Lecture Notes for Chapter 11

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1 Optimization with More than One Variable

Suppose we want to maximize the following function

$$z = f(x, y) = 10x + 10y + xy - x^2 - y^2$$

Note that there are two unknowns that must be solved for: x and y. This function is an example of a *three-dimensional dome.* (i.e. the roof of *BC Place*)

To solve this maximization problem we use **partial derivatives.** We take a partial derivative for each of the unknown choice variables and set them equal to zero

$$\frac{\partial z}{\partial x} = f_x = 10 + y - 2x = 0 \quad \textit{The slope in the "x" direction = 0}$$
$$\frac{\partial z}{\partial y} = f_y = 10 + x - 2y = 0 \quad \textit{The slope in the "y" direction = 0}$$

This gives us a set of equations, one equation for each of the unknown variables. When you have the same number of independent equations as unknowns, you can solve for each of the unknowns.

rewrite each equation as

$$y = 2x - 10$$
$$x = 2y - 10$$

substitute one into the other

$$x = 2(2x - 10) - 10$$
$$x = 4x - 30$$
$$3x = 30$$
$$x = 10$$

similarly,

$$y = 10$$

REMEMBER: To maximize (minimize) a function of many variables you use the technique of partial differentiation. This produces a set of equations, one equation for each of the unknowns. You then solve the set of equations simulaneously to derive solutions for each of the unknowns.

1.0.1 Second order Conditions (second derivative Test)

To test for a maximum or minimum we need to check the second partial derivatives. Since we have two first partial derivative equations (f_x,f_y) and two variable in each equation, we will get four *second partials* $(f_{xx}, f_{yy}, f_{xy}, f_{yx})$

Using our original first order equations and taking the partial derivatives for each of them (a second time) yields:

$$f_x = 10 + y - 2x = 0 \quad f_y = 10 + x - 2y = 0$$

$$\begin{array}{ll} f_{xx} = -2 & f_{yy} = -2 \\ f_{xy} = 1 & f_{yx} = 1 \end{array}$$

The two partials, f_{xx}, and f_{yy} are the direct effects of of a small change in x and y on the respective slopes in in the x and y direction. The partials, f_{xy} and f_{yx} are the indirect effects, or the cross effects of one variable on the slope in the other variable's direction. For both *Maximums and Minimums*, the direct effects must outweigh the cross effects

1.1 Rules for two variable Maximums and Minimums

1. Maximum

$$\begin{aligned} f_{xx} &< 0 \\ f_{yy} &< 0 \\ f_{yy}f_{xx} - f_{xy}f_{yx} &> 0 \end{aligned}$$

2. Minimum

$$\begin{aligned} f_{xx} &> 0 \\ f_{yy} &> 0 \\ f_{yy}f_{xx} - f_{xy}f_{yx} &> 0 \end{aligned}$$

3. Otherwise, we have a *Saddle Point*

From our second order conditions, above,

$$\begin{array}{ll} f_{xx} = -2 < 0 & f_{yy} = -2 < 0 \\ f_{xy} = 1 & f_{yx} = 1 \end{array}$$

and

$$f_{yy}f_{xx} - f_{xy}f_{yx} = (-2)(-2) - (1)(1) = 3 > 0$$

therefore we have a maximum.

1.2 Using Differentials Approach

Given

$$z = f(x, y)$$

Then

$$dz = f_x dx + f_y dy$$

if

$$dx \neq 0, dy \neq 0$$

and

$$dz = 0 \text{ (critical point)}$$

Then it must be true that

$$f_x = f_y = 0 \qquad or \qquad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$$

$F_x = 0$: Means z is not changing in the x-direction
$F_y = 0$: Means z is not changing in the y-direction
This is the First Order Necessary Condition for a max or min

1.3 Second Order Conditions

Given

$$z = f(x, y)$$

The first derivative (differential) is

$$dz = f_x dx + f_y dy$$

Take the total differential a second time, treating dx and dy as constants

$$\begin{aligned} d^2z &= f_{xx}dxdx + f_{yy}dydy + f_{xy}dxdy + f_{yx}dydx \\ &= f_{xx}dx^2 + f_{yy}dy^2 + f_{xy}dxdy + f_{yx}dydx \end{aligned}$$

where

$$\underbrace{\begin{array}{c} f_{xx} = \text{ \underline{2nd} partial derivative with respect to x} \\ f_{yy} = \text{ \underline{2nd} partial derivative with respect to y} \\ f_{xy} = \text{ Change in } \left(\frac{\partial z}{\partial x}\right) \text{ from a } \underline{\Delta\ in\ y} \\ f_{yx} = \text{ Change in } \left(\frac{\partial z}{\partial y}\right) \text{ from a } \underline{\Delta\ in\ x} \end{array}}_{f_{xy}, f_{yx} \text{ are cross partial derivatives}}$$

