mathbf{B}$ onto vector $\mathbf{A}$.

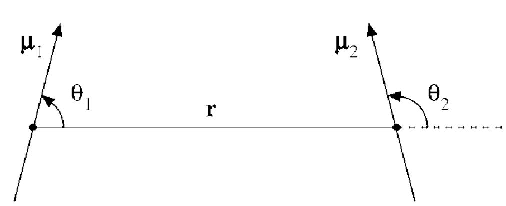

FIGURE 1.11 Two dipole moments.

1.4 VECTOR OR CROSS PRODUCT

A second form of vector multiplication employs the sine of the included angle instead of the cosine. For instance, the angular momentum of a body shown at the point of the distance vector in Fig. 1.12 is defined as

$$\begin{aligned}\text{angular momentum} &= \text{radius arm} \times \text{linear momentum}\\ &= \text{distance} \times \text{linear momentum} \times \sin\theta.\end{aligned}$$

For convenience in treating problems relating to quantities such as angular momentum, torque, and angular velocity, we define the vector product, or cross product, as

$$\boxed{\mathbf{C} = \mathbf{A} \times \mathbf{B}, \qquad \text{with } C = AB\sin\theta.} \tag{1.35}$$

Unlike the preceding case of the scalar product, $\mathbf{C}$ is now a vector, and we assign it a direction perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ form a right-handed system. With this choice of direction we have

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}, \qquad \text{anticommutation.} \tag{1.36a}$$

From this definition of cross product we have

$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = \mathbf{0}, \tag{1.36b}$$

whereas

$$\begin{aligned}&\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \qquad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \qquad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}},\\ &\hat{\mathbf{y}} \times \hat{\mathbf{x}} = -\hat{\mathbf{z}}, \qquad \hat{\mathbf{z}} \times \hat{\mathbf{y}} = -\hat{\mathbf{x}}, \qquad \hat{\mathbf{x}} \times \hat{\mathbf{z}} = -\hat{\mathbf{y}}.\end{aligned} \tag{1.36c}$$

Among the examples of the cross product in mathematical physics are the relation between linear momentum $\mathbf{p}$ and angular momentum $\mathbf{L}$, with $\mathbf{L}$ defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

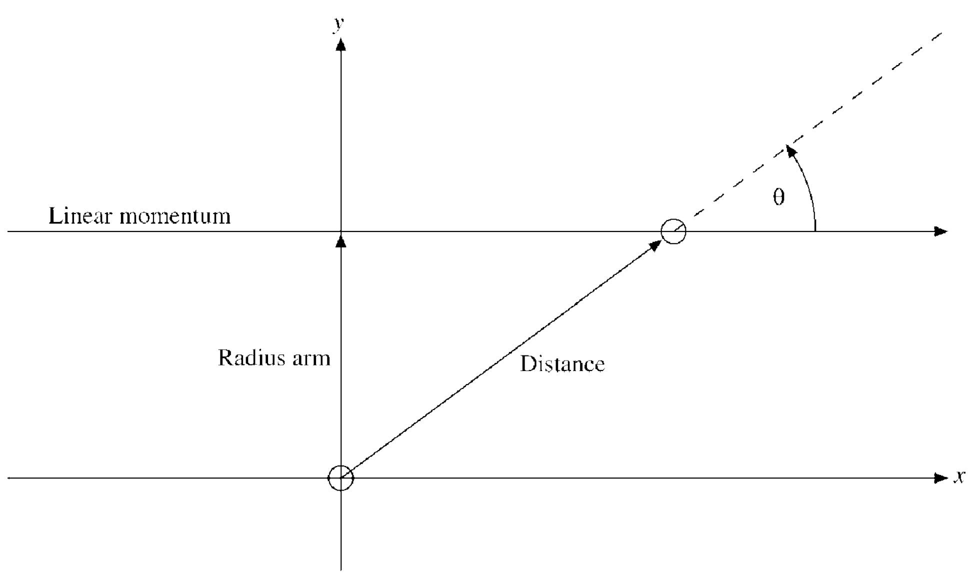

FIGURE 1.12 Angular momentum.

