Chap: 3

Definition (inner product space, Hilbert space)

⇒ An inner product space is vector space X in which norm define on X with help of norm.

⇒ Hilbert space

is a complete SP is a complete inner product space

$f: X \times X \to \mathbb{R}$
$\langle x, y \rangle \to$

inner product.

Here inner product on X is mapping of $X \times X$ into scalar fixed K of X for every pair of vector x and y there is associated a scalar

$\langle x, y \rangle \to \mathbb{R}$

is called the inner product of x and y

P_1: $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

P_2: $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

real complex inner product space

P_3: $\langle x, y \rangle = \overline{\langle y, x \rangle}$

P_4: $\langle x, x \rangle \geq 0$ $\quad \langle x, x \rangle = 0$ iff $x = 0$

Norm

$\|x\| = \sqrt{\langle x, x \rangle}$

metric defin on X

$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y \rangle}$

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hence inner product space are normed space.

$\Rightarrow$ Hilbert space are Banach spaces

for complex vector space

$$\langle x, y\rangle = \overline{\langle y, x\rangle}$$

Real vector space

$$\langle x, y\rangle = \langle y, x\rangle$$

properties

$$\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$$

proff

$$\langle \alpha x + \beta y, z\rangle = \overline{\langle z, \alpha x + \beta y\rangle} \qquad \because \langle x, y\rangle = \overline{\langle y, x\rangle}$$

$$= \overline{\langle z, \alpha x\rangle} + \overline{\langle z, \beta y\rangle}$$

$$= \langle \overline{\overline{\alpha x, z}}\rangle + \langle \overline{\overline{\beta y, z}}\rangle$$

$$= \alpha\langle x, z\rangle \cancel{+ \beta\langle y, z\rangle}$$

Remplud.

$$\langle x, \alpha y\rangle = \bar{\alpha}\langle x, y\rangle$$

$$\cancel{\langle x, \alpha y\rangle} =$$

$$\langle x, \alpha y\rangle = \overline{\langle \alpha y, x\rangle}$$

$$= \bar{\alpha}\,\overline{\langle y, x\rangle}$$

$$= \bar{\alpha}\,\overline{\langle y, x\rangle}$$

$$= \bar{\alpha}\langle x, y\rangle$$

(c)

$$\langle x, \alpha y + \beta z \rangle = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$$

proof

$$\langle x, \alpha y + \beta z \rangle = \langle x, \alpha y \rangle + \langle x, \beta z \rangle$$

$$= \overline{\langle \alpha y, x \rangle} + \overline{\langle \beta z, x \rangle}$$

$$= \overline{\alpha \langle y, x \rangle} + \overline{\beta \langle z, x \rangle}$$

$$= \bar{\alpha} \overline{\langle y, x \rangle} + \bar{\beta} \overline{\langle z, x \rangle}$$

$$\bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$$

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Example:

~~Euclidean space~~

~~po~~ **Space l^p**

Show that the space l^p with $p \neq 2$ is not an inner product ~~space~~ and show that is not hilbert space.

proof

we have to prove that l^p with $p \neq 2$ is does not satisfied the parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

let us we take

$x = (1, 1, 0, 0 \ldots) \in l^p$

$y = (1, -1, 0 \ldots) \in l^p$

$x+y = (1,1,0,0\ldots) + (1,-1,0\ldots)$

$= (2,0,0,0\ldots) \in l^p$

we know Norm defined

$\|x\| = \langle x,x\rangle^{1/2} = \left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{1/p}$

Since

$x = (1,1,0\ldots)$

$\|x\| = \langle x,x\rangle^{\frac{1}{2}} = \left(\sum_{i=1}^{\infty} |(1,1,0\ldots)|^p\right)^{\frac{1}{p}}$