1.4 Example: Two Market Monopoly with Joint Costs

A monopolist offers two different products, each having the following market demand functions

$$\begin{array}{c} q_1 = 14 - \frac{1}{4}p_1 \\ q_2 = 24 - \frac{1}{2}p_2 \end{array}$$

The monopolist's joint cost function is

$$C(q_1, q_2) = q_1^2 + 5q_1q_2 + q_2^2$$

The monopolist's profit function can be written as

$$\pi = p_1q_1 + p_2q_2 - C(q_1, q_2) = p_1q_1 + p_2q_2 - q_1^2 - 5q_1q_2 - q_2^2$$

which is the function of four variables: p_1, p_2, q_1,and q_2. Using the market demand functions, we can eliminate $p_1$and p_2 leaving us with a two variable maximization problem. First, rewrite the demand functions to get the inverse functions

$$p_1 = 56 - 4q_1$$
$$p_2 = 48 - 2q_2$$

Substitute the inverse functions into the profit function

$$\pi = (56 - 4q_1)q_1 + (48 - 2q_2)q_2 - q_1^2 - 5q_1q_2 - q_2^2$$

The first order conditions for profit maximization are

$$\frac{\partial \pi}{\partial q_1} = 56 - 10q_1 - 5q_2 = 0$$
$$\frac{\partial \pi}{\partial q_2} = 48 - 6q_2 - 5q_1 = 0$$

Solve the first order conditions using Cramer's rule. First, rewrite in matrix form

$$\begin{bmatrix} 10 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 56 \\ 48 \end{bmatrix}$$

where $|A| = 35$

$$q_1^* = \frac{\begin{vmatrix} 56 & 5 \\ 48 & 6 \end{vmatrix}}{35} = 2.75$$

$$q_2^* = \frac{\begin{vmatrix} 10 & 56 \\ 5 & 48 \end{vmatrix}}{35} = 5.7$$

Using the inverse demand functions to find the respective prices, we get

$$p_1^* = 56 - 4(2.75) = 45$$
$$p_2^* = 48 - 2(5.7) = 36.6$$

From the profit function, the maximum profit is

$$\pi = 213.94$$

Next, check the second order conditions to verify that the profit is at a maximum. The various second derivatives can be set up in a matrix called a *Hessian* The Hessian for this problem is

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -10 & -5 \\ -5 & -6 \end{bmatrix}$$

The sufficient conditions are

$$\begin{array}{ll} |H_1| = \pi_{11} = -10 < 0 & \text{(First Principle Minor of Hessian)} \\ |H_2| = \pi_{11}\pi_{22} - \pi_{12}\pi_{21} = (-10)(-6) - (-5)^2 = 35 > 0 & \text{(determinant)} \end{array}$$

Therefore the function is at a maximum. Further, since the signs of $|H_1|$ and $|H_2|$ are invariant to the values of $q_1$and q_2, we know that the profit function is strictly concave.

1.5 Example: Profit Max Capital and Labour

Suppose we have the following production function

$$q = f(K, L) = L^{\frac{1}{2}} + K^{\frac{1}{2}} \qquad \begin{array}{rl} q = & \text{Output} \\ L = & \text{Labour} \\ K = & \text{Capital} \end{array}$$

Then the profit function for a competitive firm is

$$\begin{array}{ll} \pi = Pq - wL - rK & P = \text{ Market Price} \\ or & w = \text{ Wage Rate} \\ \pi = PL^{\frac{1}{2}} + PK^{\frac{1}{2}} - wL - rK & r = \text{ Rental Rate} \end{array}$$

First order conditions

		General Form
1.	$\frac{\partial\pi}{\partial L} = \frac{P}{2}L^{\frac{-1}{2}} - w = 0$	$Pf_L - w = 0$
2.	$\frac{\partial\pi}{\partial k} = \frac{P}{2}K^{\frac{-1}{2}} - r = 0$	$Pf_K - r = 0$

Solving (1) and (2), we get

$$L^* = (\tfrac{2w}{P})^{-2} \qquad K^* = (\tfrac{2r}{P})^{-2}$$

Second order conditions (Hessian)

$$\pi_{LL} = Pf_{LL} = \tfrac{-P}{4}L^{\frac{-3}{2}} < 0$$
$$\pi_{KK} = Pf_{KK} = \tfrac{-P}{4}K^{\frac{-3}{2}} < 0$$
$$\pi_{LK} = \pi_{KL} = Pf_{LK} = Pf_{KL} = 0$$

or, in matrix form

$$H = \begin{vmatrix} \pi_{LL} & \pi_{LK} \\ \pi_{KL} & \pi_{KK} \end{vmatrix} = \begin{vmatrix} \frac{-P}{4}L^{\frac{-3}{2}} & 0 \\ 0 & \frac{-P}{4}K^{\frac{-3}{2}} \end{vmatrix}$$

$$P\left[f_{LL}f_{KK} - (f_{LK})^2\right] = \left(\frac{-P}{4}L^{\frac{-3}{2}}\right)\left(\frac{-P}{4}K^{\frac{-3}{2}}\right) - 0 > 0$$

