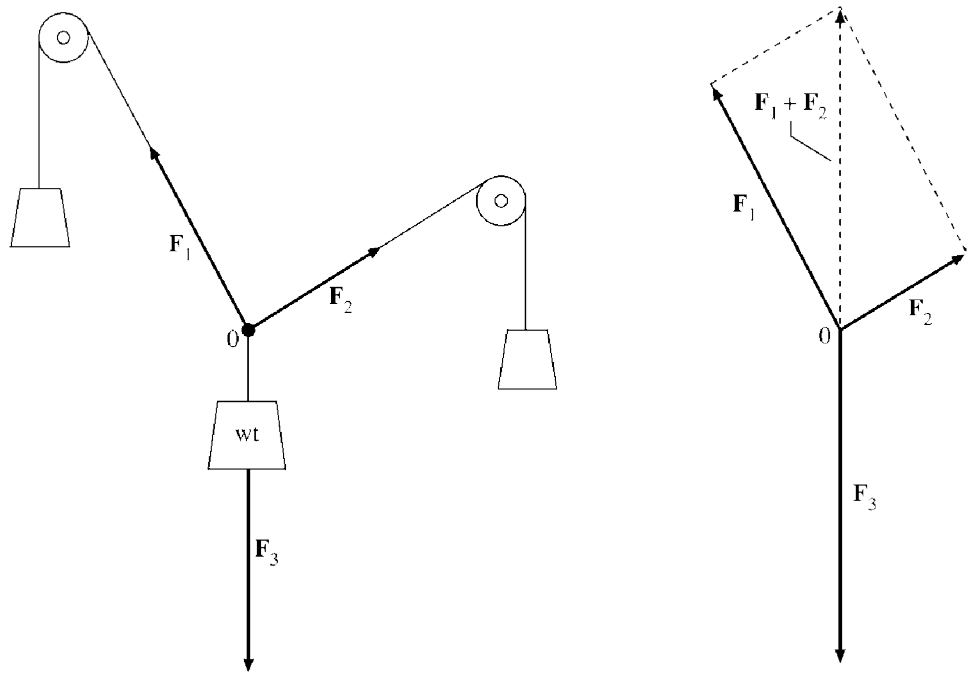

FIGURE 1.4 Equilibrium of forces: $\mathbf{F}_1 + \mathbf{F}_2 = -\mathbf{F}_3$.

sum of the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ must just cancel the downward force of gravity, $\mathbf{F}_3$. Here the parallelogram addition law is subject to immediate experimental verification.[1]

Subtraction may be handled by defining the negative of a vector as a vector of the same magnitude but with reversed direction. Then

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}).$$

In Fig. 1.3,

$$\mathbf{A} = \mathbf{E} - \mathbf{B}.$$

Note that the vectors are treated as geometrical objects that are independent of any coordinate system. This concept of independence of a preferred coordinate system is developed in detail in the next section.

The representation of vector $\mathbf{A}$ by an arrow suggests a second possibility. Arrow $\mathbf{A}$ (Fig. 1.5), starting from the origin,[2] terminates at the point (A_x, A_y, A_z). Thus, if we agree that the vector is to start at the origin, the positive end may be specified by giving the Cartesian coordinates (A_x, A_y, A_z) of the arrowhead.

Although $\mathbf{A}$ could have represented any vector quantity (momentum, electric field, etc.), one particularly important vector quantity, the displacement from the origin to the point

[1]Strictly speaking, the parallelogram addition was introduced as a definition. Experiments show that if we assume that the forces are vector quantities and we combine them by parallelogram addition, the equilibrium condition of zero resultant force is satisfied.

[2]We could start from any point in our Cartesian reference frame; we choose the origin for simplicity. This freedom of shifting the origin of the coordinate system without affecting the geometry is called **translation invariance**.

