CHAPTER 1

VECTOR ANALYSIS

1.1 DEFINITIONS, ELEMENTARY APPROACH

In science and engineering we frequently encounter quantities that have magnitude and magnitude only: mass, time, and temperature. These we label **scalar** quantities, which remain the same no matter what coordinates we use. In contrast, many interesting physical quantities have magnitude and, in addition, an associated direction. This second group includes displacement, velocity, acceleration, force, momentum, and angular momentum. Quantities with magnitude and direction are labeled **vector** quantities. Usually, in elementary treatments, a vector is defined as a quantity having magnitude and direction. To distinguish vectors from scalars, we identify vector quantities with boldface type, that is, $\mathbf{V}$.

Our vector may be conveniently represented by an arrow, with length proportional to the magnitude. The direction of the arrow gives the direction of the vector, the positive sense of direction being indicated by the point. In this representation, vector addition

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{1.1}$$

consists in placing the rear end of vector $\mathbf{B}$ at the point of vector $\mathbf{A}$. Vector $\mathbf{C}$ is then represented by an arrow drawn from the rear of $\mathbf{A}$ to the point of $\mathbf{B}$. This procedure, the triangle law of addition, assigns meaning to Eq. (1.1) and is illustrated in Fig. 1.1. By completing the parallelogram, we see that

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}, \tag{1.2}$$

as shown in Fig. 1.2. In words, vector addition is **commutative**.

For the sum of three vectors

$$\mathbf{D} = \mathbf{A} + \mathbf{B} + \mathbf{C},$$

Fig. 1.3, we may first add $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A} + \mathbf{B} = \mathbf{E}.$$

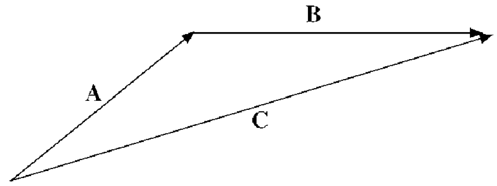

FIGURE 1.1 Triangle law of vector addition.

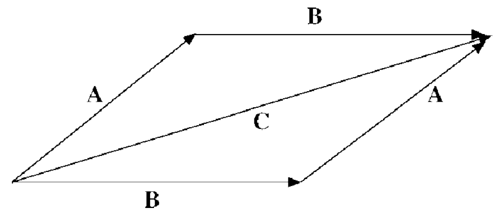

FIGURE 1.2 Parallelogram law of vector addition.

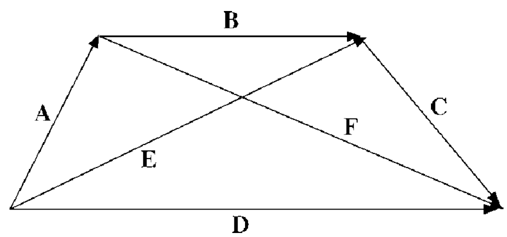

FIGURE 1.3 Vector addition is associative.

Then this sum is added to $\mathbf{C}$:

$$\mathbf{D} = \mathbf{E} + \mathbf{C}.$$

Similarly, we may first add $\mathbf{B}$ and $\mathbf{C}$:

$$\mathbf{B} + \mathbf{C} = \mathbf{F}.$$

Then

$$\mathbf{D} = \mathbf{A} + \mathbf{F}.$$

In terms of the original expression,

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).$$

Vector addition is **associative**.

A direct physical example of the parallelogram addition law is provided by a weight suspended by two cords. If the junction point (O in Fig. 1.4) is in equilibrium, the vector