Differentiate first order of conditions with respect to capital (K) and labour (L)

$\Longrightarrow$Therefore profit maximization

Example: If $P = 1000$, $w = 20$, and $r = 10$

1. Find the optimal K, L, and π
2. Check second order conditions

1.6 Example: Cobb-Douglas production function and a competitive firm

Consider a competitive firm with the following profit function

$$\pi = TR - TC = PQ - wL - rK$$

where P is price, Q is output, L is labour and K is capital, and w and r are the input prices for L and K respectively. Since the firm operates in a competitive market, the exogenous variables are P,w and r. There are three endogenous variables, K, L and Q. However output, Q, is in turn a function of K and L via the production function

$$Q = f(K, L)$$

which in this case, is the Cobb-Douglas function

$$Q = L^a K^b$$

where a and b are positive parameters. If we further assume decreasing returns to scale, then a + b < 1. For simplicity, let's consider the symmetric case where $a = b = \frac{1}{4}$

$$Q = L^{\frac{1}{4}} K^{\frac{1}{4}}$$

Substituting Equation 3 into Equation 1 gives us

$$\pi(K, L) = PL^{\frac{1}{4}} K^{\frac{1}{4}} - wL - rK$$

The first order conditions are

$$\begin{array}{l} \frac{\partial \pi}{\partial L} = P\left(\frac{1}{4}\right) L^{-\frac{3}{4}} K^{\frac{1}{4}} - w = 0 \\ \frac{\partial \pi}{\partial K} = P\left(\frac{1}{4}\right) L^{\frac{1}{4}} K^{-\frac{3}{4}} - r = 0 \end{array}$$

This system of equations define the optimal L and K for profit maximization. But first, we need to check the second order conditions to verify that we have a maximum.

The Hessian for this problem is

$$H = \begin{bmatrix} \pi_{LL} & \pi_{LK} \\ \pi_{KL} & \pi_{KK} \end{bmatrix} = \begin{bmatrix} P(-\frac{3}{16})L^{-\frac{7}{4}}K^{\frac{1}{4}} & P\left(\frac{1}{4}\right)^2 L^{-\frac{3}{4}}K^{-\frac{3}{4}} \\ P\left(\frac{1}{4}\right)^2 L^{-\frac{3}{4}}K^{-\frac{3}{4}} & P\left(-\frac{3}{16}\right) L^{\frac{1}{4}}K^{\frac{7}{4}} \end{bmatrix}$$

The sufficient conditions for a maximum are that $|H_1| < 0$ and $|H| > 0$. Therefore, the second order conditions are satisfied.

We can now return to the first order conditions to solve for the optimal K and L. Rewriting the first equation in Equation 5 to isolate K

$$P\left(\frac{1}{4}\right) L^{-\frac{3}{4}}K^{\frac{1}{4}} = w$$
$$K = (\tfrac{4w}{p}L^{\frac{3}{4}})^4$$

Substituting into the second equation of Equation 5

$$\tfrac{P}{4}L^{\frac{1}{4}}K^{-\frac{3}{4}} = \left(\tfrac{P}{4}\right) L^{\frac{1}{4}} \left[\left(\tfrac{4w}{p}L^{\frac{3}{4}}\right)^4\right]^{-\frac{3}{4}} = r$$
$$= P^4 \left(\tfrac{1}{4}\right)^4 w^{-3}L^{-2} = r$$

Re-arranging to get L by itself gives us

$$L^* = (\frac{P}{4}w^{-\frac{3}{4}}r^{-\frac{1}{4}})^2$$

Taking advantage of the symmetry of the model, we can quickly find the optimal K

$$K^* = (\frac{P}{4}r^{-\frac{3}{4}}w^{-\frac{1}{4}})^2$$

L* and K^* are the firm's factor demand equations.

1.7 Young's Theorem

For cross partial "effects" the order of differentiation is immaterial. Therefore:

$$f_{xy} = f_{yx}$$

As long as the cross partials are continuous.

In the case of GENERAL FUNCTIONS , this will always be assumed to be true!

