Lecture Notes for Chapter 12

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1 Constrained Optimization

Consider the following Utility Max problem:

Max $x_{1,}x_2$

$$U = U(x_{1,}x_2) \tag{1}$$

Subject to:

$$B = P_1x_1 + P_2x_2 \tag{2}$$

Re-write Eq. 2

$$x_2 = \frac{B}{P_2} - \frac{P_1}{P_2}x_1 \tag{Eq.2‘}$$

Now $x_2 = x_2(x_1)$ and $\frac{dx_2}{dx_1} = \frac{-P_1}{P_2}$
Sub into Eq. 1 for x_2

$$U = U(x_1, x_2(x_1)) \tag{3}$$

Eq. 3 is an unconstrained function of one variable, x_1

Differentiate, using the Chain Rule

$$\frac{dU}{dx_1} = \frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2}\frac{dx_2}{dx_1} = 0$$

From Eq. 2' we know $\frac{dx_2}{dx_1} = -\frac{P_1}{P_2}$

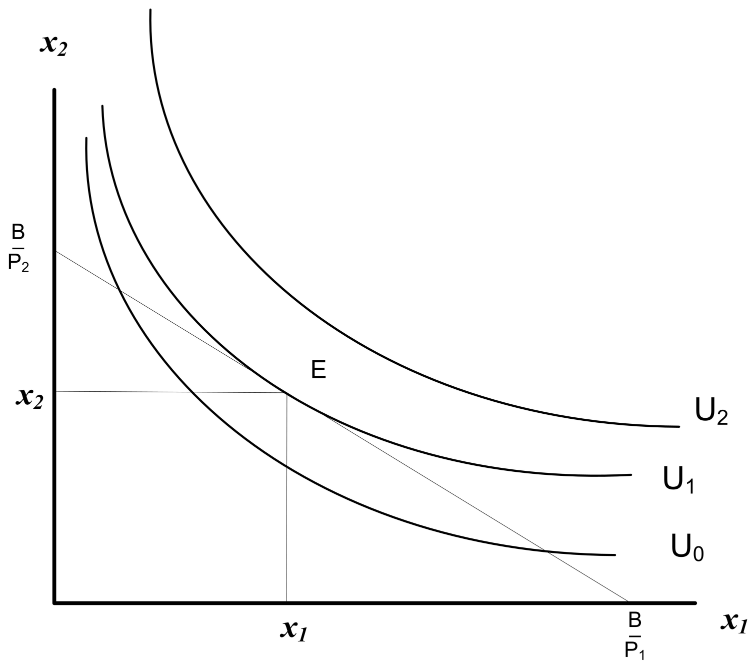
Therefore:

$$\frac{dU}{dx_1} = U_1 + U_2\left(-\frac{P_1}{P_2}\right) = 0$$

OR

$$\frac{U_1}{U_2} = \frac{P_1}{P_2}$$

This is our usual condition that MRS(x_2, x_1) $= \frac{P_1}{P_2}$ or the consumer's willingness to grade equals his ability to trade.



The More General Constrained Maximum Problem

Max:

$$y = f(x_1, x_2) \tag{4}$$

Subject to:

$$g(x_1, x_2) = 0 \tag{5}$$

Take total differentials of Eq. 4 and Eq. 5

$$dy = f_1 dx_1 + f_2 dx_2 = 0 \tag{6}$$

$$dg = g_1 dx_1 + g_2 dx_2 = 0 \tag{7}$$

or Eq.6$'$

$$dx_1 = -\frac{f_2}{f_1} dx_2$$

Eq. 7$'$

$$dx_1 = -\frac{g_2}{g_1} dx_2$$

Subtract 6$'$ from 7$'$

$dx_1 - dx_1 = \left[-\frac{g_2}{g_1} - \left(-\frac{f_2}{f_1}\right)\right] dx_2 = \left(\frac{f_2}{f_1} - \frac{g_2}{g_1}\right) dx_2 = 0$

Therefore

$$\frac{f_2}{f_1} = \frac{g_2}{g_1}$$

Eq. 8: says that the level curves of the objective function must be tangent to the level curves of the constraint

1.1 Lagrange Multiplier Approach

Create a new function called the Lagrangian:

$$L = f(x_1, x_2) + \lambda g(x_1, x_2)$$

since $g(x_1, x_2) = 0$ when the constraint is satisfied

$$L = f(x_1, x_2) + zero$$

We have created a new independent variable λ (lambda), which is called the Lagrangian Multiplier.

We now have a function of three variables; x_1,x_2,and λ

Now we Maximize

$$L = f(x_1, x_2) + \lambda g(x_1, x_2)$$

First Order Conditions

$$\begin{array}{ll} L_\lambda = \frac{\partial L}{\partial \lambda} = g\left(x_1, x_2\right) = 0 & \text{Eq. 1} \\ L_1 = \frac{\partial L}{\partial x_1} = f_1 + \lambda g_1 = 0 & \text{Eq. 2} \\ L_2 = \frac{\partial L}{\partial x_2} = f_2 + \lambda g_2 = 0 & \text{Eq. 3} \end{array}$$

From Eq. 2 and 3 we get:

$$\frac{f_1}{f_2} = \frac{-\lambda g_1}{-\lambda g_2} = \frac{g_1}{g_2}$$

From the 3 F.O.C.'s we have 3 equations and 3 unknowns $(x_{1,} x_2, \lambda)$. In principle we can solve for x_1^*, x_2^*, and λ^*.