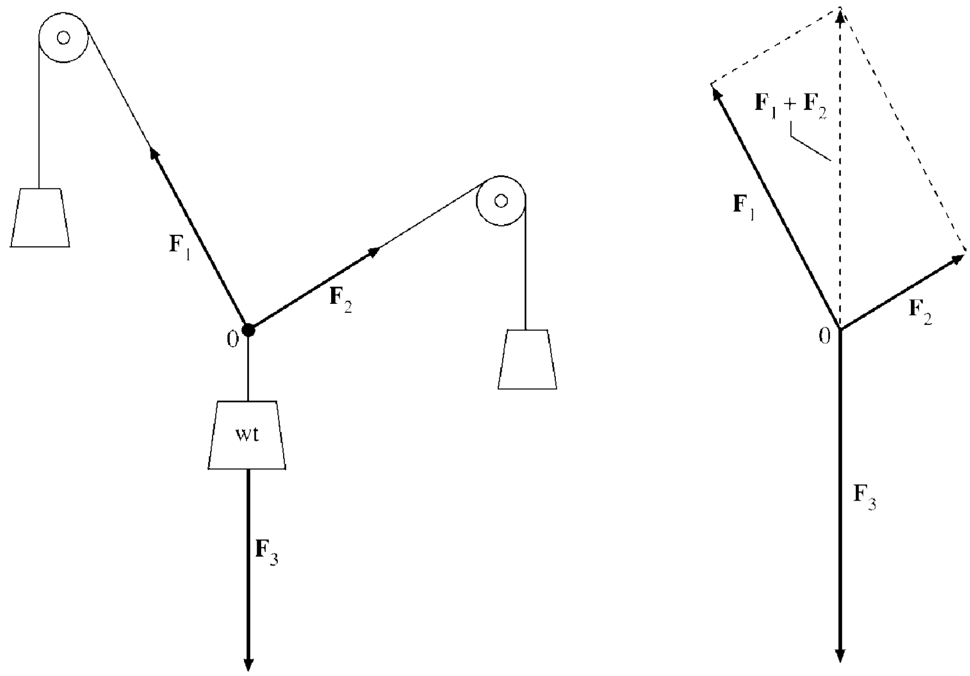

FIGURE 1.4 Equilibrium of forces: $\mathbf{F}_1 + \mathbf{F}_2 = -\mathbf{F}_3$.

sum of the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ must just cancel the downward force of gravity, $\mathbf{F}_3$. Here the parallelogram addition law is subject to immediate experimental verification.[1]

Subtraction may be handled by defining the negative of a vector as a vector of the same magnitude but with reversed direction. Then

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}).$$

In Fig. 1.3,

$$\mathbf{A} = \mathbf{E} - \mathbf{B}.$$

Note that the vectors are treated as geometrical objects that are independent of any coordinate system. This concept of independence of a preferred coordinate system is developed in detail in the next section.

The representation of vector **A** by an arrow suggests a second possibility. Arrow **A** (Fig. 1.5), starting from the origin,[2] terminates at the point (A_x, A_y, A_z). Thus, if we agree that the vector is to start at the origin, the positive end may be specified by giving the Cartesian coordinates (A_x, A_y, A_z) of the arrowhead.

Although **A** could have represented any vector quantity (momentum, electric field, etc.), one particularly important vector quantity, the displacement from the origin to the point

[1]Strictly speaking, the parallelogram addition was introduced as a definition. Experiments show that if we assume that the forces are vector quantities and we combine them by parallelogram addition, the equilibrium condition of zero resultant force is satisfied.

[2]We could start from any point in our Cartesian reference frame; we choose the origin for simplicity. This freedom of shifting the origin of the coordinate system without affecting the geometry is called **translation invariance**.