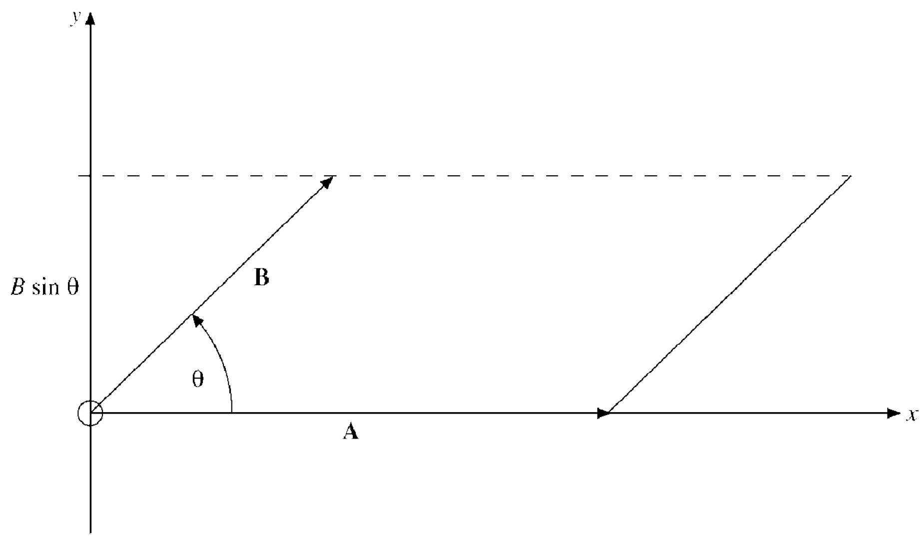

FIGURE 1.13 Parallelogram representation of the vector product.

and the relation between linear velocity **v** and angular velocity $\boldsymbol{\omega}$,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}.$$

Vectors **v** and **p** describe properties of the particle or physical system. However, the position vector **r** is determined by the choice of the origin of the coordinates. This means that $\boldsymbol{\omega}$ and **L** depend on the choice of the origin.

The familiar magnetic induction **B** is usually defined by the vector product force equation[8]

$$\mathbf{F}_M = q\mathbf{v} \times \mathbf{B} \text{ (mks units).}$$

Here **v** is the velocity of the electric charge q and $\mathbf{F}_M$ is the resulting force on the moving charge.

The cross product has an important geometrical interpretation, which we shall use in subsequent sections. In the parallelogram defined by **A** and **B** (Fig. 1.13), $B\sin\theta$ is the height if A is taken as the length of the base. Then $|\mathbf{A} \times \mathbf{B}| = AB\sin\theta$ is the **area** of the parallelogram. As a vector, $\mathbf{A} \times \mathbf{B}$ is the area of the parallelogram defined by **A** and **B**, with the area vector normal to the plane of the parallelogram. This suggests that area (with its orientation in space) may be treated as a vector quantity.

An alternate definition of the vector product can be derived from the special case of the coordinate unit vectors in Eqs. (1.36c) in conjunction with the linearity of the cross product in both vector arguments, in analogy with Eqs. (1.23) for the dot product,

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}, \tag{1.37a}$$

$$(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C}, \tag{1.37b}$$

$$\mathbf{A} \times (y\mathbf{B}) = y\mathbf{A} \times \mathbf{B} = (y\mathbf{A}) \times \mathbf{B}, \tag{1.37c}$$

[8]The electric field **E** is assumed here to be zero.

where y is a number again. Using the decomposition of $\mathbf{A}$ and $\mathbf{B}$ into their Cartesian components according to Eq. (1.5), we find

$$\boxed{\begin{aligned}\mathbf{A}\times\mathbf{B} \equiv \mathbf{C} = (C_x, C_y, C_z) &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\ &= (A_xB_y - A_yB_x)\hat{\mathbf{x}}\times\hat{\mathbf{y}} + (A_xB_z - A_zB_x)\hat{\mathbf{x}}\times\hat{\mathbf{z}} \\ &\quad + (A_yB_z - A_zB_y)\hat{\mathbf{y}}\times\hat{\mathbf{z}}\end{aligned}}$$

upon applying Eqs. (1.37a) and (1.37b) and substituting Eqs. (1.36a), (1.36b), and (1.36c) so that the Cartesian components of $\mathbf{A}\times\mathbf{B}$ become

$$C_x = A_yB_z - A_zB_y, \qquad C_y = A_zB_x - A_xB_z, \qquad C_z = A_xB_y - A_yB_x, \tag{1.38}$$

or

$$C_i = A_jB_k - A_kB_j, \qquad i, j, k \text{ all different}, \tag{1.39}$$

and with cyclic permutation of the indices i, j, and k corresponding to x, y, and z, respectively. The vector product $\mathbf{C}$ may be mnemonically represented by a determinant,[9]

$$\mathbf{C} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \equiv \hat{\mathbf{x}}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{\mathbf{y}}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{\mathbf{z}}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}, \tag{1.40}$$

which is meant to be expanded across the top row to reproduce the three components of $\mathbf{C}$ listed in Eqs. (1.38).