1.1.1 Example 1:

Let:

$$U = xy$$

Subject to:

$$10 = x + y \quad P_x = P_y = 1$$

Lagrange:

$$\begin{aligned} L &= f(x, y) + \lambda(g(x, y)) \\ L &= xy + \lambda(10 - x - y) \end{aligned}$$

F.O.C.

$$\begin{array}{ll} L_\lambda = 10 - x - y = 0 & \text{Eq. 1} \\ L_x = y - \lambda = 0 & \text{Eq. 2} \\ L_y = x - \lambda = 0 & \text{Eq. 3} \end{array}$$

From (2) and (3) we see that:

$$\tfrac{y}{x} = \tfrac{\lambda}{\lambda} = 1 \quad \underline{\text{or}} \quad y = x \qquad \text{Eq. 4}$$

From (1) and (4) we get:

$$10 - x - x = 0 \quad or \quad x^* = 5 \quad and \quad y^* = 5$$

From either (2) or (3) we get:

$$\lambda^* = 5$$

1.1.2 Example 2: Utility Maximization

Maximize

$$u = 4x^2 + 3xy + 6y^2$$

subject to

$$x + y = 56$$

Set up the Lagrangian Equation:

$$L = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y)$$

Take the first-order partials and set them to zero

$$\begin{aligned} L_x &= 8x + 3y - \lambda = 0 \\ L_y &= 3x + 12y - \lambda = 0 \\ L_\lambda &= 56 - x - y = 0 \end{aligned}$$

From the first two equations we get

$$\begin{aligned} 8x + 3y &= 3x + 12y \\ x &= 1.8y \end{aligned}$$

Substitute this result into the third equation

$$\begin{aligned} 56 - 1.8y - y &= 0 \\ y &= 20 \end{aligned}$$

therefore

$$x = 36 \qquad \lambda = 348$$

1.1.3 Example 3: Cost minimization

A firm produces two goods, x and y. Due to a government quota, the firm must produce subject to the constraint $x + y = 42$. The firm's cost functions is

$$c(x, y) = 8x^2 - xy + 12y^2$$

The Lagrangian is

$$L = 8x^2 - xy + 12y^2 + \lambda(42 - x - y)$$

The first order conditions are

$$\begin{aligned} L_x &= 16x - y - \lambda = 0 \\ L_y &= -x + 24y - \lambda = 0 \\ L_\lambda &= 42 - x - y = 0 \end{aligned} \tag{8}$$

Solving these three equations simultaneously yields

$$x = 25 \qquad y = 17 \qquad \lambda = 383$$

1.1.4 Example 4:

Max:

$$U = x_1 x_2$$

Subject to:

$$B = P_1 x_1 + P_2 x_2$$

Langrange:

$$L = x_1 x_2 + \lambda \left(B - P_1 x_1 - P_2 x_2\right)$$

F.O.C.

$$\begin{aligned} L_\lambda = B - P_1 x_1 - P_2 x_2 = 0 &\qquad \text{Eq. 1} \\ L_1 = x_2 - \lambda P_1 = 0 &\qquad \text{Eq. 2} \\ L_2 = x_1 - \lambda P_2 = 0 &\qquad \text{Eq. 3} \end{aligned}$$

From Eq. (2) and (3) $\left(\frac{x_2}{x_1} = \frac{P_1}{P_2} = MRS\right)$

Solve for x_1^*
From (2) and (3)

$$x_2 = \frac{P_1}{P_2}x_1$$

Sub into (1)

$$B = P_1x_1 + P_2\left(\frac{P_1}{P_2}x_1\right) = 2P_1x_1$$

$$x_1^* = \frac{B}{2P_1} \quad \text{and} \quad x_2^* = \frac{B}{2P_2}$$

The solution to $x_1^*$and x_2^* are the Demand Functions for x_1 and x_2

1.1.5 Properties of Demand Functions

1. "Homogenous of degree zero" multiply prices and income by α

$$x_1^* = \frac{\alpha B}{2(\alpha P_1)} = \frac{B}{2P_1}$$

2. "For normal goods demand has a negative slope"

$$\frac{\partial x_1^*}{\partial P_1} = -\frac{B}{2P_1^2} < 0$$

3. "For normal goods Engel curve positive slope"

$$\frac{\partial x_1^*}{\partial B} = \frac{1}{2P_1} > 0$$

In this example x_1^* and $x_2^*$are both normal goods (rather than inferior or giffen)

Given:

$$U = x_1 x_2$$

And:

$$x_1^* = \frac{B}{2P_1} \quad \text{and} \quad x_2^* = \frac{B}{2P_2}$$

Substituting into the utility function we get:

$$\begin{aligned} U &= x_1^*, x_2^* = \left(\frac{B}{2P_1}\right)\left(\frac{B}{2P_2}\right) \\ U &= \left(\frac{B^2}{4P_1P_2}\right) \end{aligned}$$

Now we have the utility expressed as a function of Prices and Income

$U^* = U(P_1P_2, B)$ is "The Indirect Utility Function"

At $U = U_0 = \frac{B^2}{4P_1P_2}$ we can re-arrange to get:

$$\underbrace{B = 2P_1^{\frac{1}{2}}P_2^{\frac{1}{2}}U_0^{\frac{1}{2}}}_{\text{This is the "Expenditure Function"}}$$