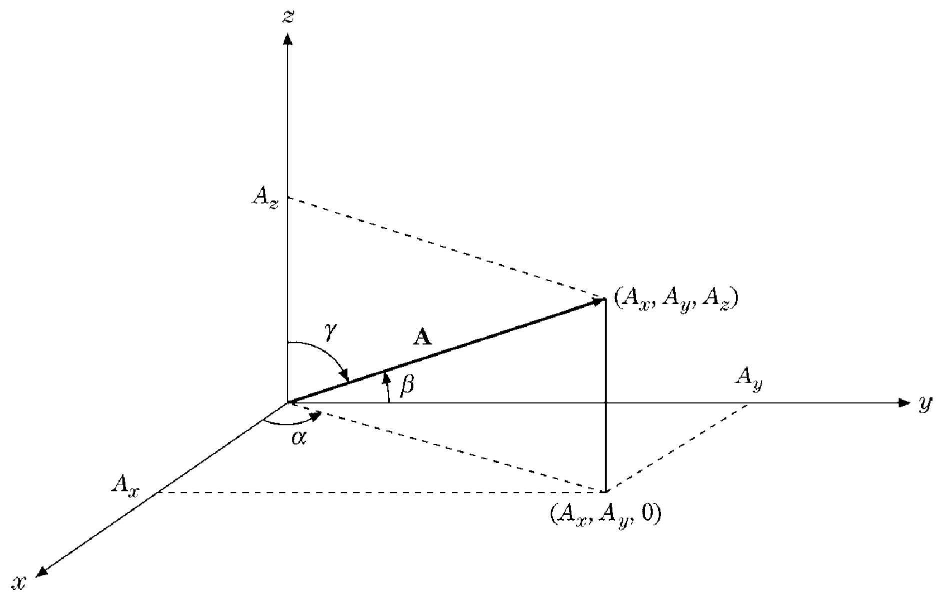

FIGURE 1.5 Cartesian components and direction cosines of **A**.

(x, y, z), is denoted by the special symbol **r**. We then have a choice of referring to the displacement as either the vector **r** or the collection (x, y, z), the coordinates of its endpoint:

$$\mathbf{r} \leftrightarrow (x, y, z). \tag{1.3}$$

Using r for the magnitude of vector **r**, we find that Fig. 1.5 shows that the endpoint coordinates and the magnitude are related by

$$x = r\cos\alpha, \qquad y = r\cos\beta, \qquad z = r\cos\gamma. \tag{1.4}$$

Here $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the **direction cosines**, α being the angle between the given vector and the positive x-axis, and so on. One further bit of vocabulary: The quantities A_x, A_y, and A_z are known as the (Cartesian) **components** of **A** or the **projections** of **A**, with $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Thus, any vector **A** may be resolved into its components (or projected onto the coordinate axes) to yield $A_x = A\cos\alpha$, etc., as in Eq. (1.4). We may choose to refer to the vector as a single quantity **A** or to its components (A_x, A_y, A_z). Note that the subscript x in A_x denotes the x component and not a dependence on the variable x. The choice between using **A** or its components (A_x, A_y, A_z) is essentially a choice between a geometric and an algebraic representation. Use either representation at your convenience. The geometric "arrow in space" may aid in visualization. The algebraic set of components is usually more suitable for precise numerical or algebraic calculations.

Vectors enter physics in two distinct forms. (1) Vector **A** may represent a single force acting at a single point. The force of gravity acting at the center of gravity illustrates this form. (2) Vector **A** may be defined over some extended region; that is, **A** and its components may be functions of position: $A_x = A_x(x, y, z)$, and so on. Examples of this sort include the velocity of a fluid varying from point to point over a given volume and electric and magnetic fields. These two cases may be distinguished by referring to the vector defined over a region as a **vector field**. The concept of the vector defined over a region and being a function of position will become extremely important when we differentiate and integrate vectors.

At this stage it is convenient to introduce unit vectors along each of the coordinate axes. Let $\hat{\mathbf{x}}$ be a vector of unit magnitude pointing in the positive x-direction, $\hat{\mathbf{y}}$, a vector of unit magnitude in the positive y-direction, and $\hat{\mathbf{z}}$ a vector of unit magnitude in the positive z-direction. Then $\hat{\mathbf{x}}A_x$ is a vector with magnitude equal to $|A_x|$ and in the x-direction. By vector addition,

$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z. \tag{1.5}$$

Note that if $\mathbf{A}$ vanishes, all of its components must vanish individually; that is, if

$$\mathbf{A} = 0, \qquad \text{then } A_x = A_y = A_z = 0.$$

This means that these unit vectors serve as a **basis**, or complete set of vectors, in the three-dimensional Euclidean space in terms of which any vector can be expanded. Thus, Eq. (1.5) is an assertion that the three unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ span our real three-dimensional space: Any vector may be written as a linear combination of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. Since $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are linearly independent (no one is a linear combination of the other two), they form a **basis** for the real three-dimensional Euclidean space. Finally, by the Pythagorean theorem, the magnitude of vector $\mathbf{A}$ is

$$\boxed{|\mathbf{A}| = \left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}.} \tag{1.6}$$

Note that the coordinate unit vectors are not the only complete set, or basis. This resolution of a vector into its components can be carried out in a variety of coordinate systems, as shown in Chapter 2. Here we restrict ourselves to Cartesian coordinates, where the unit vectors have the coordinates $\hat{\mathbf{x}} = (1, 0, 0)$, $\hat{\mathbf{y}} = (0, 1, 0)$ and $\hat{\mathbf{z}} = (0, 0, 1)$ and are all constant in length and direction, properties characteristic of Cartesian coordinates.