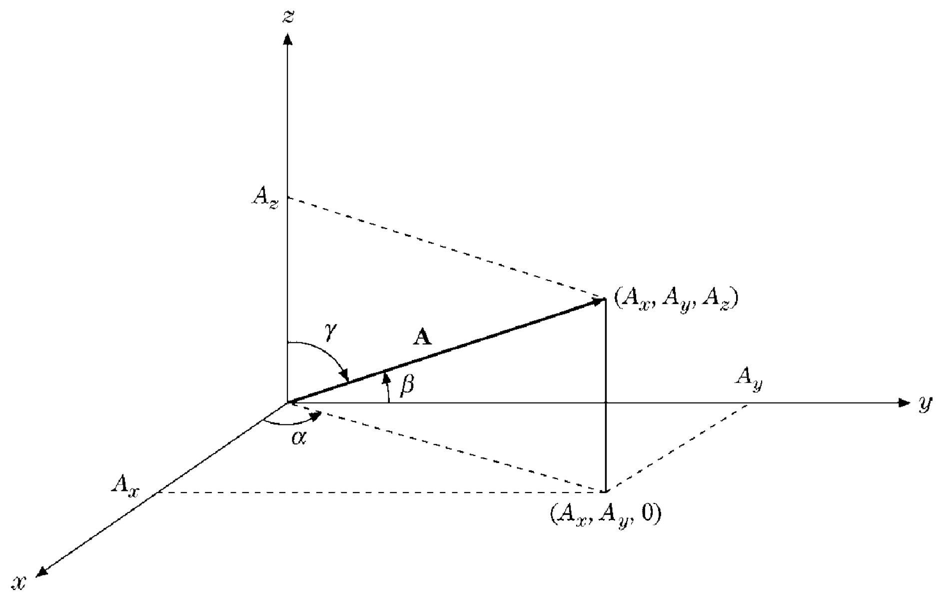

FIGURE 1.5 Cartesian components and direction cosines of **A**.

(x, y, z), is denoted by the special symbol **r**. We then have a choice of referring to the displacement as either the vector **r** or the collection (x, y, z), the coordinates of its endpoint:

$$\mathbf{r} \leftrightarrow (x, y, z). \tag{1.3}$$

Using r for the magnitude of vector **r**, we find that Fig. 1.5 shows that the endpoint coordinates and the magnitude are related by

$$x = r\cos\alpha, \qquad y = r\cos\beta, \qquad z = r\cos\gamma. \tag{1.4}$$

Here $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the **direction cosines**, α being the angle between the given vector and the positive x-axis, and so on. One further bit of vocabulary: The quantities A_x, A_y, and A_z are known as the (Cartesian) **components** of **A** or the **projections** of **A**, with $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Thus, any vector **A** may be resolved into its components (or projected onto the coordinate axes) to yield $A_x = A\cos\alpha$, etc., as in Eq. (1.4). We may choose to refer to the vector as a single quantity **A** or to its components (A_x, A_y, A_z). Note that the subscript x in A_x denotes the x component and not a dependence on the variable x. The choice between using **A** or its components (A_x, A_y, A_z) is essentially a choice between a geometric and an algebraic representation. Use either representation at your convenience. The geometric "arrow in space" may aid in visualization. The algebraic set of components is usually more suitable for precise numerical or algebraic calculations.

Vectors enter physics in two distinct forms. (1) Vector **A** may represent a single force acting at a single point. The force of gravity acting at the center of gravity illustrates this form. (2) Vector **A** may be defined over some extended region; that is, **A** and its components may be functions of position: $A_x = A_x(x, y, z)$, and so on. Examples of this sort include the velocity of a fluid varying from point to point over a given volume and electric and magnetic fields. These two cases may be distinguished by referring to the vector defined over a region as a **vector field**. The concept of the vector defined over a region and being a function of position will become extremely important when we differentiate and integrate vectors.