$= (|1|^p + |1|^p + |0|^p \ldots)^{\frac{1}{p}}$

$= (1+1+0+0\ldots)^{1/p}$

$\|x\| = (2)^{\frac{1}{p}}$

$\|y\| = \langle y,y\rangle = \left[\sum_{j=1}^{\infty} |(1,-1,0\ldots)|^p\right]^{\frac{1}{p}}$

$= (|1|^p + |-1|^p + |0|^p \ldots)^{\frac{1}{p}}$

$= (1+1+0+0\ldots)^{1/p}$

$(2)^{1/p}$

$\|x+y\| = \left[\sum_{j=1}^{\infty} |(2,0,0\ldots)|^p\right]^{\frac{1}{p}}$

$= (|2|^p + |0|^p + |0|^p \ldots)^{\frac{1}{p}}$

$(2^p)^{\frac{1}{p}} = 2$ (2)

$x-y = (1,1,0\ldots) - (1,-1,0\ldots)$

$= (0,2,0\ldots)$

$\|x-y\| = \left[\sum_{j=1}^{\infty} |(0,2,0\ldots)|^p\right]^{\frac{1}{p}}$

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$$= (|0|^p + |2|^p + |0|^p \dots)^{1/p}$$

$$= (2^p)^{1/p}$$

$$= 2$$

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

$$(2)^2 + (2)^2 = 2(2^{2/p} + 2^{2/p})$$

$$8 = 2 \cdot (2 \cdot 2^{2/p})$$

$$8 = 4 \cdot 2^{2/p}$$

$$8 = 2^2 \cdot 2^{2/p} = 2^{2+\frac{2}{p}} = 2^{\frac{2p+2}{p}}$$

$$2^3 = 2^{\frac{2p+2}{p}}$$

$$3p = 2p+2$$

$$3p - 2p = 2 \qquad \boxed{p = 2}$$

$\boxed{p=2}$ this equality hold when $p=2$ which is impossible $p \neq 2$

Since l^p is comple we alred proved and Not inner product space Since, complet inner product space is called the Hilbert space;

Since l^p is Not inner product space if $p=2$ so l^p is not the Hilbert space.

Space $C[a,b]$

Show that space $C[a,b]$ is not an inner product space hence not Hilbert

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we have to prove that norm does not satisfied the parallelogram inequality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

we know that Normed defined on $C[a,b]$

$$\|x\| = \max_{t \in [a,b]} |x(t)|$$

we take

$$x(t) = 1, \quad y(t) = \frac{t-a}{b-a}$$

$C[a, b]$

since

$$\|x(t)\| = 1$$

$$\|y(t)\| = \max_{t \in [a,b]} \left| \frac{t-a}{b-a} \right|$$

$b = \max$

$$= \left| \frac{b-a}{b-a} \right| = |1| = 1$$

$$\|x\| = \|y(t)\| = 1$$

$$\|x+y\| = \max_{t \in [a,b]} |x+y|$$

$$= \max_{t \in [a,b]} \left| 1 + \frac{t-a}{b-a} \right| = \max_{t \in [a,b]} \left| \frac{b-a+t-a}{b-a} \right|$$

$$= \left| 1 + \frac{b-a}{b-a} \right|$$

$$= |1+1| = 2$$

Since $C[a,b]$

$$\|x-y\| = \max_{t \in [a,b]} \left| 1 - \frac{t-a}{b-a} \right| = \left| 1 - \frac{a-a}{b-a} \right|$$

$$= |1-0|$$

$$= 1$$

$$\|x-y\|^2 + \|x+y\|^2 = 2\left[\|x\|^2 + \|y\|^2\right]$$

$$1^2 + 9^2 = 2[1+1]$$

$$2(2)$$

$$1+4 \neq$$

$$5 \neq 4$$

hence equality does not hold ∴ not inner product space hence is not a Hilbert space.

for inner product space

$$\langle x, y \rangle = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2\right]$$

for complex inner product space

$$\mathrm{Re}\langle x, y \rangle = \frac{1}{4}\left[\|x+y\|^2 + \|x-y\|^2\right]$$

$$\mathrm{Img}\langle x, y \rangle = \frac{1}{4}\left[\|x+iy\|^2 + \|x-iy\|^2\right] \quad - (1)$$

equation (1) is called the polarization identity

E:

Lemma:

An inner product and Norm satisfy the schwarz inequality and triangular inequality

Proof:

(a) Schwarz inequality

$$|\langle x, y\rangle| \leq \|x\| \|y\|$$

where equality sign hold iff $\{x, y\}$ is linearly dependent set

Proof

First we proved

$$|\langle x, y\rangle| \leq \|x\| \|y\| \quad \text{—(A)}$$

Case i if $y = 0$

L.H.S

$|\langle x, y\rangle| = |\langle x, 0\rangle| = |0| = 0$

R.H.S

$\|x\|\|y\| = \|x\| \|0\| = \|x\|(0) = 0$

here

$|\langle x, y\rangle| = \|x\| \|y\|$

hence if $y = 0$ then (A) hold

Case ii

if $y \neq 0$ for any scalar we have

$\|x - \alpha y\| \geq 0$

$0 \leq \|x - \alpha y\|^2 = \langle (x - \alpha y), (x - \alpha y)\rangle$

$|\langle x, y\rangle| \leq \|x\| \|y\|$

$\|x - y\| \geq 0$

$$= \langle x,x\rangle + \langle x,-\alpha y\rangle + \langle -\alpha y,x\rangle + \langle -\alpha y,-\alpha y\rangle$$

$$= \langle x,x\rangle - \bar{\alpha}\langle x,y\rangle - \alpha\langle y,x\rangle + \alpha\bar{\alpha}\langle y,y\rangle \quad (1)$$

if ~~we chose~~ $= \langle x,x\rangle - \bar{\alpha}\langle x,y\rangle - \alpha[\langle y,x\rangle - \bar{\alpha}\langle y,y\rangle]$

we take

$$\langle y,x\rangle - \bar{\alpha}\langle y,y\rangle = 0$$

$$\langle y,x\rangle = \bar{\alpha}\langle y,y\rangle$$

$$\bar{\alpha} = \frac{\langle y,x\rangle}{\langle y,y\rangle} = \frac{\langle y,x\rangle}{\|y\|^2}$$

puting in (1)

$$0 \le \|x-\alpha y\|^2 = \langle x,x\rangle - \frac{\langle y,x\rangle}{\|y\|^2}\langle x,y\rangle - \alpha\left[\langle y,x\rangle - \frac{\langle y,x\rangle}{\cancel{\langle y,y\rangle}}\cancel{\langle y,y\rangle}\right]$$

$$0 \le \langle x,x\rangle - \frac{\langle y,x\rangle\langle x,y\rangle}{\|y\|^2} - \alpha(0)$$

$$0 \le \|x\|^2 - \frac{\langle x,y\rangle\langle y,x\rangle}{\|y\|^2}$$

$$z\cdot\bar{z} = |z|^2$$

$$0\|y\|^2 \le \|x\|^2\|y\|^2 - \frac{\langle x,y\rangle\overline{\langle x,y\rangle}}{\cancel{\|y\|^2}} \times \cancel{\|y\|^2}$$

$$|\langle x,y\rangle|$$

$$0 \le \|x\|^2\|y\|^2 - \langle x,y\rangle\overline{\langle x,y\rangle}$$

$$0 \le \|x\|^2\|y\|^2 - \cancel{\|z\|}\ |\langle x,y\rangle|^2$$

$$|\langle x,y\rangle| \le \|x\|\|y\|$$

$$|\langle x,y\rangle|^2 \le \|x\|^2\|y\|^2$$

here equality hold if $y \ne 0$

hold iff $\|x-\alpha y\| = 0$ and $y \ne 0$

$$(x-\alpha y) = 0$$

$$x = \alpha y$$

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vector in given set $\{x, y\}$ are scalar multiple of other which show that set $\{x, y\}$ is linearly dependent