As a replacement of the graphical technique, addition and subtraction of vectors may now be carried out in terms of their components. For $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$ and $\mathbf{B} = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z$,

$$\mathbf{A} \pm \mathbf{B} = \hat{\mathbf{x}}(A_x \pm B_x) + \hat{\mathbf{y}}(A_y \pm B_y) + \hat{\mathbf{z}}(A_z \pm B_z). \tag{1.7}$$

It should be emphasized here that the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are used for convenience. They are not essential; we can describe vectors and use them entirely in terms of their components: $\mathbf{A} \leftrightarrow (A_x, A_y, A_z)$. This is the approach of the two more powerful, more sophisticated definitions of vector to be discussed in the next section. However, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ emphasize the **direction**.

So far we have defined the operations of addition and subtraction of vectors. In the next sections, three varieties of multiplication will be defined on the basis of their applicability: a scalar, or inner, product, a vector product peculiar to three-dimensional space, and a direct, or outer, product yielding a second-rank tensor. Division by a vector is not defined.

Exercises

1.1.1 Show how to find $\mathbf{A}$ and $\mathbf{B}$, given $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} - \mathbf{B}$.

1.1.2 The vector $\mathbf{A}$ whose magnitude is 1.732 units makes equal angles with the coordinate axes. Find A_x, A_y, and A_z.

1.1.3 Calculate the components of a unit vector that lies in the xy-plane and makes equal angles with the positive directions of the x- and y-axes.

1.1.4 The velocity of sailboat A relative to sailboat B, $\mathbf{v}_{\text{rel}}$, is defined by the equation $\mathbf{v}_{\text{rel}} = \mathbf{v}_A - \mathbf{v}_B$, where $\mathbf{v}_A$ is the velocity of A and $\mathbf{v}_B$ is the velocity of B. Determine the velocity of A relative to B if

$$\mathbf{v}_A = 30 \text{ km/hr east}$$
$$\mathbf{v}_B = 40 \text{ km/hr north.}$$

ANS. $\mathbf{v}_{\text{rel}} = 50$ km/hr, 53.1° south of east.

1.1.5 A sailboat sails for 1 hr at 4 km/hr (relative to the water) on a steady compass heading of 40° east of north. The sailboat is simultaneously carried along by a current. At the end of the hour the boat is 6.12 km from its starting point. The line from its starting point to its location lies 60° east of north. Find the x (easterly) and y (northerly) components of the water's velocity.

ANS. $v_{\text{east}} = 2.73$ km/hr, $v_{\text{north}} \approx 0$ km/hr.

1.1.6 A vector equation can be reduced to the form $\mathbf{A} = \mathbf{B}$. From this show that the one vector equation is equivalent to **three** scalar equations. Assuming the validity of Newton's second law, $\mathbf{F} = m\mathbf{a}$, as a **vector** equation, this means that a_x depends only on F_x and is independent of F_y and F_z.

1.1.7 The vertices A, B, and C of a triangle are given by the points $(-1, 0, 2)$, $(0, 1, 0)$, and $(1, -1, 0)$, respectively. Find point D so that the figure $ABCD$ forms a plane parallelogram.

ANS. $(0, -2, 2)$ or $(2, 0, -2)$.

1.1.8 A triangle is defined by the vertices of three vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ that extend from the origin. In terms of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ show that the **vector** sum of the successive sides of the triangle $(AB + BC + CA)$ is zero, where the side AB is from A to B, etc.

1.1.9 A sphere of radius a is centered at a point $\mathbf{r}_1$.

(a) Write out the algebraic equation for the sphere.
(b) Write out a **vector** equation for the sphere.

ANS. (a) $(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = a^2$.
(b) $\mathbf{r} = \mathbf{r}_1 + \mathbf{a}$, with $\mathbf{r}_1 =$ center.
($\mathbf{a}$ takes on all directions but has a fixed magnitude a.)