At this stage it is convenient to introduce unit vectors along each of the coordinate axes. Let $\hat{\mathbf{x}}$ be a vector of unit magnitude pointing in the positive x-direction, $\hat{\mathbf{y}}$, a vector of unit magnitude in the positive y-direction, and $\hat{\mathbf{z}}$ a vector of unit magnitude in the positive z-direction. Then $\hat{\mathbf{x}}A_x$ is a vector with magnitude equal to $|A_x|$ and in the x-direction. By vector addition,

$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z. \tag{1.5}$$

Note that if $\mathbf{A}$ vanishes, all of its components must vanish individually; that is, if

$$\mathbf{A} = 0, \qquad \text{then } A_x = A_y = A_z = 0.$$

This means that these unit vectors serve as a **basis**, or complete set of vectors, in the three-dimensional Euclidean space in terms of which any vector can be expanded. Thus, Eq. (1.5) is an assertion that the three unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ span our real three-dimensional space: Any vector may be written as a linear combination of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. Since $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are linearly independent (no one is a linear combination of the other two), they form a **basis** for the real three-dimensional Euclidean space. Finally, by the Pythagorean theorem, the magnitude of vector $\mathbf{A}$ is

$$\boxed{|\mathbf{A}| = \left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}.} \tag{1.6}$$

Note that the coordinate unit vectors are not the only complete set, or basis. This resolution of a vector into its components can be carried out in a variety of coordinate systems, as shown in Chapter 2. Here we restrict ourselves to Cartesian coordinates, where the unit vectors have the coordinates $\hat{\mathbf{x}} = (1, 0, 0)$, $\hat{\mathbf{y}} = (0, 1, 0)$ and $\hat{\mathbf{z}} = (0, 0, 1)$ and are all constant in length and direction, properties characteristic of Cartesian coordinates.

As a replacement of the graphical technique, addition and subtraction of vectors may now be carried out in terms of their components. For $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$ and $\mathbf{B} = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z$,

$$\mathbf{A} \pm \mathbf{B} = \hat{\mathbf{x}}(A_x \pm B_x) + \hat{\mathbf{y}}(A_y \pm B_y) + \hat{\mathbf{z}}(A_z \pm B_z). \tag{1.7}$$

It should be emphasized here that the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are used for convenience. They are not essential; we can describe vectors and use them entirely in terms of their components: $\mathbf{A} \leftrightarrow (A_x, A_y, A_z)$. This is the approach of the two more powerful, more sophisticated definitions of vector to be discussed in the next section. However, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ emphasize the **direction**.

So far we have defined the operations of addition and subtraction of vectors. In the next sections, three varieties of multiplication will be defined on the basis of their applicability: a scalar, or inner, product, a vector product peculiar to three-dimensional space, and a direct, or outer, product yielding a second-rank tensor. Division by a vector is not defined.

Exercises

1.1.1 Show how to find $\mathbf{A}$ and $\mathbf{B}$, given $\mathbf{A}+\mathbf{B}$ and $\mathbf{A}-\mathbf{B}$.

1.1.2 The vector $\mathbf{A}$ whose magnitude is 1.732 units makes equal angles with the coordinate axes. Find A_x, A_y, and A_z.

1.1.3 Calculate the components of a unit vector that lies in the xy-plane and makes equal angles with the positive directions of the x- and y-axes.

1.1.4 The velocity of sailboat A relative to sailboat B, $\mathbf{v}_{\text{rel}}$, is defined by the equation $\mathbf{v}_{\text{rel}} = \mathbf{v}_A - \mathbf{v}_B$, where $\mathbf{v}_A$ is the velocity of A and $\mathbf{v}_B$ is the velocity of B. Determine the velocity of A relative to B if

$$\mathbf{v}_A = 30 \text{ km/hr east}$$
$$\mathbf{v}_B = 40 \text{ km/hr north.}$$

ANS. $\mathbf{v}_{\text{rel}} = 50$ km/hr, 53.1° south of east.

1.1.5 A sailboat sails for 1 hr at 4 km/hr (relative to the water) on a steady compass heading of 40° east of north. The sailboat is simultaneously carried along by a current. At the end of the hour the boat is 6.12 km from its starting point. The line from its starting point to its location lies 60° east of north. Find the x (easterly) and y (northerly) components of the water's velocity.

ANS. $v_{\text{east}} = 2.73$ km/hr, $v_{\text{north}} \approx 0$ km/hr.

1.1.6 A vector equation can be reduced to the form $\mathbf{A} = \mathbf{B}$. From this show that the one vector equation is equivalent to **three** scalar equations. Assuming the validity of Newton's second law, $\mathbf{F} = m\mathbf{a}$, as a **vector** equation, this means that a_x depends only on F_x and is independent of F_y and F_z.