1.2 Minimization and Lagrange

Min x, y

$$P_xX + P_yY$$

Subject to

$$U_0 = U(x, y)$$

Lagrange

$$L = P_xX + P_yY + \lambda(U_0 - U(x, y))$$

F.O.C.

$$\begin{aligned} L_\lambda = U_0 - U(x,y) = 0 \quad & \text{Eq. } 1 \\ L_x = P_x - \lambda \frac{\partial U}{\partial x} = 0 \quad & \text{Eq. } 2 \\ L_y = P_y - \lambda \frac{\partial U}{\partial y} = 0 \quad & \text{Eq. } 3 \end{aligned}$$

From (2) and (3) we get

$$\underbrace{\frac{P_x}{P_y} = \frac{\lambda U_x}{\lambda U_y} = \frac{U_x}{U_y} = MRS}_{\text{(The same result as in the MAX problem)}}$$

Solving (1), (2), and (3) by Cramer's Rule, or some other method, we get:

$$x^* = x(P_x, P_y, U_0) \qquad y^* = y(P_x, P_y, U_0) \qquad \lambda^* = \lambda(P_x, P_y, U_0)$$

1.3 Second Order Conditions

1. (a) To determine whether the Lagrangian is at a Max or Min we use an approach similar to the Hessian in unconstrained cases.
   (b) Second order conditions are determined from the Bordered Hessian
   (c) There are two ways of setting up a bordered Hessian
   (d) We will look at both ways since both forms are used equally in economic literature
   (e) Both ways are equally good.

Given Max:

$$f(x,y) + \lambda(g(x,y)$$

F.O.C.'s

$$L_\lambda = g(x,y) = 0 \quad \text{Eq. 1}$$
$$L_x = f_x - \lambda g_x = 0 \quad \text{Eq. 2}$$
$$L_y = f_y - \lambda g_y = 0 \quad \text{Eq. 3}$$

For the 2nd order conditions, totally differentiate the F.O.C.'s with respect to x, y, and λ

$$g_x dx + g_y dy = 0 \qquad \text{No } \lambda \text{ in Eq. 1} \tag{1'}$$

$$(f_{xx} + \lambda g_{xx})dx + (f_{xy} + \lambda g_{xy})dy + g_x d\lambda = 0 \tag{2'}$$

$$(f_{yy} + \lambda g_{yy})dy + (f_{yx} + \lambda g_{yx})dx + g_y d\lambda = 0 \tag{3'}$$

Matrix From

$$\overbrace{\begin{bmatrix} 0 & g_x & g_y \\ g_x & (f_{xx} + \lambda g_{xx}) & (f_{xy} + \lambda g_{xy}) \\ g_y & (f_{yx} + \lambda g_{yx}) & (f_{yy} + \lambda g_{yy}) \end{bmatrix}}^{\text{Bordered Hessian}} \begin{pmatrix} d\lambda \\ dx \\ dy \end{pmatrix}$$

Or written as

$$\underbrace{\begin{bmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{bmatrix}}_{\text{(Where } L_{xx} = f_{xx} + \lambda g_{xx} \text{ etc...)}} \quad \begin{pmatrix} d\lambda \\ dx \\ dy \end{pmatrix}$$

Notice that the Bordered Hessian is the ordinary Hessian bordered by the first partial derivatives of the constraint.

$$|H| = \begin{vmatrix} L_{xx} & L_{xy} \\ L_{yx} & L_{yy} \end{vmatrix} \quad \text{Where} \quad \left|\bar{H}\right| = \begin{bmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{bmatrix}$$

H is ordinary (unconstrained) Hessian
H̄ is bordered (constrained) Hessian

1.4 Determining Max or Min with a Single Constraint

2 Variable Case

$$\left|\bar{H}_\alpha\right| = \begin{bmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{bmatrix}$$

is Max if $\left|\bar{H}_\alpha\right| > 0$
is Min if $\left|\bar{H}_\alpha\right| < 0$

3 Variabl Case

$$\bar{H}_3 = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & L_{11} & L_{12} & L_{13} \\ g_2 & L_{21} & L_{22} & L_{23} \\ g_3 & L_{31} & L_{32} & L_{33} \end{vmatrix}$$

is Max if $\left|\bar{H}_2\right| > 0, \ \left|\bar{H}_3\right| < 0$
is Min if $\left|\bar{H}_2\right| < 0, \ \left|\bar{H}_3\right| > 0$

n-Variable case
Max: $\left|\bar{H}_2\right| > 0, \left|\bar{H}_3\right| < 0, \bar{H}_4 > 0 ... (-1)^n \left|\bar{H}_n\right| > 0$
Min: $\left|\bar{H}_2\right| < 0, \left|\bar{H}_3\right| < 0, ... \left|\bar{H}_n\right| < 0$

1.5 Altrnative form of Bordered Hessian

Given Max x,y

$$f(x,y) + \lambda g(x,y)$$

F.O.C's

$$\begin{gathered} L_x = f_x - \lambda g_x = 0 \\ L_y = f_y - \lambda g_y = 0 \\ L_\lambda = g(x,y) = 0 \end{gathered}$$

Bordered Hessian

$$\begin{vmatrix} f_{xx} + \lambda g_{xx} & f_{xy} + \lambda g_{xy} & g_x \\ f_{yx} + \lambda g_{yx} & f_{yy} + \lambda g_{yy} & g_y \\ g_x & g_y & 0 \end{vmatrix} \begin{pmatrix} d_x \\ d_y \\ d_\lambda \end{pmatrix}$$

Rules for Max or Min are the same for this form as well.