Equation (1.35) might be called a geometric definition of the vector product. Then Eqs. (1.38) would be an algebraic definition.

To show the equivalence of Eq. (1.35) and the component definition, Eqs. (1.38), let us form $\mathbf{A}\cdot\mathbf{C}$ and $\mathbf{B}\cdot\mathbf{C}$, using Eqs. (1.38). We have

$$\begin{aligned}\mathbf{A}\cdot\mathbf{C} &= \mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) \\ &= A_x(A_yB_z - A_zB_y) + A_y(A_zB_x - A_xB_z) + A_z(A_xB_y - A_yB_x) \\ &= 0. \end{aligned}\tag{1.41}$$

Similarly,

$$\mathbf{B}\cdot\mathbf{C} = \mathbf{B}\cdot(\mathbf{A}\times\mathbf{B}) = 0. \tag{1.42}$$

Equations (1.41) and (1.42) show that $\mathbf{C}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$ ($\cos\theta = 0, \theta = \pm 90°$) and therefore perpendicular to the plane they determine. The positive direction is determined by considering special cases, such as the unit vectors $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ ($C_z = +A_xB_y$).

The magnitude is obtained from

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{B}) &= A^2B^2 - (\mathbf{A}\cdot\mathbf{B})^2 \\ &= A^2B^2 - A^2B^2\cos^2\theta \\ &= A^2B^2\sin^2\theta. \end{aligned}\tag{1.43}$$

[9] See Section 3.1 for a brief summary of determinants.

Hence

$$C = AB\sin\theta. \tag{1.44}$$

The first step in Eq. (1.43) may be verified by expanding out in component form, using Eqs. (1.38) for $\mathbf{A} \times \mathbf{B}$ and Eq. (1.24) for the dot product. From Eqs. (1.41), (1.42), and (1.44) we see the equivalence of Eqs. (1.35) and (1.38), the two definitions of vector product.

There still remains the problem of verifying that $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is indeed a vector, that is, that it obeys Eq. (1.15), the vector transformation law. Starting in a rotated (primed system),

$$\begin{aligned} C_i' &= A_j' B_k' - A_k' B_j', \qquad i, j, \text{ and } k \text{ in cyclic order,} \\ &= \sum_l a_{jl} A_l \sum_m a_{km} B_m - \sum_l a_{kl} A_l \sum_m a_{jm} B_m \\ &= \sum_{l,m} (a_{jl} a_{km} - a_{kl} a_{jm}) A_l B_m. \end{aligned} \tag{1.45}$$

The combination of direction cosines in parentheses vanishes for $m = l$. We therefore have j and k taking on fixed values, dependent on the choice of i, and six combinations of l and m. If $i = 3$, then $j = 1, k = 2$ (cyclic order), and we have the following direction cosine combinations:[10]

$$\begin{aligned} a_{11}a_{22} - a_{21}a_{12} &= a_{33}, \\ a_{13}a_{21} - a_{23}a_{11} &= a_{32}, \\ a_{12}a_{23} - a_{22}a_{13} &= a_{31} \end{aligned} \tag{1.46}$$

and their negatives. Equations (1.46) are identities satisfied by the direction cosines. They may be verified with the use of determinants and matrices (see Exercise 3.3.3). Substituting back into Eq. (1.45),

$$\begin{aligned} C_3' &= a_{33}A_1B_2 + a_{32}A_3B_1 + a_{31}A_2B_3 - a_{33}A_2B_1 - a_{32}A_1B_3 - a_{31}A_3B_2 \\ &= a_{31}C_1 + a_{32}C_2 + a_{33}C_3 \\ &= \sum_n a_{3n}C_n. \end{aligned} \tag{1.47}$$

By permuting indices to pick up C_1' and C_2', we see that Eq. (1.15) is satisfied and $\mathbf{C}$ is indeed a vector. It should be mentioned here that this **vector nature** of the **cross product** is an accident associated with the **three-dimensional** nature of ordinary space.[11] It will be seen in Chapter 2 that the cross product may also be treated as a second-rank antisymmetric tensor.

[10]Equations (1.46) hold for rotations because they preserve volumes. For a more general orthogonal transformation, the r.h.s. of Eqs. (1.46) is multiplied by the determinant of the transformation matrix (see Chapter 3 for matrices and determinants).

[11]Specifically Eqs. (1.46) hold only for three-dimensional space. See D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus* (Dordrecht: Reidel, 1984) for a far-reaching generalization of the cross product.