(b) Norm also satisfies

$$\|x+y\| \le \|x\| + \|y\|$$

where inequality only hold iff $y=0$ or $x = cy$ c is Real ($\mathbb{C}$)

proof

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$

$$= \langle x,x \rangle + \langle x,y \rangle + \langle y,x \rangle + \langle y,y \rangle$$

$$\|x+y\|^2 = \|x\|^2 + |\langle x,y \rangle| + \langle y,x \rangle + \|y\|^2 \quad -②$$

$\|\cancel{x+y}\|^2 =$ we know that by schwarz inequality

$$|\langle x,y \rangle| \le \|x\|\|y\| \quad -Ⓐ$$

$$\| \langle y,x \rangle \| \le \|y\|\|x\| \quad -Ⓑ$$

using Ⓐ and Ⓑ in ②

$$\|x+y\|^2 \le \|x\|^2 + \|x\|\|y\| + \|y\|\|x\| + \|y\|^2$$

$$\|x+y\|^2 \le \|x\|^2 + 2\|y\|\|x\| + \|y\|^2$$

$$\|x+y\|^2 \le (\|x\| + \|y\|)^2$$

taking sqring

$$\|x+y\| \le \|x\| + \|y\| \quad \text{hence proved}$$

since

equality ② hold if

$$\langle x,y \rangle + \langle y,x \rangle = 2\|x\|\|y\|$$

$$\langle x,y \rangle + \overline{\langle x,y \rangle} = 2\|x\|\|y\|$$

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$a+bi+a\cdot bi$

$a+ai+a-ai$

$R(2a)$

~~Real~~ $\langle x,y\rangle = \cancel{x}\,\|x\|\,\|y\|$

Real $\langle x,y\rangle = \|x\|\,\|y\|$

Since $|\langle x,y\rangle| \le \|x\|\,\|y\|$

$\Rightarrow \|x\|\,\|y\| \ge |\langle x,y\rangle|$

Real $\langle x,y\rangle = \|x\|\,\|y\| \ge |\langle x,y\rangle|$

$z = x+yi$

$|z| = \sqrt{x^2+y^2}$

Real $= x$

if $y=0$

we know that

$\Rightarrow \|x\|\,\|y\| = |\langle x,y\rangle|$ if x and y are the linear dependent or $y=0$, $x=cy$, c should be $c \ne 0$ and Real

Since

$|\langle x,y\rangle| =$ Real $\langle x,y\rangle \ge 0$

$0 \le \langle x,y\rangle = \langle cy,y\rangle = c\langle y,y\rangle = c\|y\|^2$

$\Rightarrow c\|y\|^2 \ge 0$

$\Rightarrow c \ge 0$ and c is Real

Lemma:

if in an inner product space $x_n \longrightarrow x$ and $y_n \longrightarrow y$

then $\langle x_n, y_n\rangle \longrightarrow \langle x,y\rangle$

<u>proof</u>

if $x_n \longrightarrow x$ and $y_n \longrightarrow y$

$|\langle x_n, y_n\rangle - \langle x,y\rangle| = |\langle x_n, y_n\rangle + \langle x_n, -y\rangle + \langle x_n, y\rangle - \langle x,y\rangle|$

$= |\langle x_n, y_n\rangle + \langle x_n, -y\rangle + \langle x_n, y\rangle + \langle -x, y\rangle|$

We know

$\langle \alpha x + \beta y, z\rangle = \langle \alpha x, z\rangle + \langle \beta y, z\rangle$

$= | \langle x_n, y_n - y \rangle + \langle x_n - x, y \rangle |$

we know Schwarz inequality

$|\langle x, y \rangle| \leq \|x\| \|y\|$

$\leq |\langle x_n, y_n - y \rangle| + |\langle x_n - x, y \rangle|$

$\leq \|x_n\| \|y_n - y\| + \|x_n - x\| \|y\|$

Since $y_n \longrightarrow y \Rightarrow \|y_n - y\| \longrightarrow 0$ as $n \to \infty$

