Exercises

1.1.1 Show how to find $\mathbf{A}$ and $\mathbf{B}$, given $\mathbf{A}+\mathbf{B}$ and $\mathbf{A}-\mathbf{B}$.

1.1.2 The vector $\mathbf{A}$ whose magnitude is 1.732 units makes equal angles with the coordinate axes. Find A_x, A_y, and A_z.

1.1.3 Calculate the components of a unit vector that lies in the xy-plane and makes equal angles with the positive directions of the x- and y-axes.

1.1.4 The velocity of sailboat A relative to sailboat B, $\mathbf{v}_{\text{rel}}$, is defined by the equation $\mathbf{v}_{\text{rel}} = \mathbf{v}_A - \mathbf{v}_B$, where $\mathbf{v}_A$ is the velocity of A and $\mathbf{v}_B$ is the velocity of B. Determine the velocity of A relative to B if

$$\mathbf{v}_A = 30 \text{ km/hr east}$$
$$\mathbf{v}_B = 40 \text{ km/hr north.}$$

ANS. $\mathbf{v}_{\text{rel}} = 50$ km/hr, 53.1° south of east.

1.1.5 A sailboat sails for 1 hr at 4 km/hr (relative to the water) on a steady compass heading of 40° east of north. The sailboat is simultaneously carried along by a current. At the end of the hour the boat is 6.12 km from its starting point. The line from its starting point to its location lies 60° east of north. Find the x (easterly) and y (northerly) components of the water's velocity.

ANS. $v_{\text{east}} = 2.73$ km/hr, $v_{\text{north}} \approx 0$ km/hr.

1.1.6 A vector equation can be reduced to the form $\mathbf{A} = \mathbf{B}$. From this show that the one vector equation is equivalent to **three** scalar equations. Assuming the validity of Newton's second law, $\mathbf{F} = m\mathbf{a}$, as a **vector** equation, this means that a_x depends only on F_x and is independent of F_y and F_z.

1.1.7 The vertices A, B, and C of a triangle are given by the points $(-1, 0, 2)$, $(0, 1, 0)$, and $(1, -1, 0)$, respectively. Find point D so that the figure $ABCD$ forms a plane parallelogram.

ANS. $(0, -2, 2)$ or $(2, 0, -2)$.

1.1.8 A triangle is defined by the vertices of three vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ that extend from the origin. In terms of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ show that the **vector** sum of the successive sides of the triangle $(AB + BC + CA)$ is zero, where the side AB is from A to B, etc.

1.1.9 A sphere of radius a is centered at a point $\mathbf{r}_1$.

(a) Write out the algebraic equation for the sphere.
(b) Write out a **vector** equation for the sphere.

ANS. (a) $(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 = a^2$.
(b) $\mathbf{r} = \mathbf{r}_1 + \mathbf{a}$, with $\mathbf{r}_1$ = center.
($\mathbf{a}$ takes on all directions but has a fixed magnitude a.)

1.1.10 A corner reflector is formed by three mutually perpendicular reflecting surfaces. Show that a ray of light incident upon the corner reflector (striking all three surfaces) is reflected back along a line parallel to the line of incidence.
Hint. Consider the effect of a reflection on the components of a vector describing the direction of the light ray.

1.1.11 *Hubble's law*. Hubble found that distant galaxies are receding with a velocity proportional to their distance from where we are on Earth. For the ith galaxy,

$$\mathbf{v}_i = H_0 \mathbf{r}_i,$$

with us at the origin. Show that this recession of the galaxies from us does **not** imply that we are at the center of the universe. Specifically, take the galaxy at $\mathbf{r}_1$ as a new origin and show that Hubble's law is still obeyed.

1.1.12 Find the diagonal vectors of a unit cube with one corner at the origin and its three sides lying along Cartesian coordinates axes. Show that there are four diagonals with length $\sqrt{3}$. Representing these as vectors, what are their components? Show that the diagonals of the cube's faces have length $\sqrt{2}$ and determine their components.

1.2 ROTATION OF THE COORDINATE AXES[3]

In the preceding section vectors were defined or represented in two equivalent ways: (1) geometrically by specifying magnitude and direction, as with an arrow, and (2) algebraically by specifying the components relative to Cartesian coordinate axes. The second definition is adequate for the vector analysis of this chapter. In this section two more refined, sophisticated, and powerful definitions are presented. First, the vector field is defined in terms of the behavior of its components under rotation of the coordinate axes. This transformation theory approach leads into the tensor analysis of Chapter 2 and groups of transformations in Chapter 4. Second, the component definition of Section 1.1 is refined and generalized according to the mathematician's concepts of vector and vector space. This approach leads to function spaces, including the Hilbert space.

The definition of vector as a quantity with magnitude and direction is incomplete. On the one hand, we encounter quantities, such as elastic constants and index of refraction in anisotropic crystals, that have magnitude and direction **but** that are not vectors. On the other hand, our naïve approach is awkward to generalize to extend to more complex quantities. We seek a new definition of vector field using our coordinate vector $\mathbf{r}$ as a prototype.

There is a physical basis for our development of a new definition. We describe our physical world by mathematics, but it and any physical predictions we may make must be **independent** of our mathematical conventions.

In our specific case we assume that space is isotropic; that is, there is no preferred direction, or all directions are equivalent. Then the physical system being analyzed or the physical law being enunciated cannot and must not depend on our choice or **orientation** of the coordinate axes. Specifically, if a quantity S does not depend on the orientation of the coordinate axes, it is called a scalar.

[3]This section is optional here. It will be essential for Chapter 2.