1.1.7 The vertices A, B, and C of a triangle are given by the points $(-1,0,2)$, $(0,1,0)$, and $(1,-1,0)$, respectively. Find point D so that the figure $ABCD$ forms a plane parallelogram.

ANS. $(0,-2,2)$ or $(2,0,-2)$.

1.1.8 A triangle is defined by the vertices of three vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ that extend from the origin. In terms of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ show that the **vector** sum of the successive sides of the triangle $(AB + BC + CA)$ is zero, where the side AB is from A to B, etc.

1.1.9 A sphere of radius a is centered at a point $\mathbf{r}_1$.

(a) Write out the algebraic equation for the sphere.
(b) Write out a **vector** equation for the sphere.

ANS. (a) $(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 = a^2$.
(b) $\mathbf{r} = \mathbf{r}_1 + \mathbf{a}$, with $\mathbf{r}_1 =$ center.
($\mathbf{a}$ takes on all directions but has a fixed magnitude a.)

1.1.10 A corner reflector is formed by three mutually perpendicular reflecting surfaces. Show that a ray of light incident upon the corner reflector (striking all three surfaces) is reflected back along a line parallel to the line of incidence.
Hint. Consider the effect of a reflection on the components of a vector describing the direction of the light ray.

1.1.11 *Hubble's law*. Hubble found that distant galaxies are receding with a velocity proportional to their distance from where we are on Earth. For the ith galaxy,

$$\mathbf{v}_i = H_0 \mathbf{r}_i,$$

with us at the origin. Show that this recession of the galaxies from us does **not** imply that we are at the center of the universe. Specifically, take the galaxy at $\mathbf{r}_1$ as a new origin and show that Hubble's law is still obeyed.

1.1.12 Find the diagonal vectors of a unit cube with one corner at the origin and its three sides lying along Cartesian coordinates axes. Show that there are four diagonals with length $\sqrt{3}$. Representing these as vectors, what are their components? Show that the diagonals of the cube's faces have length $\sqrt{2}$ and determine their components.

1.2 Rotation of the Coordinate Axes[3]

In the preceding section vectors were defined or represented in two equivalent ways: (1) geometrically by specifying magnitude and direction, as with an arrow, and (2) algebraically by specifying the components relative to Cartesian coordinate axes. The second definition is adequate for the vector analysis of this chapter. In this section two more refined, sophisticated, and powerful definitions are presented. First, the vector field is defined in terms of the behavior of its components under rotation of the coordinate axes. This transformation theory approach leads into the tensor analysis of Chapter 2 and groups of transformations in Chapter 4. Second, the component definition of Section 1.1 is refined and generalized according to the mathematician's concepts of vector and vector space. This approach leads to function spaces, including the Hilbert space.

The definition of vector as a quantity with magnitude and direction is incomplete. On the one hand, we encounter quantities, such as elastic constants and index of refraction in anisotropic crystals, that have magnitude and direction **but** that are not vectors. On the other hand, our naïve approach is awkward to generalize to extend to more complex quantities. We seek a new definition of vector field using our coordinate vector $\mathbf{r}$ as a prototype.

There is a physical basis for our development of a new definition. We describe our physical world by mathematics, but it and any physical predictions we may make must be **independent** of our mathematical conventions.

In our specific case we assume that space is isotropic; that is, there is no preferred direction, or all directions are equivalent. Then the physical system being analyzed or the physical law being enunciated cannot and must not depend on our choice or **orientation** of the coordinate axes. Specifically, if a quantity S does not depend on the orientation of the coordinate axes, it is called a scalar.

[3] This section is optional here. It will be essential for Chapter 2.

