2.1.2 Derivative of a Function

Let f be a real valued function continuous in the interval $(x, x_1) \subseteq D_f$ (the domain of f), then

difference quotient $\quad \dfrac{f(x_1) - f(x)}{x_1 - x} \qquad$ (i)

represents the **average rate** of change in the value of f with respect to the change $x_1 - x$ in the value of independent variable x.

If x_1 approaches to x, then

$$\lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x}$$

provided this limit exists, is called the **instantaneous** rate of change of f with respect to x at x and is written as $f'(x)$.

If $x_1 = x + \delta x$ i.e., $x_1 - x = \delta x$, then the expression (i) can be expressed as

$$\frac{f(x + \delta x) - f(x)}{\delta x} \qquad \text{(ii)}$$

and

$$\lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \qquad \text{(iii)}$$

provided the limit exists, is defined to be the **derivative** of f (or **differential coefficient** of f) with respect to x at x and is denoted by $f'(x)$ (read as "f–prime of x"). The domain of f' consists of all x for which the limit exists. If $x \in D_f$ and $f'(x)$ exists, then f is said to be differentiable at x. The process of finding f' is called **differentiation**.

Notation for Derivative

Several notations are used for derivatives. We have used the functional symbol $f'(x)$, for the derivative of f at x. For the function $y = f(x)$.

$$y + \delta y = f(x + \delta x)$$

where δy is the increment of y (change in the value of y) corresponding to δx, the change in the value of x, then (iv)

$$\delta y = f(x + \delta x) - f(x)$$

Dividing both the sides of (iv) by δx, we get (v)

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Taking limit of both the sides of (v) as $\delta x \to 0$, we have (vi)

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x}$ is denoted by $\frac{dy}{dx}$, so (vi) is written as $\frac{dy}{dx} = f'(x)$.

Note: The symbol $\frac{dy}{dx}$ is used for the derivative of y with respect to x and here it is not a quotient of dy and dx. $\frac{dy}{dx}$ is also denoted by y'.

Now we write, in a table the notations for the derivative of $y = f(x)$ used by different mathematicians:

Name of Mathematician	Leibniz	Newton	Lagrange	Cauchy
Notation used for derivative	$\frac{dy}{dx}$ or $\frac{df}{dx}$	$f(x)$	$f'(x)$	$D f(x)$

If we replace $x + \delta x$ by x and x by a, then the expression $f(x + \delta x) - f(x)$ becomes $f(x) - f(a)$. and the change δx in the independent variable, in this case, is $x - a$.

So the expression $\frac{f(x + \delta x) - f(x)}{\delta x}$ is written as $\frac{f(x) - f(a)}{x - a}$ (vii)

Taking the limit of the expression (vii) when $x \to a$, gives

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a).$$ Here $f'(a)$ is called the derivative of f at $x = a$.

Example 1: **Find the derivative of the following functions by definition**

(a) $f(x) = c$ (b) $f(x) = x^2$

Solution: (a) For $f(x) = c$

(i) $f(x + \delta x) = c$

(ii) $f(x + \delta x) - f(x) = c - c = 0$

(iii) $$\frac{f(x + \delta x) - f(x)}{\delta x} = \frac{0}{\delta x} = 0$$

(iv) $$\lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} (0) = 0$$

Thus $f'(x) = 0$, that is, $\frac{d}{dx}(c) = 0$

(i) $f(x + \delta x) = (x + \delta x)^2$

(ii) $f(x + \delta x) - f(x) = (x + \delta x)^2 - x^2 = x^2 + 2x\,\delta x + (\delta x)^2 - x^2$

$= 2x\,\delta x + (\delta x)^2 = (2x + \delta x)\,\delta x.$

(iii) $$\frac{f(x + \delta x) - f(x)}{\delta x} = \frac{(2x + \delta x)\,\delta x}{\delta x} = 2x + \delta x \qquad (\delta x \neq 0)$$

(iv) $$\lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} (2x + \delta x) = 2x$$

i.e., $f'(x) = 2x$

Example 4: Find the derivative of $x^{\frac{2}{3}}$ and also calculate the value of derivative at $x = 8$.

Solution: Let $f(x) = x^{\frac{2}{3}}$. Then

$$f(x+\delta x) = (x+\delta x)^{\frac{2}{3}}$$

and

$$f(x+\delta x) - f(x) = (x+\delta x)^{\frac{2}{3}} - x^{\frac{2}{3}} = \frac{\left((x+\delta x)^{\frac{2}{3}} - x^{\frac{2}{3}}\right)\left[(x+\delta x)^{\frac{4}{3}} + (x+\delta x)^{\frac{2}{3}} \cdot x^{\frac{2}{3}} + x^{\frac{4}{3}}\right]}{(x+\delta x)^{\frac{4}{3}} + (x+\delta x)^{\frac{2}{3}} \cdot x^{\frac{2}{3}} + x^{\frac{4}{3}}}$$