1.7.1 Example:

$$z = x^3 + 5xy = y^2$$

$$\underbrace{\begin{array}{cc} \underline{f_x = 3x^2 + 5y} & \underline{f_y = 5x - 2y} \\ f_{xx} = 6x & f_{yy} = -2 \\ f_{xy} = 5 & f_{yx} = 5 \end{array}}_{\text{f}_{xy}=\text{f}_{yx}}$$

1.8 Quadratic Form

The function

$$q = ax^2 + 2bxy + cy^2$$

is a quadratic form. A quadratic can be written in matrix form as:

$$\underset{1x1}{q} = \underset{1x2}{\begin{bmatrix} x & y \end{bmatrix}} \underset{2x2}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \underset{2x1}{\begin{bmatrix} x \\ y \end{bmatrix}}$$

$Det = (ac - b^2)$

1.9 Positive and Negative Definiteness

q is said to be

1. (a) i. positive definite
       ii. positive semi-definite
       iii. negative semi-definite
       iv. negative definite

IF q is always $> 0, \geq 0, \leq 0, < 0$ (for all x, y)

$$\text{q is } \begin{Bmatrix} \text{positive definite} \\ \text{negative definite} \end{Bmatrix} \text{ if } \begin{Bmatrix} a > 0 \\ a < 0 \end{Bmatrix} \text{ and } (ac - b^2) > 0$$

The second order total differential
$d^2z = (f_{xx})dx^2 + (2f_{xy})dxdy + (f_{yy})dy^2 \quad \{Young\prime sTheorem \quad f_{xy} = f_{yx}\}$
Therefore

$$d^2z = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

The matrix of 2nd partial derivatives is called the Hessian

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \text{ where } |H| = f_{xx}f_{yy} - f_{xy}^2$$

Second Order Conditions

$$\text{q is } \begin{Bmatrix} \text{positive definite} \\ \text{negative definite} \end{Bmatrix}$$

if $\{f_{xx} > 0, f_{xx} < 0\}$, and $\left\{f_{xz}f_{yy} - f_{xy}^2 > 0\right\}$

Note: $f_{xz}f_{yy} - f_{xy}^2 > 0$ implies that f_{yy} must have the same sign as f_{xx}

Therefore if:

1. $f_x = f_y = 0$ (FOC), and if:
2. SOC

   $$d^2z \text{ is } \begin{Bmatrix} \text{negative definite} \\ \text{positive definite} \end{Bmatrix} \text{ then z is } \begin{cases} \text{a maximum} \\ \text{a minimum} \end{cases}$$

1.10 n-Variable Case

Given

$$z = f(x_1, x_2, ...x_n)$$

For an Extremum (max or min):

$$f_1 = f_2 = f_3 = ... = f_n = 0 \quad \text{(First Order Conditions)}$$

Then d^2z in Matrix Form is

$$\underset{(1xn)}{\begin{bmatrix} dx_1 & dx_2 & ... & dx_n \end{bmatrix}} \underset{(nxn)}{\begin{bmatrix} f_{11} & f_{12} & f_{13} & ... & f_{1n} \\ f_{21} & f_{22} & ... & ... & f_{2n} \\ f_{31} & & f_{33} & ... & ... \\ ... & ... & ... & ... & ... \\ f_{n1} & ... & ... & ... & f_{nn} \end{bmatrix}} \underset{(nx1)}{\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \\ ... \\ dx_n \end{bmatrix}}$$

For a Max: (Principal minors alternate signs)
$|H_1| = f_{11} < 0, |H_2| = f_{11}f_{22} - f_{12}^2 > 0, |H_3| < 0, ...(-1)^n |H_n| > 0$
For a Min: (Principal minors have the same sign)
$|H_1| > 0, |H_2| > 0, ... |H_n| > 0$

1.10.1 Example: Output Maximization

Let

$$Q = 10L + 10K + LK - L^2 - K^2$$

F.O.C.'s

$$\begin{bmatrix} \frac{\partial Q}{\partial L} = 10 + K - 2L = 0 \\ \frac{\partial Q}{\partial K} = 10 + L - 2K = 0 \end{bmatrix} \text{ OR } \begin{Bmatrix} 2L - K = 10 \\ -L + 2K = 10 \end{Bmatrix}$$

2 Equations with 2 unknowns from FOC. Matrix Form:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} L \\ K \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix} \qquad \{Det = 3\}$$

Cramer's Rule

$$L = \frac{\begin{vmatrix} 10 & -1 \\ 10 & 2 \end{vmatrix}}{3} = \frac{20 + 10}{3} = 10$$

$$K = \frac{\begin{vmatrix} 2 & 10 \\ -1 & 10 \end{vmatrix}}{3} = \frac{20 + 10}{3} = 10$$

Now check 2nd order conditions from F.O.C

$$\left.\begin{matrix} 10 + K - 2L = 0 \\ 10 + L - 2K = 0 \end{matrix}\right\} \begin{matrix} \text{when K,L=10 FOC's} \\ \text{are identities} \end{matrix}$$

S.O.C.