If we define a vector as an ordered triplet of numbers (or functions), as in the latter part of Section 1.2, then there is no problem identifying the cross product as a vector. The cross-product operation maps the two triples **A** and **B** into a third triple, **C**, which by definition is a vector.

We now have two ways of multiplying vectors; a third form appears in Chapter 2. But what about division by a vector? It turns out that the ratio $\mathbf{B}/\mathbf{A}$ is not uniquely specified (Exercise 3.2.21) unless **A** and **B** are also required to be parallel. Hence division of one vector by another is not defined.

Exercises

1.4.1 Show that the medians of a triangle intersect in the center, which is $2/3$ of the median's length from each corner. Construct a numerical example and plot it.

1.4.2 Prove the law of cosines starting from $\mathbf{A}^2 = (\mathbf{B} - \mathbf{C})^2$.

1.4.3 Starting with $\mathbf{C} = \mathbf{A} + \mathbf{B}$, show that $\mathbf{C} \times \mathbf{C} = 0$ leads to

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}.$$

1.4.4 Show that

(a) $(\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B}) = A^2 - B^2$,
(b) $(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} + \mathbf{B}) = 2\mathbf{A} \times \mathbf{B}$.

The distributive laws needed here,

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C},$$

and

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C},$$

may easily be verified (if desired) by expansion in Cartesian components.

1.4.5 Given the three vectors,

$$\mathbf{P} = 3\hat{\mathbf{x}} + 2\hat{\mathbf{y}} - \hat{\mathbf{z}},$$

$$\mathbf{Q} = -6\hat{\mathbf{x}} - 4\hat{\mathbf{y}} + 2\hat{\mathbf{z}},$$

$$\mathbf{R} = \hat{\mathbf{x}} - 2\hat{\mathbf{y}} - \hat{\mathbf{z}},$$

find two that are perpendicular and two that are parallel or antiparallel.

1.4.6 If $\mathbf{P} = \hat{\mathbf{x}}P_x + \hat{\mathbf{y}}P_y$ and $\mathbf{Q} = \hat{\mathbf{x}}Q_x + \hat{\mathbf{y}}Q_y$ are any two nonparallel (also nonantiparallel) vectors in the xy-plane, show that $\mathbf{P} \times \mathbf{Q}$ is in the z-direction.

1.4.7 Prove that $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) = (AB)^2 - (\mathbf{A} \cdot \mathbf{B})^2$.

1.4.8 Using the vectors

$$\mathbf{P} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta,$$
$$\mathbf{Q} = \hat{\mathbf{x}}\cos\varphi - \hat{\mathbf{y}}\sin\varphi,$$
$$\mathbf{R} = \hat{\mathbf{x}}\cos\varphi + \hat{\mathbf{y}}\sin\varphi,$$

prove the familiar trigonometric identities

$$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi,$$
$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi.$$

1.4.9 (a) Find a vector **A** that is perpendicular to

$$\mathbf{U} = 2\hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}},$$
$$\mathbf{V} = \hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}.$$

(b) What is **A** if, in addition to this requirement, we demand that it have unit magnitude?

1.4.10 If four vectors **a**, **b**, **c**, and **d** all lie in the same plane, show that

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = 0.$$

Hint. Consider the directions of the cross-product vectors.

1.4.11 The coordinates of the three vertices of a triangle are $(2, 1, 5)$, $(5, 2, 8)$, and $(4, 8, 2)$. Compute its area by vector methods, its center and medians. Lengths are in centimeters.
Hint. See Exercise 1.4.1.

1.4.12 The vertices of parallelogram $ABCD$ are $(1, 0, 0)$, $(2, -1, 0)$, $(0, -1, 1)$, and $(-1, 0, 1)$ in order. Calculate the vector areas of triangle ABD and of triangle BCD. Are the two vector areas equal?

ANS. $\text{Area}_{ABD} = -\frac{1}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}} + 2\hat{\mathbf{z}})$.

1.4.13 The origin and the three vectors **A**, **B**, and **C** (all of which start at the origin) define a tetrahedron. Taking the outward direction as positive, calculate the total vector area of the four tetrahedral surfaces.
Note. In Section 1.11 this result is generalized to any closed surface.

1.4.14 Find the sides and angles of the spherical triangle ABC defined by the three vectors

$$\mathbf{A} = (1, 0, 0),$$
$$\mathbf{B} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),$$
$$\mathbf{C} = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Each vector starts from the origin (Fig. 1.14).