1.5.1 Example

Max

$$xy + \lambda(B - P_x x - P_y y)$$

F.O.C.'s

$$\underbrace{\begin{gathered} L_x = y - \lambda P_x = 0 \\ L_y = x - \lambda P_y = 0 \\ L_\lambda = B - P_x x - P_y y = 0 \end{gathered}}$$

$$x^* = \frac{B}{2P_x} \qquad y^* = \frac{B}{2P_y} \qquad \lambda^* = \frac{B}{2P_x P_y}$$

S.O.C.'s

$$\left|\bar{H}_2\right| = \begin{vmatrix} 0 & 1 & -P_x \\ 1 & 0 & -P_y \\ -P_x & -P_y & 0 \end{vmatrix}$$

$$Det = 0 + (-1)\begin{vmatrix} 1 & -P_x \\ -P_y & 0 \end{vmatrix} + -P_x\begin{vmatrix} 1 & -P_x \\ 0 & -P_y \end{vmatrix} = P_xP_y + P_xP_y = 2P_xP_y > 0$$

Therefore L* is a Max

1.5.2 Example

Min

$$P_xx + P_yy + \lambda(U_0 - xy)$$

F.O.C.'s

$$L_x = P_x - \lambda y = 0$$
$$L_y = P_y - \lambda x = 0$$
$$L_\lambda = U_0 - xy = 0$$

$$x^* = \frac{P_y^{\frac{1}{2}}U_0^{\frac{1}{2}}}{P_x^{\frac{1}{2}}} \qquad y^* = \frac{P_x^{\frac{1}{2}}U_0^{\frac{1}{2}}}{P_y^{\frac{1}{2}}} \qquad \lambda^* = \frac{U_0^{\frac{1}{2}}}{P_x^{\frac{1}{2}}P_y^{\frac{1}{2}}}$$

S.O.C.'s

$$\begin{bmatrix} 0 & -\lambda & -y \\ -\lambda & 0 & -x \\ -y & -x & 0 \end{bmatrix}\begin{pmatrix} d_x \\ d_y \\ d_\lambda \end{pmatrix}$$

$$\left|\bar{H}\right| = \lambda\begin{vmatrix} -\lambda & -y \\ -x & 0 \end{vmatrix} + (-y)\begin{vmatrix} -\lambda & -y \\ 0 & -x \end{vmatrix} = -\lambda xy + -\lambda xy$$

$$\left|\bar{H}\right| = -2\lambda xy < 0$$

Therefore L* is a Min

1.6 Interpreting λ

Given Max

$$U(x, y) + \lambda(B - P_xx - P_yy)$$

By solving the F.O.C.'s we get

$$x^* = x(P_x, P_y, B) \qquad y^* = y(P_x, P_y, B) \qquad \lambda^* = \lambda(P_x, P_y, B)$$

Sub x^*, y^*, λ^* back into the Lagrange

$$L^* = U\left(x^*, y^*\right) + \lambda^*\left(B - P_x x^* - P_y y^*\right)$$

Differentiate with respect to the constant,B

$$\frac{\partial L^*}{\partial B} = U_x \frac{dx^*}{dB} + U_y \frac{dy^*}{dB} - \lambda^* P_x \frac{dx^*}{dB} - \lambda^* Py \frac{dy^*}{dB} + \lambda^* \frac{dB}{dB} + \left(B - P_x x^* - P_y y^*\right) \frac{d\lambda^*}{dB}$$

Or

$$\frac{\partial L^*}{\partial B} = \underbrace{(U_x - \lambda^* P_x)}_{=0} \frac{dx^*}{dB} + \underbrace{(U_y - \lambda^* P_y)}_{=0} \frac{dy^*}{dB} + \underbrace{(B - P_x x^* - P_y y^*)}_{=0} \frac{d\lambda^*}{dB} + \lambda^*$$

$\frac{\partial L^*}{\partial B} = \lambda^*$ $= \Delta$ in utility from Δ in the constant
$=$ Marginal Utility of Money

2 Extensions and Applications of Constrained Optimization

2.1 Income and Substitution Effects (The Slutsky Equation)

Consider:
Max

$$U = U(x_1, x_2)$$

Subject to

$$B = P_\lambda x_1 + P_2 x_2$$

The FOC's

$$\begin{aligned} L_1 &= U_1 - \lambda P_1 = 0 \\ L_2 &= U_2 - \lambda P_2 = 0 \\ L_3 &= B - P_1 x_1 - P_2 x_2 = 0 \end{aligned}$$

Solving the FOC's gives x_1^*, x_2^*, λ^*

$\Rightarrow$ Totally differentiate the FOC's with respect to EVERY variable

$\dot{U}_{11} dx_1^* + U_{12} dx_2^* - P_1 d\lambda - \lambda dP_1 = 0$

$\dot{U}_{21} dx_1^* + U_{22} dx_2^* - P_2 d\lambda - \lambda dP_2 = 0$

$P_1 dx_1^* - P_2 dx_2^* - x_1^* dP_1 - x_2^* dP_2 + d\beta = 0$

Take exogenous differentials $(dP_1, dP_2, d\beta)$ to the other side and set up matrix

$$\begin{bmatrix} U_{11} & U_{12} & -P_1 \\ U_{21} & U_{22} & -P_2 \\ -P_1 & -P_2 & 0 \end{bmatrix} \begin{pmatrix} dx_1^* \\ dx_2^* \\ d\lambda^* \end{pmatrix} = \begin{pmatrix} \lambda dP_1 \\ \lambda dP_2 \\ -d\beta + x_1 dP_1 + x_2 dP_2 \end{pmatrix}$$