$$dK - 2dL = 0$$
$$dL - 2dK = 0$$

Find the Hessian

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} dL \\ dK \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$H = \begin{bmatrix} Q_{LL} & Q_{LK} \\ Q_{KL} & Q_{KK} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\underbrace{Q_{LL} = -2 < 0 \quad Q_{LL}Q_{KK} - Q_{KL}^2 = (-2)(-2) - (1) > 0}_{\underline{\text{Therefore Q is Max at K=10, L=10}}}$$

1.11 Economic Interpretation of the 2nd Order Conditions

Given the production function

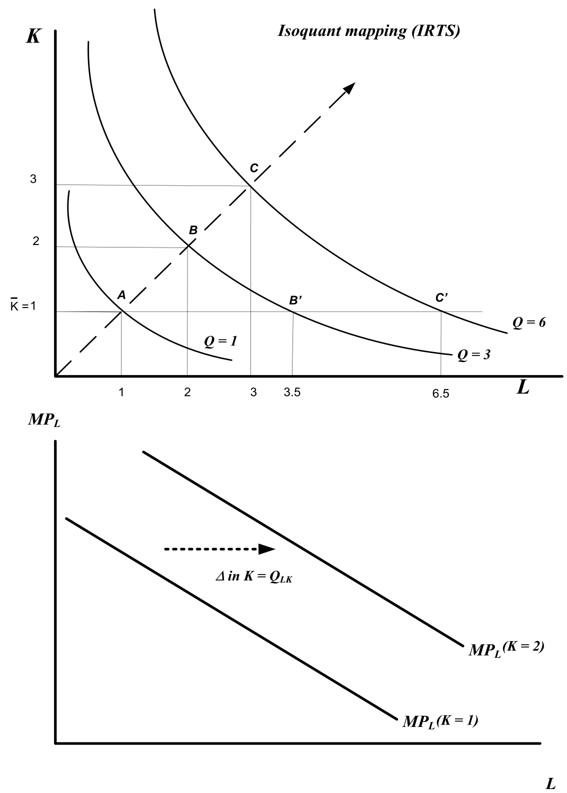
$$Q = Q(K, L)$$

$Q_L > 0, Q_{LL} < 0$ implies the "Law of Diminishing Returns"

The condition

$$Q_{LL}Q_{KK} - Q_{KL}^2 > 0$$

1. (a) says that for a Maximum the direct effects (Q_{LL},Q_{KK}) must outweigh the indirect effects ($Q_{KL}Q_{LK}$)
   (b) a production function can have the properties of "the law of diminishing returns" and "increasing returns to scale" <u>at the same time</u>
   (c) Therefore Q(K,L) has no unconstrained maximum



1.12 Profit Maximization and Comparative Statics

Let $q = f(x_1, x_2)$be the production function.

The profit function is

$$\pi = pf(x_1, x_2) - w_1 x_1 - w_2 x_2$$

where p = output price, w_1,w_2 = factor prices of x_1 and x_2 respectively

The FOC's for profit max

$$\left.\begin{array}{l} \frac{\partial \pi}{\partial x_1} = pf_1 - w_1 = 0 \\ \frac{\partial \pi_2}{\partial x_2} = pf_2 - w_2 = 0 \end{array}\right\} \text{This gives us 2 equations and 2 unknowns, } x_1^*, x_2^*$$

Solving the F.O.C's we get:

$$x_1^* = x_1^*(w_1, w_2, P) \qquad x_2^* = x_2^*(w_1, w_2, P)$$

{x_1^*,x_2^* as functions of exogenous variables}

2nd order conditions

From the F.O.C.'s

$$\left.\begin{array}{l} pf_1 - w_1 = 0 \\ pf_2 - w_2 = 0 \end{array}\right\} \text{ at } x_1 = x_1^*, \ x_2 = x_2^*$$

Differentiate again for the Hessian

$$\begin{bmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{bmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$H = \begin{bmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{bmatrix} = p \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

For Maximum

$$|H_1| = f_{11} < 0 \qquad |H_2| = f_{11}f_{22} - f_{12}^2 > 0$$

1.13 Comparative Statics

By substituting

$$x_1^* = x_1^*(w_1, w_2, P) \qquad x_2^* = x_2^*(w_1, w_2, P)$$

back into the F.O.C.'s

$$\begin{aligned} Pf_1 &= (x_1^*(w_1, w_2, P), x_2^*(w_1, w_2, P)) - w_1 = 0 \\ Pf_2 &= (x_1^*(w_1, w_2, P), x_2^*(w_1, w_2, P)) - w_2 = 0 \end{aligned}$$

The F.O.C.'s become identities that implicitly define x_1,x_2 as functions of w_1,w_2,and P. Therefore to find $\frac{\partial x_1^*}{\partial w_1}$, $\frac{\partial x_2^*}{\partial w_1}$ etc. we can use the implicit function theorem by finding the Jacobian of the F.O.C.'s