$x_n \longrightarrow x \Rightarrow \|x_n - x\| \longrightarrow 0$ as $n \to \infty$

$\Rightarrow \|x_n\| \|y_n - y\| \longrightarrow 0$ and $\|x_n - x\| \|y\| \longrightarrow 0$

$\Rightarrow \|x_n\| \|y_n - y\| + \|x_n - x\| \|y\| \longrightarrow 0$

$\Rightarrow$

$| \langle x_n, y_n \rangle - \langle x, y \rangle | \longrightarrow 0$

$\Rightarrow \langle x_n, y_n \rangle - \langle x, y \rangle \longrightarrow 0$

$\Rightarrow \langle x_n, y_n \rangle \longrightarrow \langle x, y \rangle$ ✓

hence proved

Weak Convergence and strong

→ strong Convergence:

A Segence (x_n) in Normed Space 'X' is Said to be strong convergent or convergent if there is an $x \in X$ s.t

$$\lim_{n\to\infty} \|x_n - x\| = 0$$

written $\lim_{n\to\infty} x_n = x$

or $x_n \to x$ as $n \to \infty$

then x_n is called strong limit of (x_n) and segenee (x_n) converges strongly to 'x'

→ Weak converges

A segence (x_n) in Norm Space is Said to be weakly Convergent if there exist $x \in X$ s.t For every $f \in X'$

$$f : X \to X'$$

$$\lim_{n\to\infty} f(x_n) = f(x)$$

$x_n \xrightarrow{w} x$ as $n \to \infty$

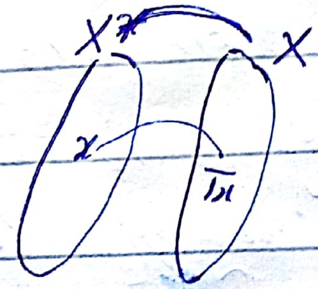
Then limit $x \in X$ is called weak limit of (x_n) and sequence (x_n) weakly converges to 'x'

written as $x_n \xrightarrow{w} x$

Chapter: 5

ix point A Fixed point of a Mapping $T: X \to X$ of set X into itself is an $x \in X$, which is mapped onto itself

$$Tx = x$$



Tx is image of 'x'

x is called Fixed point of Mapping 'T'

Contraction:

Let (X, d) be Metric space

A Mapping $T: X \to X$ is called contraction on X if there exis $\alpha < 1$ such that For all $x, y \in X$

$$d(Tx, Ty) \le \alpha\, d(x,y)$$

or

$$\frac{d(Tx, Ty)}{d(x,y)} \le \alpha \quad \text{--- (1)}$$

Geometrically this Mean that any point 'x' and 'y' have images that are closer together than those points 'x' and 'y' Ration in (1) does Not exceed a constant $\alpha < 1$

Lipschitz condition:

A mapping $T: [a,b] \to [a,b]$ is said to satisfy a Lipschitz condition with Lipschitz constant ~~k~~ on $[a,b]$ if there is a constant 'k' for all $x, y \in [a,b]$

$$|Tx - Ty| \le k\, |x - y|$$

Banch Fixed point theorem:

Consider a metric space (X, d) where $X \ne 0$ suppose that 'X' is complete and let $T: X \to X$ be contraction on X. Then T has Fixed point.