$\{\text{Where } \bar{H} > 0\}$

Set $dP_1 = dP_2 = 0$ find $\frac{dx_1}{d\beta}$

$$\frac{dx_1^*}{d\beta} = \frac{\begin{vmatrix} 0 & U_{12} & -P_1 \\ 0 & U_{22} & -P_2 \\ -1 & -P_2 & 0 \end{vmatrix}}{|\bar{H}|} = (-1) \frac{\begin{vmatrix} U_{12} & -P_1 \\ U_{22} & -P_2 \end{vmatrix}}{|\bar{H}|} = \frac{|H_{31}|}{|\bar{H}|}$$

Where

$$|H_{31}| = \begin{vmatrix} U_{12} & -P_1 \\ U_{22} & -P_2 \end{vmatrix} = (-U_{12} P_1 + U_{22} P_1)$$

Therefore

$$\frac{dx_1^*}{d\beta} = \frac{|\bar{H}_{31}|}{|\bar{H}|} \gtreqless 0 \ (?)$$

Now set $dP_2 = d\beta = 0$

$$\begin{bmatrix} U_{11} & U_{12} & -P_1 \\ U_{21} & U_{22} & -P_2 \\ -P_1 & -P_2 & 0 \end{bmatrix} \begin{pmatrix} \frac{dx_1^*}{dP_1} \\ \frac{dx_2^*}{dP_1} \\ \frac{d\lambda^*}{dP_1} \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ x_1 \end{pmatrix}$$

$$\frac{dx_1^*}{dP_1} = \frac{\overbrace{\begin{vmatrix} \lambda & U_{12} & -P_1 \\ 0 & U_{22} & -P_2 \\ x_1 & -P_2 & 0 \end{vmatrix}}^{\substack{\text{Cramer's Rule} \\ \text{Expand Column 1}}}}{|\bar{H}|} = \lambda \frac{\overset{(H_{11})}{\begin{vmatrix} U_{22} & -P_2 \\ -P_2 & 0 \end{vmatrix}}}{|\bar{H}|} + x_1 \frac{\overset{(H_{31})}{\begin{vmatrix} U_{12} & -P_1 \\ U_{22} & -P_2 \end{vmatrix}}}{|\bar{H}|}$$

$$\frac{dx_1^*}{dP_1} = \lambda \frac{H_{11}}{|\bar{H}|} + x_1 \frac{H_{31}}{|\bar{H}|}$$

$H_{11} = -P_2^2 < 0$
$H_{31} = -U_{12}P_2 + U_{22}P_1$

But

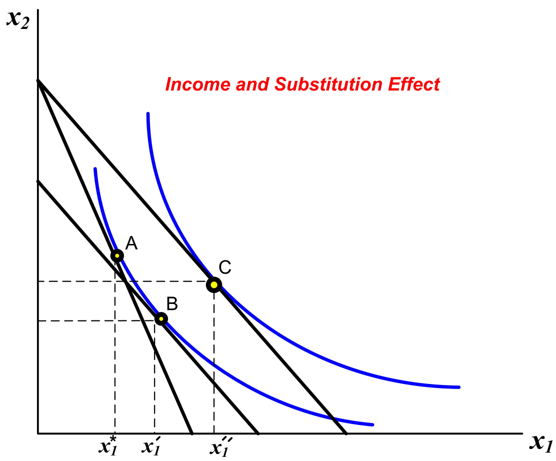
$$\frac{dx_1^*}{d\beta} = -\frac{|H_{31}|}{|\bar{H}|}$$

So

$$\frac{dx_1^*}{dP_1} = \lambda \frac{H_{11}}{|\bar{H}|} - x_1 \frac{dx_1^*}{d\beta} = \underbrace{\frac{dx_1^*}{dP_1}}_{\text{U held constant}} + (-x_1)\frac{dx_1^*}{d\beta}$$

2.2 The Slutsky Equation

$$\begin{aligned}\frac{dx_1^*}{dP_1} &= \lambda\frac{|H_{11}|}{|\bar{H}|}+x_1\frac{|H_{31}|}{|\bar{H}|}\\ &= \frac{dx_1^*}{dP_1}+(-x_1)\frac{dx^*}{d\beta}\\ &= \{\text{Pure Substitution Effect}\}+\{\text{Income Effect}\}\end{aligned}$$



A to B = $\lambda\frac{|H_{11}|}{|H|}$"Substitution Effect"

B to C = $x_1\frac{|H_{31}|}{|H|}$"Income Effect"

3 Homogenous Functions

3.1 Constant Returns to Scale

$\Longrightarrow$ Given

$$y=f(x_1,x_2,...x_n)$$

if we change all the inputs by a factor of t, then

$$f(tx_1, tx_2, ...tx_n) = tf(x_1, x_2, ...x_n) = tY$$

ie. if we double inputs, we double output

$\Longrightarrow$ A constant returns to scale production function is said to be:

HOMOGENOUS of DEGREE ONE or **LINEARLY HOMOGENOUS**

3.2 Homogenous of Degree r

A function, $Y = f(x_1, ..., x_n)$ is said to be Homogenous of Degree r if

$$f(tx_1, tx_2, ...tx_n) = t^r f(x_1, x_2, ...x_n)$$

Example

Let f$(x_1, x_2) = x_1x_2$

change all $x_i's$ by t

$$\begin{aligned} f(tx_1, tx_2) &= (tx_1)(tx_2) \\ &= t^2(x_1x_2) \\ &= t^2 f(x_1x_2) \end{aligned}$$

Therefore $f(x_1, x_2) = x_1x_2$ is homogenous of degree 2

3.3 Cobb-Douglas

Let output, $Y = f(K, L) = L^\alpha K^{1-\alpha}$ $\{$where $0 \leq 1\}$

Multiply K, L by t

$$
\begin{aligned}
f(tL, tK) &= (tL)^{\alpha}(tK)^{1-\alpha} \\
&= t^{\alpha+1-\alpha}L^{\alpha}K^{1-\alpha} \\
& \quad tL^{\alpha}K^{1-\alpha}
\end{aligned}
$$

Therefore L$^{\alpha}K^{1-\alpha}$is H.O.D one.
General Cobb-Douglas: y=L$^{\alpha}K^{\beta}$

$$
\begin{aligned}
f(tL, tK) &= (tL)^{\alpha}(tK)^{\beta} \\
&= t^{\alpha+\beta}L^{\alpha}K^{\beta}
\end{aligned}
$$

L$^{\alpha}K^{\beta}$ is homogenous of degree $\alpha + \beta$

3.4 Further properties of Cobb-Douglas

Given

$$y = L^{\alpha}K^{1-\alpha}$$

$$
\begin{aligned}
MP_L &= \frac{dY}{dL} = dL^{\alpha-1}K^{1-\alpha} = \alpha\left(\frac{K}{L}\right)^{1-\alpha} \\
MP_K &= \frac{dY}{dK} = (1-\alpha)L^{\alpha}K^{-\alpha} = (1-\alpha)\left(\frac{K}{L}\right)^{-\alpha}
\end{aligned}
$$

MP$_L$ and MP$_K$ are homogenous of degree zero

$$MP_L(tL, tK) = \alpha\left(\frac{tK}{tL}\right)^{1-\alpha} = \alpha\left(\frac{K}{L}\right)^{1-\alpha}$$

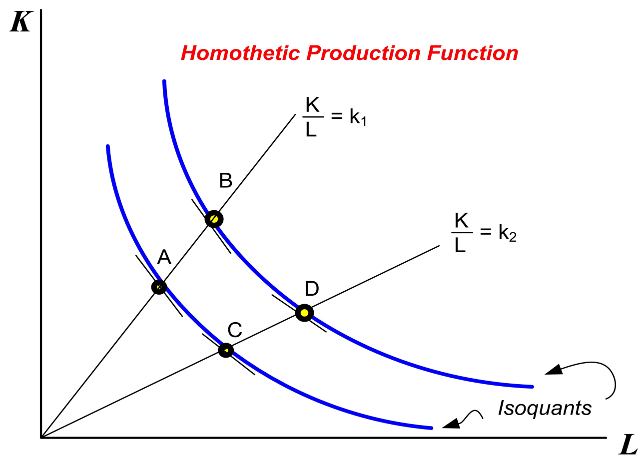
MP$_L$ and MP$_K$ depend only on the $\frac{K}{L}$ratio

3.5 The Marginal Rate of Technical Substitution

$$MRTS = \frac{MP_L}{MP_K} = \frac{\alpha(\frac{K}{L})^{1-\alpha}}{(1-\alpha)(\frac{K}{L})^{-\alpha}} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{K}{L}\right)$$

MRTS is homogenous of degree zero

The slope of the isoquant (MRTS) depends only on the $\frac{K}{L}$ ratio, not the absolute levels of K and L



Along any ray from the origin the isoquants are parallel. This is true for all homogenous functions regardless of the degree.

Given:

$$f(tx_1, ...tx_n) = t^r f(x_1, ...x_n)$$

Differentiate both sides with respect to x_1

$$\frac{df}{d(tx)}\frac{d(tx_1)}{dx_1} = t^r \frac{df}{dx_1}$$

But

$$\frac{d(tx_1)}{dx_1} = t$$
$$\frac{df}{d(tx_1)}t = t^r \frac{df}{dx_1}$$

$$\frac{df}{d(tx_1)} = \frac{t^r}{t}\frac{df}{dx_1} = t^{r-1}\frac{df}{dx_1}$$

Therefore: For any function homogenous of degree r, that function's first partial derivatives are homogenous of degree $r-1$.

3.6 Monotonic Transformations and Homothetic Functions

Let $y = f(x_1, x_2)$and Let $z = g(y)$
{where $g`(y) > 0$ and $f(x_1, x_2)$ is H.O.D. r}
$g(y)$ is a monotonic transformation of y

We know:

$$MRTS = -\frac{f_1}{fx} = \frac{dx_2}{dx_1}$$

Totally differentiate $z = g(y)$ and set $dz = 0$

$$dz = \frac{dg}{dy}\frac{dy}{dx_1}dx_1 + \frac{dg}{dy}\frac{dy}{dx_2}dx_2 = 0$$

or

$$\frac{dx_2}{dx_1} = \frac{-\left(\frac{dg}{dy_1}\right)\left(\frac{dy}{dx_1}\right)}{\left(\frac{dg}{dy_1}\right)\left(\frac{dy}{dx_2}\right)} = \frac{-\left(\frac{dy}{dx_1}\right)}{\left(\frac{dy}{dx_2}\right)} = \frac{-f_1}{f_2}$$

The slope of the level curves (isoquants) are invariant to monotonic transformations.