Find: $\frac{\partial x_1^*}{\partial w_1}, \frac{\partial x_2^*}{\partial w_1}$
Totally differentiate with respect to w_1

$$\begin{aligned} Pf_{11}\frac{\partial x_1^*}{\partial w_1} + Pf_{12}\frac{\partial x_2^*}{\partial w_1} - \frac{dw_1}{dw_1} &= 0 \qquad \left\{\frac{dw_1}{dw_1} = 1\right\} \\ Pf_{21}\frac{\partial x_1^*}{\partial w_1} + Pf_{22}\frac{\partial x_2^*}{\partial w_1} &= 0 \end{aligned}$$

Matrix Form:

$$\begin{bmatrix} Pf_{11} & Pf_{12} \\ Pf_{21} & Pf_{22} \end{bmatrix} \begin{pmatrix} \frac{\partial x_1^*}{\partial w_1} \\ \frac{\partial x_2^*}{\partial w_1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The Jacobian determinant

$$|J| = P(f_{11}f_{22} - f_{12}^2) > 0$$

The Jacobian of the F.O.C.'s is also the Hessian of the S.O.C.'s

1.13.1 Solving by Cramer's Rule

$$\begin{aligned} \frac{\partial x_1^*}{\partial w_1} &= \frac{\begin{vmatrix} 1 & Pf_{12} \\ 0 & Pf_{22} \end{vmatrix}}{|H|} = \frac{Pf_{22}}{P(f_{11}f_{22} - f_{12}^2)} = \frac{f_{22}}{f_{11}f_{22} - f_{12}^2} < 0 \\ \frac{\partial x_2^*}{\partial w_1} &= \frac{\begin{vmatrix} Pf_{11} & 1 \\ Pf_{21} & 0 \end{vmatrix}}{|H|} == \frac{-f_{22}}{f_{11}f_{22} - f_{12}^2} \gtrless 0? \end{aligned}$$

$\frac{\partial x_1^*}{\partial w_1} < 0$ implies downward sloping factor demand curve. For $\frac{\partial x_2^*}{\partial w_1}$ this sign depends on the relationship in production between x_1 and x_2

1.13.2 Example: Profit Maximization

Suppose we have the following production

$$q = f(K,L) = L^{\frac{1}{2}}K^{\frac{1}{2}} \qquad \left\{\begin{matrix} q = output \\ L = labour \\ K = capital \end{matrix}\right\}$$

Then the profit function for a competitive firm is

$$\begin{aligned} \pi &= Pq - wL - rK \qquad \left\{\begin{matrix} P = market\ price \\ w = wage\ rate \\ r = rental\ rate \end{matrix}\right\} \\ or \ \ \pi &= PL^{\frac{1}{2}} + PK^{\frac{1}{2}} - wL - rK \end{aligned}$$

First Order Conditions

$$\begin{aligned} (1) \quad \frac{\partial \pi}{\partial L} &= \frac{P}{2}L^{\frac{1}{2}} - w = 0 \qquad \overbrace{\{Pf_L - w = 0\}}^{\text{General Form}} \\ (2) \quad \frac{\partial \pi}{\partial K} &= \frac{P}{2}K^{\frac{1}{2}} - r = 0 \qquad \{Pf_K - r = 0\} \end{aligned}$$

Solving (1) and (2) we get

$$L^* = \left(\tfrac{2w}{P}\right)^{-2} \qquad K^* = \left(\tfrac{2n}{P}\right)^{-2}$$

Second Order Conditions (Hessian)
Differentiate First Order Conditions with respect to K, L
General

$$\begin{aligned} Pf_{LL}dL + Pf_{LK}dK &= 0 \\ Pf_{KL}dL + Pf_{KK}dK &= 0 \end{aligned}$$

Hessian

$$\begin{pmatrix} Pf_{LL} & Pf_{LK} \\ Pf_{KL} & Pf_{KK} \end{pmatrix} \begin{pmatrix} dL \\ dK \end{pmatrix}$$

$$\begin{aligned} |H_1| &= Pf_{LL} < 0 \\ |H_2| &= P\left[f_{LL}f_{KK} - (f_{KK})^2\right] > 0 \end{aligned}$$

<u>Specific</u>

$$\begin{aligned} -\frac{P}{4}L^{\frac{-3}{2}}dL + (0)dK &= 0 \\ -\frac{P}{4}K^{\frac{-3}{2}}dL + (0)dL &= 0 \end{aligned}$$

