The table of values of $f(x)$ for different values of x as x approaches 2 from left and right is as follows:

from left of 2 ⟶ **2** ⟵ from right of 2

x	1	1.5	1.8	1.9	1.99	1.999	1.9999	2.0001	2.001	2.01	2.1	2.2	2.5	3
$f(x)=x^3$	1	3.375	5.832	6.859	7.8806	7.988	7.9988	8.0012	8.012	8.1206	9.261	10.648	15.625	27

The table shows that, as x gets closer and closer to 2 (sufficiently close to 2), from both sides, $f(x)$ gets closer and closer to 8.

We say that 8 is the limit of $f(x)$ when x approaches 2 and is written as:

$$f(x) \to 8 \text{ as } x \to 2 \quad \text{or} \quad \lim_{x\to 2}(x^3) = 8$$

$e = 2.718$

1.4.5 Limit of a Function

Let a function $f(x)$ be defined in an open interval near the number "a" (need not at a).

⇒ If, as x approaches "a" from both left and right side of "a", $f(x)$ approaches a specific number "L" then "L", is called the limit of $f(x)$ as x approaches a.

Symbolically it is written as:

$\boxed{\lim_{x\to a} f(x) = L}$ read as "limit of $f(x)$, as $x \to a$, is L".

It is neither desirable nor practicable to find the limit of a function by numerical approach. We must be able to evaluate a limit in some mechanical way. The theorems on limits will serve this purpose. Their proofs will be discussed in higher classes.

1.4.6 Theorems on Limits of Functions

Let f and g be two functions, for which $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

Theorem 1: **The limit of the sum of two functions is equal to the sum of their limits.**

imp.

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$$

For example, $\lim_{x\to 1}(x+5) = \lim_{x\to 1} x + \lim_{x\to 1} 5 = 1+5 = 6$.

imp

Theorem 2: The limit of the difference of two functions is equal to the difference of their limits.

$$\lim_{x\to a}[f(x)-g(x)]=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)=L-M$$

imp For example, $\lim_{x\to 3}(x-5)=\lim_{x\to 3}x-\lim_{x\to 3}5=3-5=-2$

Theorem 3: If k is any real number, then

$$\lim_{x\to a}[kf(x)]=k\lim_{x\to a}f(x)=kL$$

imp For example, $\lim_{x\to 2}(3x)=3\lim_{x\to 2}(x)=3(2)=6$

Theorem 4: The limit of the product of the functions is equal to the product of their limits.

$$\lim_{x\to a}[f(x)g(x)]=\left[\lim_{x\to a}f(x)\right]\left[\lim_{x\to a}g(x)\right]=LM$$

For example: $\lim_{x\to 1}(2x)(x+4)=\left[\lim_{x\to 1}(2x)\right]\left[\lim_{x\to 1}(x+4)\right]=(2)(5)=10$

Theorem 5: The limit of the quotient of the functions is equal to the quotient of their limits provided the limit of the denominator is non-zero.

imp

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}=\frac{L}{M},\quad g(x)\neq 0, M\neq 0$$

For example: $\lim_{x\to 2}\left(\frac{3x+4}{x+3}\right)=\frac{\lim_{x\to 2}(3x+4)}{\lim_{x\to 2}(x+3)}=\frac{6+4}{2+3}=\frac{10}{5}=2$

imp

Theorem 6: Limit of $[f(x)]^n$, where n is an integer

$$\lim_{x\to a}[f(x)]^n=\left(\lim_{x\to a}f(x)\right)^n=L^n$$

For example $\lim_{x\to 4}(2x-3)^3=\left(\lim_{x\to 4}(2x-3)\right)^3=(5)^3=125$

We conclude from the theorems on limits that limits are evaluated by merely substituting the number that x approaches into the function.

imp

Example : If $P(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$ is a polynomial function of degree n, then show that $\lim_{x\to c}P(x)=P(c)$

Solution: Using the theorems on limits, we have

$$\lim_{x\to c}P(x)=\lim_{x\to c}(a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0)$$

$x<a$

2 Definition We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

$x>a$

Notice that Definition 2 differs from Definition 1 only in that we require x to be less than a. Similarly, if we require that x be greater than a, we get "the **right-hand limit of $f(x)$ as x approaches a** is equal to L" and we write

$$\lim_{x \to a^+} f(x) = L$$

Thus the symbol "$x \to a^+$" means that we consider only $x > a$. These definitions are illustrated in Figure 9.