A monotonic transformation of a homogenous function creates a **homothetic function**

Homothetic functions have the same slope properties along a ray from the origin as the homogenous function.

However, homothetic functions are NOT homogenous.

Example: Let $f(x_1, x_2) = x_1, x_2$ {where $r = 2$}

Let:

$$\begin{aligned} z &= g(y) = \ln(x_1, x_2) \\ &= \ln x_1 + \ln x_2 \\ g(f(tx_1, tx_2)) &= \ln(tx_1) + \ln(tx_2) \\ &= 2\ln t + \ln x_1 + \ln x_2 \\ &\neq t^r \ln(x_1, x_2) \end{aligned}$$

Properties of Homothetic Functions

1. A homothetic function has the same shaped level curves as the homogenous function that was transformed to create it.

2. Homogenous production functions cannot produce U-shaped average cost curves, but a homothetic function can.

3. Slopes of Level Curves (ie. Indifference Curves)

   For homothetic functions the slope of their level curves only depend on the ratio of quantities.

   ie. <u>If</u>: $y = f(x_1, x_2)$ is homothetic

   Then: $\frac{f_1}{f_2} = g\left(\frac{x_2}{x_1}\right)$

3.7 Euler's Theorem

Let $f(x_1, x_2)$ be homogenous of degree r

Then $f(tx_1, tx_2) = t^r f(x_1, x_2)$

Differentiate with respect to t

$$\frac{df}{d(tx_1)}\frac{d\,(tx_1)}{dt} + \frac{df}{d(tx_2)}\frac{d(tx_2)}{dt} = rt^{r-1}f(tx_1, tx_2)$$

Since: $\frac{dtx_i}{dt} = x_i$ for all i

$$\frac{df}{d(tx_1)}x_1 + \frac{df}{d(tx_2)}x_2 = rt^{r-1}f(tx_1, tx_2)$$

This is true for all values of t, so let $t = 1$

$$\underbrace{\frac{df}{dx_1}x_1 + \frac{df}{dx_2}x_2 = f_1x_1 + f_2x_2 = rf(x_1, x_2)}_{\text{"Euler's Theorm"}}$$

If $y = f(L, K)$ is constant returns to scale

Then y=MP$_L L + MP_K K$ (Euler's Theorm)

Example: Let

$$y = L^\alpha K^{1-\alpha}$$

Where:

$$MP_L = \alpha L^{\alpha-1}K^{1-\alpha}$$

$$MP_K = (1-\alpha)L^\alpha K^{-\alpha}$$

From Euler's Theorm

$$
\begin{aligned}
y &= MP_L L + MP_K K = \left(\alpha L^{\alpha-1}K^{1-\alpha}\right) L + \left((1-\alpha)L^{\alpha}K^{-\alpha}\right) K \\
&= \alpha L^{\alpha-1}K^{1-\alpha} + (1-\alpha)L^{\alpha}K^{-\alpha} \\
&= \left[d + (1-\alpha)\right] L^{\alpha}K^{1-\alpha} \\
&= L^{\alpha}K^{1-\alpha} \\
&= y
\end{aligned}
$$

3.7.1 Euler's Theorm and Long Run Equilibrium

Suppose $q = f(K, L)$ is H.O.D 1

Then the profit function for a perfectly competitive firm is

$$
\begin{aligned}
\pi &= pq - rK - wL \\
\pi &= pf(K, L) - rK - wL
\end{aligned}
$$

<u>F.O.C's</u>

$$
\begin{aligned}
\frac{d\pi}{dL} &= pf_L - w = 0 \\
\frac{d\pi}{dK} &= pf_K - r = 0
\end{aligned}
$$

$\{f_L = MP_L \qquad f_K = MP_K\}$

or $MP_L = \frac{w}{p}, MP_K = \frac{r}{p}$ are necessary conditions for Profit Maximization

Therefore, at the optimum

$$\pi^* = pf(K^*L^*) - wL^* - rK^*$$

From Euler's Theorem

$$f(K^*L^*) = MP_K K^* + MP_L L^*$$

Substitute into π^*

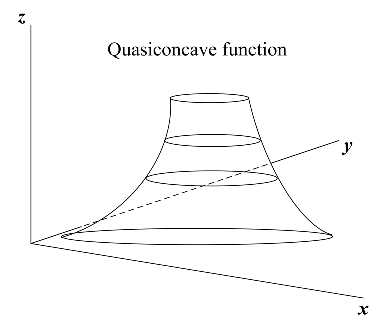
$$\pi^* = P\left[MP_K K^* + MP_L L^*\right] - wL^* - rK^*$$