$$= \frac{\left[(x+\delta x)^{\frac{2}{3}}\right]^3 - \left(x^{\frac{2}{3}}\right)^3}{(x+\delta x)^{\frac{4}{3}} + (x+\delta x)^{\frac{2}{3}} \cdot x^{\frac{2}{3}} + x^{\frac{4}{3}}} = \frac{(x+\delta x)^2 - x^2}{(x+\delta x)^{\frac{4}{3}} + (x+\delta x)^{\frac{2}{3}} \cdot x^{\frac{2}{3}} + x^{\frac{4}{3}}}$$

i.e., $$f(x+\delta x) - f(x) = \frac{\delta x(2x+\delta x)}{(x+\delta x)^{\frac{4}{3}} + (x+\delta x)^{\frac{2}{3}} \cdot x^{\frac{2}{3}} + x^{\frac{4}{3}}} \qquad \text{(i)}$$

Dividing both the sides of (i) by δx, we get

$$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{2x+\delta x}{(x+\delta x)^{\frac{4}{3}} + (x+\delta x)^{\frac{2}{3}} \cdot x^{\frac{2}{3}} + x^{\frac{4}{3}}} \qquad \text{(ii)}$$

Taking limit of both the sides of (ii) as $\delta x \to 0$, we get

$$f'(x) = \frac{2x}{x^{\frac{4}{3}} + x^{\frac{2}{3}} \cdot x^{\frac{2}{3}} + x^{\frac{4}{3}}} = \frac{2x}{3x^{\frac{4}{3}}} = \frac{2}{3x^{\frac{1}{3}}}$$

and $$f'(8) = \frac{2}{3 \cdot (8)^{\frac{1}{3}}} = \frac{1}{3}.$$

(i)

Example 5: Find the derivative of $x^3 + 2x + 3$.

Solution: Let $y = x^3 + 2x + 3$. Then

(i) $$y + \delta y = (x+\delta x)^3 + 2(x+\delta x) + 3$$

and (ii) $$\delta y = [(x+\delta x)^3 + 2(x+\delta x) + 3] - [x^3 + 2x + 3]$$

$$= [(x+\delta x)^3 - x^3] + 2[(x+\delta x) - x] + (3-3)$$

$$= [(x+\delta x) - x][(x+\delta x)^2 + (x+\delta x)\,x + x^2] + 2\delta x$$

(iii) $$\frac{\delta y}{\delta x} = \frac{\delta x[(x+\delta x)^2 + (x+\delta x)\,x + x^2] + 2\delta x}{\delta x}$$

$$= (x+\delta x)^2 + (x+\delta x)\,x + x^2 + 2$$

(iv) $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} [(x + \delta x)^2 + (x + \delta x)\, x + x^2 + 2]$

$$\frac{dy}{dx} = (x)^2 + (x)\, x + x^2 + 2$$

i.e., $\frac{d}{dx}(x^3 + 2x + 3) = 3x^2 + 2$

2.2.1 Derivation of x^n where $n \in Z$.

(a) We find the derivative of x^n when n is positive integer.

(a) Let $y = x^n$. Then

$$y + \delta y = (x + \delta x)^n$$

and $\delta y = (x + \delta x)^n - x^n$

Using the binomial theorem, we have

$$\delta y = \left[x^n + nx^{n-1}.\,\delta x + \frac{n(n-1)}{\underline{|2}} x^{n-2} (\delta x)^2 + \ldots + (\delta x)^n\right] - x^n$$

i.e. $\delta y = \delta x \left[nx^{n-1} + \frac{n(n-1)}{\underline{|2}} x^{n-2}\,.\,\delta x + \ldots + (\delta x)^{n-1}\right]$ (i)

Dividing both sides of (i) by δx, gives

$$\frac{\delta y}{\delta x} = nx^{n-1} + \frac{n(n-1)''}{\underline{|2}} x^{n-2}\,.\,\delta x + \ldots + (\delta x)^{n-1} \qquad \text{(ii)}$$

Note that each term on the right hand side of (ii) involves δx except the first term, so taking the limit as $\delta x \to 0$, we get $\frac{dy}{dx} = nx^{n-1}$

As $y = x^n$, so $\frac{d}{dx}(x^n) = n\,.\,x^{n-1}$.

Note: If $n = 0$, then the formula $\frac{d}{dx}(x^n) = nx^{n-1}$ reduces to $\frac{d}{dx}(x^0) = 0x^{0-1} = 0$ i.e. $\frac{d}{dx}(1) = 0$ which is correct by example 1 part (a).

(b) Let $y = x^n$ where n is a negative integer.

Let $n = -m$ (m is a positive integer). Then

$$y = x^{-m} = \frac{1}{x^m} \qquad \text{(i)}$$

and $y + \delta y = \frac{1}{(x + \delta x)^m}$ (ii)