$x<a$

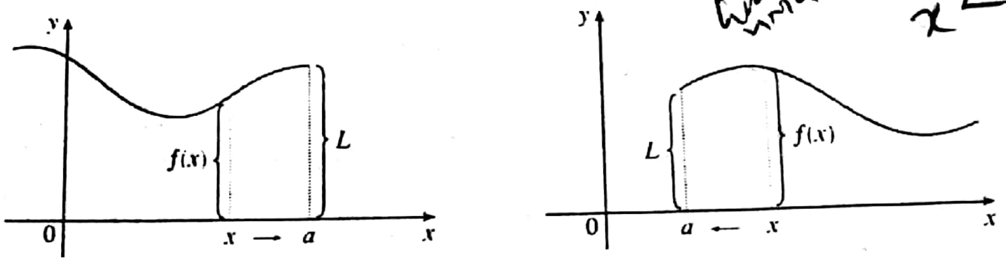


FIGURE 9 (a) $\lim_{x \to a^-} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$

By comparing Definition 1 with the definitions of one-sided limits, we see that the following is true.

$$\boxed{3} \quad \lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L$$

V EXAMPLE 7 The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$

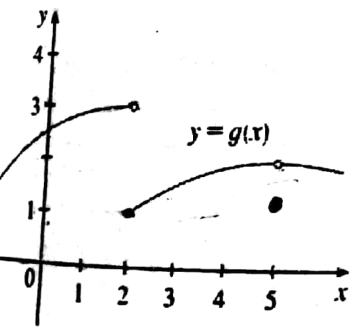


FIGURE 10

SOLUTION From the graph we see that the values of $g(x)$ approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right. Therefore

$$\text{(a)} \lim_{x \to 2^-} g(x) = 3 \quad \text{and} \quad \text{(b)} \lim_{x \to 2^+} g(x) = 1$$

(c) Since the left and right limits are different, we conclude from $\boxed{3}$ that $\lim_{x \to 2} g(x)$ does not exist.

lim

The graph also shows that

$$\text{(d)} \lim_{x \to 5^-} g(x) = 2 \quad \text{and} \quad \text{(e)} \lim_{x \to 5^+} g(x) = 2$$

ating Limits Using the Limit Laws

In Section 1.5 we used calculators and graphs to guess the values of limits, but we saw that such methods don't always lead to the correct answer. In this section we use the following properties of limits, called the *Limit Laws*, to calculate limits.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

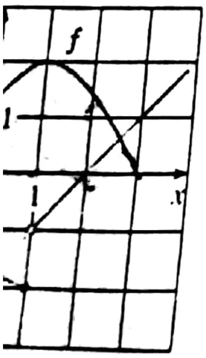
exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

These five laws can be stated verbally as follows:

1. The limit of a sum is the sum of the limits.
2. The limit of a difference is the difference of the limits.
3. The limit of a constant times a function is the constant times the limit of the function.
4. The limit of a product is the product of the limits.
5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

It is easy to believe that these properties are true. For instance, if $f(x)$ is close to L and $g(x)$ is close to M, it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$. This gives us an intuitive basis for believing that Law 1 is true. In Section 1.7 we give a precise definition of a limit and use it to prove this law. The proofs of the remaining laws are given in Appendix F.



EXAMPLE 1 Use the Limit Laws and the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$ (c) $\lim_{x \to 2} \dfrac{f(x)}{g(x)}$

SOLUTION

(a) From the graphs of f and g we see that

As the first example illustrates, it is often necessary to use the problem-solving principle of *taking cases* when dealing with absolute values.

EXAMPLE 1 Solve the inequality $|x - 3| + |x + 2| < 11$.

SOLUTION Recall the definition of absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

It follows that

$$|x - 3| = \begin{cases} x - 3 & \text{if } x - 3 \geq 0 \\ -(x - 3) & \text{if } x - 3 < 0 \end{cases}$$

$$= \begin{cases} x - 3 & \text{if } x \geq 3 \\ -x + 3 & \text{if } x < 3 \end{cases}$$

Similarly

$$|x + 2| = \begin{cases} x + 2 & \text{if } x + 2 \geq 0 \\ -(x + 2) & \text{if } x + 2 < 0 \end{cases}$$

$$= \begin{cases} x + 2 & \text{if } x \geq -2 \\ -x - 2 & \text{if } x < -2 \end{cases}$$

Take cases

These expressions show that we must consider three cases:

$$x < -2 \qquad -2 \leq x < 3 \qquad x \geq 3$$

CASE I If $x < -2$, we have

$$|x - 3| + |x + 2| < 11$$
$$-x + 3 - x - 2 < 11$$
$$-2x < 10$$
$$x > -5$$

CASE II If $-2 \leq x < 3$, the given inequality becomes

$$-x + 3 + x + 2 < 11$$
$$5 < 11 \quad \text{(always true)}$$

CASE III If $x \geq 3$, the inequality becomes

$$x - 3 + x + 2 < 11$$
$$2x < 12$$
$$x < 6$$

Combining cases I, II, and III, we see that the inequality is satisfied when $-5 < x < 6$. So the solution is the interval $(-5, 6)$.