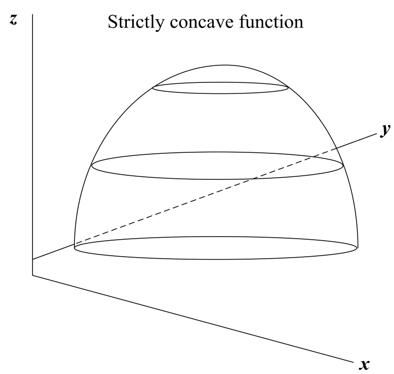
OR

$$\pi^* = \underbrace{[wL^* + rK^*] - wL^* - rK^* = 0}_{\text{Long Run } \pi=0}$$

3.7.2

Concavity and Quasiconcavity





3.7.3 Concavity:

· Convex level curves and concave in scale
  · Necessary for unconstrained optimum

3.7.4 Quasi-Concavity:

· Only has convex level curves
  · Necessary for constrained optimum

Example:

1. Concave: $y = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$ is H.O.D. 2/3 (diminishing returns)

$$MRTS = \frac{x_2}{x_1}$$

2. Quasi -Concave: $y = x_1^2 x_2^2$ is H.O.D. 4 (increasing returns)

$$MRTS = \frac{x_2}{x_1}$$

REVIEW: When to use the Implicit Function Theorem (Jacobian)

GENERAL FORM:
Max

$$U(x,y) + \lambda(\beta - P_x x - P_y y)$$

F.O.C.

$$\begin{aligned} L_x = U_x - \lambda P_x = 0 & \quad \text{(Eq 1)} \\ L_y = U_y - \lambda P y = 0 & \quad \text{(Eq 2)} \\ L_\lambda = \beta - P_x x - P_y y & \quad \text{(Eq 3)} \end{aligned}$$

Equations 1, 2, and 3 IMPLICITLY DEFINE

$x^* = x^*(\beta, P_x, P_y)$
$y^* = y^*(\beta, P_x, P_y)$
$\lambda^* = \lambda^*(\beta, P_x, P_y)$

S.O.C.

$$\left|\bar{H}\right| = \begin{vmatrix} 0 & -P_x & -P_y \\ -P_x & U_{xx} & U_{xy} \\ -P_y & U_{yx} & U_{yy} \end{vmatrix} > 0 \text{ (by assumption)}$$

Find $\frac{dx^*}{dPx}$: use Implicit Function Theorem

SPECIFIC FORM:

Max

$$xy + \lambda(\beta - P_x x - P_y y)$$

F.O.C

$$\begin{aligned} L_x = y - \lambda P_x = 0 &\quad \text{(Eq 1)} \\ L_y = x - \lambda Py = 0 &\quad \text{(Eq 2)} \\ L_\lambda = \beta - P_x x - P_y y &\quad \text{(Eq 3)} \end{aligned}$$

Equations 1, 2, and 3 EXPLICITLY DEFINE

$$x^* = \tfrac{\beta}{\alpha Px} \quad y^* = \tfrac{\beta}{\alpha Py} \quad \lambda^* = \tfrac{\beta}{\alpha PxPy}$$

S.O.C.

$$\left|\bar{H}\right| = \begin{vmatrix} 0 & -P_x & -P_y \\ -P_x & 0 & 1 \\ -P_y & 1 & 0 \end{vmatrix} = 2PxPy > 0$$

To find: $\frac{dx^*}{dPx}$ Differentiate x^* directly

$$\frac{dx^*}{dPx} = -\frac{\beta}{\alpha Px^2} < 0$$

3.8 Review: When to use the Implicit Function Theorem (Jacobian)??

3.8.1 General Form

Max

$$U(x, y) + \lambda(B - P_x x + P_y y)$$

F.O.C.

$$\begin{aligned} L_x : U_x - \lambda P_x = 0 &\qquad \text{Eq. 1} \\ L_y : U_y - \lambda P_y = 0 &\qquad \text{Eq. 2} \\ L_\lambda : B - P_x x + P_y y = 0 &\qquad \text{Eq. 3} \end{aligned}$$

Equations 1, 2, and 3 IMPLICITY define

$$x^* = x^*(B, P_x, P_y)$$
$$y^* = y^*(B, P_x, P_y)$$
$$\lambda^* = \lambda^*(B, P_x, P_y)$$

S.O.C.

$$\left|\bar{H}\right| = \begin{vmatrix} 0 & -P_x & -P_y \\ -P_x & U_{xx} & U_{xy} \\ -P_y & U_{yx} & U_{yy} \end{vmatrix} \underset{\text{(By Assumption)}}{> 0}$$

Find $\frac{dx^*}{dP_x}$: use Implicit Function Theorem

3.8.2 Specific Form

Max

$$xy + \lambda(B - P_x x + P_y y)$$

F.O.C

$$L_x : y - \lambda P_x = 0 \qquad \text{Eq. 1}$$
$$L_y : x - \lambda P_y = 0 \qquad \text{Eq. 2}$$
$$L_\lambda : B - P_x x + P_y y = 0 \qquad \text{Eq. 3}$$

Equations 1, 2, and 3 EXPLICITLY define

$$x^* = \frac{B}{2P_x}$$
$$y^* = \frac{B}{2P_y}$$
$$\lambda^* = \frac{B}{2P_x P_y}$$

S.O.C.

$$\left|\bar{H}\right| = \begin{vmatrix} 0 & -P_x & -P_y \\ -P_x & 0 & 1 \\ -P_y & 1 & 0 \end{vmatrix} = 2P_x P_y > 0$$

Find $\frac{dx^*}{dP_x}$: Differentiate x^* directly

$\frac{dx^*}{dP_x} = -\frac{B}{2P_x^2} < 0$