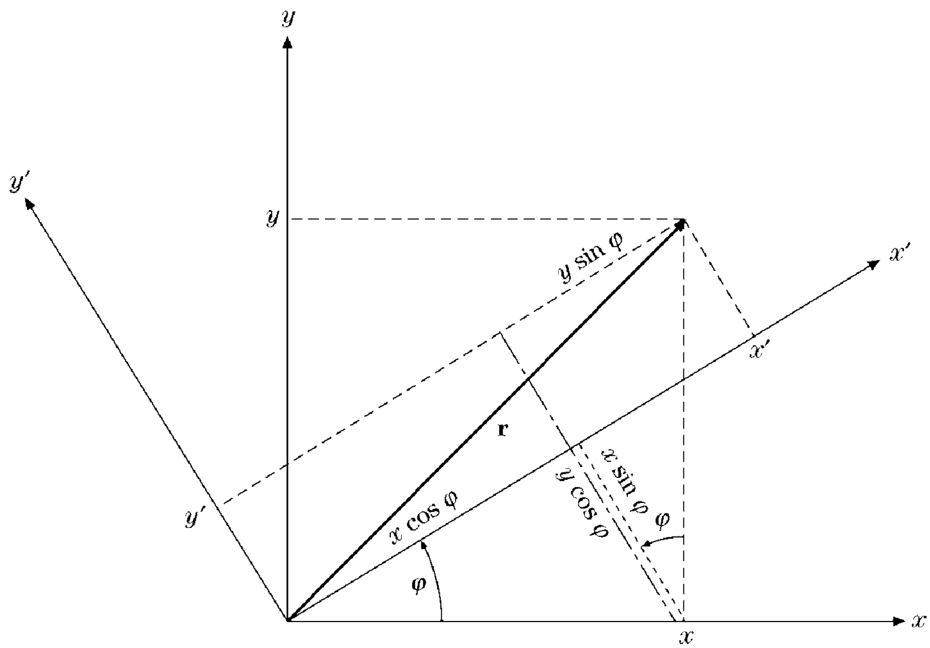

FIGURE 1.6 Rotation of Cartesian coordinate axes about the z-axis.

Now we return to the concept of vector $\mathbf{r}$ as a geometric object independent of the coordinate system. Let us look at $\mathbf{r}$ in two different systems, one rotated in relation to the other.

For simplicity we consider first the two-dimensional case. If the x-, y-coordinates are rotated counterclockwise through an angle φ, **keeping r, fixed** (Fig. 1.6), we get the following relations between the components resolved in the original system (unprimed) and those resolved in the new rotated system (primed):

$$\begin{aligned} x' &= x\cos\varphi + y\sin\varphi, \\ y' &= -x\sin\varphi + y\cos\varphi. \end{aligned} \tag{1.8}$$

We saw in Section 1.1 that a vector could be represented by the coordinates of a point; that is, the coordinates were proportional to the vector components. Hence the components of a vector must transform under rotation as coordinates of a point (such as $\mathbf{r}$). Therefore whenever any pair of quantities A_x and A_y in the xy-coordinate system is transformed into (A'_x, A'_y) by this rotation of the coordinate system with

$$\begin{aligned} A'_x &= A_x\cos\varphi + A_y\sin\varphi, \\ A'_y &= -A_x\sin\varphi + A_y\cos\varphi, \end{aligned} \tag{1.9}$$

we **define**[4] A_x and A_y as the components of a vector $\mathbf{A}$. Our vector now is defined in terms of the transformation of its components under rotation of the coordinate system. If A_x and A_y transform in the same way as x and y, the components of the general two-dimensional coordinate vector $\mathbf{r}$, they are the components of a vector $\mathbf{A}$. If A_x and A_y do not show this

[4] A scalar quantity does not depend on the orientation of coordinates; $S' = S$ expresses the fact that it is invariant under rotation of the coordinates.

form invariance (also called **covariance**) when the coordinates are rotated, they do not form a vector.

The vector field components A_x and A_y satisfying the defining equations, Eqs. (1.9), associate a magnitude A and a direction with each point in space. The magnitude is a scalar quantity, invariant to the rotation of the coordinate system. The direction (relative to the unprimed system) is likewise invariant to the rotation of the coordinate system (see Exercise 1.2.1). The result of all this is that the components of a vector may vary according to the rotation of the primed coordinate system. This is what Eqs. (1.9) say. But the variation with the angle is just such that the components in the rotated coordinate system A'_x and A'_y define a vector with the same magnitude and the same direction as the vector defined by the components A_x and A_y relative to the x-, y-coordinate axes. (Compare Exercise 1.2.1.) The components of **A** in a particular coordinate system constitute the **representation** of **A** in that coordinate system. Equations (1.9), the transformation relations, are a guarantee that the entity **A** is independent of the rotation of the coordinate system.