Hessian

$$\begin{pmatrix} -\frac{P}{4}L^{\frac{-3}{2}} & 0 \\ 0 & -\frac{P}{4}K^{\frac{-3}{2}} \end{pmatrix} \begin{pmatrix} dL \\ dK \end{pmatrix}$$

$$\begin{aligned} H_1 &= -\frac{P}{4}L^{\frac{-3}{2}} \\ |H_2| &= \left(-\frac{P}{4}L^{\frac{-3}{2}}\right)\left(-\frac{P}{4}K^{\frac{-3}{2}}\right) - 0 > 0 \end{aligned}$$

$|H_2|$ for both general and specific >0, therefore Profit Max
From the FOC's we know:

$$L^* = \left(\tfrac{2w}{P}\right)^{-2} \qquad K^* = \left(\tfrac{2r}{P}\right)^{-2}$$

by subbing K* and L* into the profit function, we get:

$$\begin{aligned} \pi^* &= PL^{\frac{1}{2}} + PK^{\frac{1}{2}} - wL - rK \\ \pi^* &= P\left[\left(\frac{2w}{P}\right)^{-2}\right]^{\frac{1}{2}} + P\left[\left(\frac{2r}{P}\right)^{-2}\right]^{\frac{1}{2}} - w\left(\frac{2w}{P}\right)^{-2} - r\left(\frac{2r}{P}\right)^{-2} \\ \pi^* &= \frac{P^2}{2w} + \frac{P^2}{2r} - \frac{P^2}{4w} - \frac{P^2}{4r} \end{aligned}$$

Finally:

$$\pi^* = \pi^*(w,r,P) = \frac{P^2}{4w} + \frac{P^2}{4r}$$

where $\pi^*(w,r,P)$ is "Maximum profits as a function of w,r, and P"

1.14 Hotelling's Lemma

Hotelling's Lemma states the following conditions about the profit function:

$$1. \quad \left(\frac{\partial \pi^*(w,r,P)}{\partial P}\right) = q^*$$

$$2a. \quad -\frac{\partial \pi^*(w,r,P)}{\partial w} = L^* \qquad 2b. \quad -\frac{\partial \pi^*(w,r,P)}{\partial r} = K^*$$

Using the profit function:

$$\pi^*(w,r,P) = \frac{P^2}{4w} + \frac{P^2}{4r}$$

Condition 1:

$$\frac{\partial \pi^*}{\partial P} = \frac{2P}{4w} + \frac{2P}{4r} = \underline{\underline{\frac{P}{2w} + \frac{P}{2r}}}$$

Check:

$$\begin{aligned} q &= L^{\frac{1}{2}}K^{\frac{1}{2}} = \left[\left(\frac{2w}{P}\right)^{-2}\right]^{\frac{1}{2}} + \left[\left(\frac{2r}{P}\right)^{-2}\right]^{\frac{1}{2}} \\ &= \left(\frac{2w}{P}\right)^{-1} + \left(\frac{2r}{P}\right)^{-1} = \underline{\underline{\frac{P}{2w} + \frac{P}{2r}}} \end{aligned}$$

Condition 2a

$$-\frac{\partial \pi^*(w,r,P)}{\partial w} = -\frac{\partial}{\partial w}\left[\frac{P^2}{4w} + \frac{P^2}{4r}\right] = -\left(-\frac{P^2}{4w^2}\right) = \underline{\left(\frac{2w}{P}\right)^{-2}}$$

Therefore $-\frac{\partial \pi^*}{\partial w} = L^*$

Condition 2b

$$-\frac{\partial \pi^*(w,r,P)}{\partial r} = -\left(-\frac{P^2}{4r^2}\right) = \underline{\left(\frac{2r}{P}\right)^{-2}} = K^*$$

1.14.1 Factor Demand Curves

L* and K* are the firms demand curves for labour and capital

$$\begin{aligned} L^* &= \frac{P^2}{4w^2} \implies \frac{\partial h^*}{\partial w} = -\frac{P^2}{4w^3} < 0 \\ K^* &= \frac{P^2}{4r^2} \implies \frac{\partial K^*}{\partial r} - \frac{P^2}{4r^3} < 0 \end{aligned}$$