Fredholm equation

An integral equation

$$x(t) - \mu \int_a^b K(t,\tau)\, x(\tau)\, d\tau = v(t)$$

is called Fredholm equation of second kind. here $[a,b]$ is given interval, x is a function on $[a,b]$

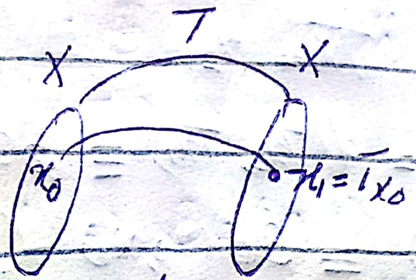
$U \rightarrow$ unknown μ is parameter

$K(t,\tau) \rightarrow$ called kernal of equation in given function on ~~sq~~ Square $[a,b] \times [a,b]$

[Volterra integral equation]

$$x(t) - \mu \int_a^t K(t,\tau)\, x(\tau)\, d\tau = v(t)$$

Iteration:



Calculate Recursively seqence x_0, x_1, x_2 ..

we use Relation

$$x_{n+1} = Tx_n \qquad n = 0, 1, 2$$

$x_1 = Tx_0$, $x_2 = Tx_1$ ...... $x_{n+1} = Tx_n$

Banach Fixed point Theorem

proof

taking interation sequence we choose a point $x_0 \in X$

$x_1 = Tx_0$ or $x_1 = Tx_0$

$x_2 = Tx_1$ or $x_2 = Tx_1 = T(Tx_0) = T^2x_0$

$\vdots$ $\qquad x_3 = Tx_2 = T(Tx_2) = T^3x_0$

$x_n = Tx_{n-1}$ or $x_n = T^n x_0$

By definition of contraction

$$d(Tx, Ty) \le \beta d(x, y) \qquad \boxed{\beta < 1}$$

we obtain the ~~seqen~~ interation sequence from ~~recursively~~ recursive Relation (x_n) we have to show (x_n) is cauchy or $d(x_n, x_m) < \varepsilon$

$Tx_0 = x_1, \quad Tx_1 = x_2 \; \ldots$

$d(x_1, x_2) = d(Tx_0, Tx_1) \leq \beta d(x_0, x_1)$

$d(x_1, x_2) \leq \beta d(x_0, x_1)$ — (1)

$d(x_2, x_3) \leq d(Tx_1, Tx_2) \leq \beta d(x_1, x_2)$

(1) $\Rightarrow$

$d(x_2, x_3) \leq \beta(\beta d(x_0, x_1))$

$d(x_2, x_3) \leq \beta^2 d(x_0, x_1)$ — (11)

Similarly

$d(x_3, x_4) \leq \beta^3 d(x_0, x_1)$

For $m > n$

~~d(x_n~~ Using Triangular inequality

$\cancel{d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \leq \beta^n d(x_0, x_1) + \beta^{n+1}}$

$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})$
$+ \ldots + d(x_{m-1}, x_m)$

$\leq \beta^n d(x_0, x_1) + \beta^{n+1} d(x_0, x_1) + \ldots$
$\ldots + \beta^{m-1} d(x_0, x_1)$

$\leq \beta^n [1 + \beta^1 + \beta^2 + \ldots \beta^{m-n-1}] d(x_0, x_1)$

$$\leq \beta^m \left[\frac{1-\beta^{m-n}}{1-\alpha}\right] d(x_0, x_1)$$

as $0 < B < 1 \Rightarrow B \in (0,1)$

as $m \to \infty$ the $B^m \to 0$

so

$$d(x_n, x_m) \leq 0$$

$\Rightarrow$

$$\lim_{n\to\infty} d(x_n, x_m) = 0$$

or

$$\lim_{n\to\infty} |x_n - x_m| = 0$$

hence (x_n) is cauchy sequence

Since 'X' is complete so

$$x_n \to x \in X$$

we have to show that

'x' is Fixed point of 'T'

$$d(Tx, x) \leq d(Tx, \cancel{T}x_{n+1}) + d(x_{n+1}, x)$$

$$d(Tx, x) \leq d(Tx, Tx_n) + d(x_{n+1}, x)$$

$$d(Tx, x) \leq \beta\, d(x, x_n) + d(x_{n+1}, x)$$

As $n \to \infty$ then

$x_n \to x$ so $x_{n+1} \to x$

$d(x, Tx) \leq 0$

$\Rightarrow$

$d(x, Tx) = 0$

$Tx = x$

hence 'x' is Fixed point of 'T'