To go on to three and, later, four dimensions, we find it convenient to use a more compact notation. Let

$$\begin{aligned} x &\to x_1 \\ y &\to x_2 \end{aligned} \tag{1.10}$$

$$\begin{aligned} a_{11} &= \cos\varphi, & a_{12} &= \sin\varphi, \\ a_{21} &= -\sin\varphi, & a_{22} &= \cos\varphi. \end{aligned} \tag{1.11}$$

Then Eqs. (1.8) become

$$\begin{aligned} x'_1 &= a_{11}x_1 + a_{12}x_2, \\ x'_2 &= a_{21}x_1 + a_{22}x_2. \end{aligned} \tag{1.12}$$

The coefficient a_{ij} may be interpreted as a direction cosine, the cosine of the angle between x'_i and x_j; that is,

$$\begin{aligned} a_{12} &= \cos(x'_1, x_2) = \sin\varphi, \\ a_{21} &= \cos(x'_2, x_1) = \cos\left(\varphi + \tfrac{\pi}{2}\right) = -\sin\varphi. \end{aligned} \tag{1.13}$$

The advantage of the new notation[5] is that it permits us to use the summation symbol $\sum$ and to rewrite Eqs. (1.12) as

$$x'_i = \sum_{j=1}^{2} a_{ij}x_j, \qquad i = 1, 2. \tag{1.14}$$

Note that i remains as a parameter that gives rise to one equation when it is set equal to 1 and to a second equation when it is set equal to 2. The index j, of course, is a summation index, a dummy index, and, as with a variable of integration, j may be replaced by any other convenient symbol.

[5] You may wonder at the replacement of one parameter φ by four parameters a_{ij}. Clearly, the a_{ij} do not constitute a minimum set of parameters. For two dimensions the four a_{ij} are subject to the three constraints given in Eq. (1.18). The justification for this redundant set of direction cosines is the convenience it provides. Hopefully, this convenience will become more apparent in Chapters 2 and 3. For three-dimensional rotations (9 a_{ij} but only three independent) alternate descriptions are provided by: (1) the Euler angles discussed in Section 3.3, (2) quaternions, and (3) the Cayley–Klein parameters. These alternatives have their respective advantages and disadvantages.

The generalization to three, four, or N dimensions is now simple. The set of N quantities V_j is said to be the components of an N-dimensional vector $\mathbf{V}$ if and only if their values relative to the rotated coordinate axes are given by

$$V_i' = \sum_{j=1}^{N} a_{ij} V_j, \qquad i = 1, 2, \ldots, N. \tag{1.15}$$

As before, a_{ij} is the cosine of the angle between x_i' and x_j. Often the upper limit N and the corresponding range of i will not be indicated. It is taken for granted that you know how many dimensions your space has.

From the definition of a_{ij} as the cosine of the angle between the positive x_i' direction and the positive x_j direction we may write (Cartesian coordinates)[6]

$$a_{ij} = \frac{\partial x_i'}{\partial x_j}. \tag{1.16a}$$

Using the inverse rotation ($\varphi \to -\varphi$) yields

$$x_j = \sum_{i=1}^{2} a_{ij} x_i' \qquad \text{or} \qquad \frac{\partial x_j}{\partial x_i'} = a_{ij}. \tag{1.16b}$$

Note that these are **partial derivatives**. By use of Eqs. (1.16a) and (1.16b), Eq. (1.15) becomes

$$V_i' = \sum_{j=1}^{N} \frac{\partial x_i'}{\partial x_j} V_j = \sum_{j=1}^{N} \frac{\partial x_j}{\partial x_i'} V_j. \tag{1.17}$$

The direction cosines a_{ij} satisfy an **orthogonality condition**

$$\sum_i a_{ij} a_{ik} = \delta_{jk} \tag{1.18}$$

or, equivalently,

$$\sum_i a_{ji} a_{ki} = \delta_{jk}. \tag{1.19}$$

Here, the symbol δ_{jk} is the Kronecker delta, defined by

$$\begin{aligned} \delta_{jk} &= 1 \qquad \text{for} \qquad j = k, \\ \delta_{jk} &= 0 \qquad \text{for} \qquad j \neq k. \end{aligned} \tag{1.20}$$

It is easily verified that Eqs. (1.18) and (1.19) hold in the two-dimensional case by substituting in the specific a_{ij} from Eqs. (1.11). The result is the well-known identity $\sin^2\varphi + \cos^2\varphi = 1$ for the nonvanishing case. To verify Eq. (1.18) in general form, we may use the partial derivative forms of Eqs. (1.16a) and (1.16b) to obtain

$$\sum_i \frac{\partial x_j}{\partial x_i'} \frac{\partial x_k}{\partial x_i'} = \sum_i \frac{\partial x_j}{\partial x_i'} \frac{\partial x_i'}{\partial x_k} = \frac{\partial x_j}{\partial x_k}. \tag{1.21}$$

[6] Differentiate x_i' with respect to x_j. See discussion following Eq. (1.21).

The last step follows by the standard rules for partial differentiation, assuming that x_j is a function of x'_1, x'_2, x'_3, and so on. The final result, $\partial x_j/\partial x_k$, is equal to δ_{jk}, since x_j and x_k as coordinate lines ($j \neq k$) are assumed to be perpendicular (two or three dimensions) or orthogonal (for any number of dimensions). Equivalently, we may assume that x_j and x_k ($j \neq k$) are totally independent variables. If $j = k$, the partial derivative is clearly equal to 1.

In redefining a vector in terms of how its components transform under a rotation of the coordinate system, we should emphasize two points:

1. This definition is developed because it is useful and appropriate in describing our physical world. Our vector equations will be independent of any particular coordinate system. (The coordinate system need not even be Cartesian.) The vector equation can always be expressed in some particular coordinate system, and, to obtain numerical results, we must ultimately express the equation in some specific coordinate system.
2. This definition is subject to a generalization that will open up the branch of mathematics known as tensor analysis (Chapter 2).

A qualification is in order. The behavior of the vector components under rotation of the coordinates is used in Section 1.3 to prove that a scalar product is a scalar, in Section 1.4 to prove that a vector product is a vector, and in Section 1.6 to show that the gradient of a scalar ψ, $\nabla\psi$, is a vector. The remainder of this chapter proceeds on the basis of the less restrictive definitions of the vector given in Section 1.1.

Summary: Vectors and Vector Space

It is customary in mathematics to label an ordered triple of real numbers (x_1, x_2, x_3) a **vector** $\mathbf{x}$. The number x_n is called the nth component of vector $\mathbf{x}$. The collection of all such vectors (obeying the properties that follow) form a three-dimensional real **vector space**. We ascribe five properties to our vectors: If $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$,

1. Vector equality: $\mathbf{x} = \mathbf{y}$ means $x_i = y_i$, $i = 1, 2, 3$.
2. Vector addition: $\mathbf{x} + \mathbf{y} = \mathbf{z}$ means $x_i + y_i = z_i$, $i = 1, 2, 3$.
3. Scalar multiplication: $a\mathbf{x} \leftrightarrow (ax_1, ax_2, ax_3)$ (with a real).
4. Negative of a vector: $-\mathbf{x} = (-1)\mathbf{x} \leftrightarrow (-x_1, -x_2, -x_3)$.
5. Null vector: There exists a null vector $\mathbf{0} \leftrightarrow (0, 0, 0)$.

Since our vector components are real (or complex) numbers, the following properties also hold:

1. Addition of vectors is commutative: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
2. Addition of vectors is associative: $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.
3. Scalar multiplication is distributive:

$$a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}, \qquad \text{also} \qquad (a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}.$$

4. Scalar multiplication is associative: $(ab)\mathbf{x} = a(b\mathbf{x})$.