Therefore: Downward sloping factor demand curves

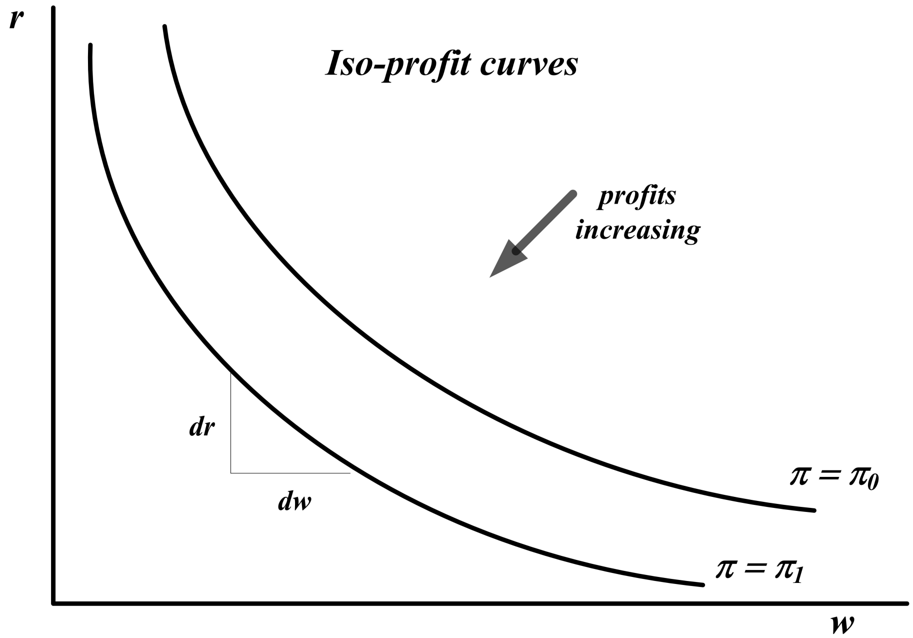
1.15 Iso-Profit Curves (Level Curves)

Take the total differential of $\pi^*(w,r,P)$; let d$\pi^* = 0$

$$\begin{aligned} d\pi^* &= -\frac{P^2}{4w^2}dw + -\frac{P^2}{4r^2}dr = 0 \\ \frac{dr}{dw} &= -\frac{\frac{P^2}{4w^2}}{\frac{P^2}{4r^2}} = -\frac{r^2}{w^2} < 0 \quad \text{(slope of Iso-Profit Curve)} \end{aligned}$$

Concave or Convex?

$$\frac{d}{dw}\left(\frac{dr}{dw}\right) = -\left(-2\frac{r^2}{w^3}\right) = 2\frac{r^2}{w^3} > 0$$



Therefore the slope of the Iso-Profit curve is negative $\left(\frac{dr}{dw}\right)$ but the slope is becoming less negative: $\left(\frac{d^2r}{dw^2}\right) > 0$ Therefore: Convex

1.16 Profit Maximization

Developing the profit function

$$\pi = TR - TC$$

where

$$\pi = PQ - C(Q)$$

Therefore profit max is:

$$\frac{\partial \pi}{\partial Q} = \frac{\partial TR}{\partial Q} - \frac{\partial C}{\partial Q} = MR - MC = 0 \qquad \text{Q is the choice variable}$$

Now suppose

$$Q = f(K, L)$$

Then

$$\pi = P \cdot f(K, L) - (wL + rK) \qquad \text{where TC=wL+rK}$$

Now profit max is:

$$\begin{array}{ll} (1) & \frac{\partial \pi}{\partial L} = Pf_L - w = 0 \\ (2) & \frac{\partial \pi}{\partial K} = Pf_K - r = 0 \end{array}$$

Now K,L are the choice variables.

The solution $\begin{Bmatrix} K^* = K^*(w, r, P) \\ L^* = L^*(w, r, P) \end{Bmatrix}$ are demand curves

Now suppose the firm is a monopolist, then he faces a downward sloping demand curve

$$P = D(Q)$$

Profit function is

$$\pi = D(Q)Q - wL - rK$$

where

$$Q = f(K, L)$$

Differentiate using the Chain Rule
F.O.C.

$$\begin{aligned}\pi_L &: \left[D(Q) + QD'(Q)\right] f_L - w = 0 \\ \pi_K &: \left[D(Q) + QD'(Q)\right] f_K - r = 0\end{aligned}$$

OR

$$\begin{aligned}MR \cdot MP_L - w &= 0 \\ MR \cdot MP_L - r &= 0\end{aligned}$$

OR

$$\begin{aligned}\left[D(f(K,L)) + f(K,L)D'(f(K,L))\right] f_L - w &= 0 \\ \left[D(f(K,L)) + f(K,L)D'(f(K,L))\right] f_K - r &= 0\end{aligned}$$

Giving

$$K^* = K^*(w, r) \qquad L^* = L^*(w, r)$$

2 Chapter 11: Part 2 - Price Disc.

2.0.1 Example

Let

$$P_1 = 100 - q_1 \quad P_2 = 150 - 2q_2 \quad \text{Mkt. AR Functions}$$

Let

$$\begin{aligned}TC &= 100 + (q_1 + q_2)^2 \\ \pi &= P_1 q_1 + P_2 q_2 - 100 - (q_1 + q_2)^2 \\ \pi &= 100q_1 - q_1^2 + 150q_2 - 2q_2^2 - 100 - (q_1 + q_2)^2\end{aligned}$$

FOC's

$$\begin{aligned} \pi_1 &= 100 - 2q_1 - 2(q_1 + q_2) = 100 - 4q_1 - 2q_2 = 0 \\ \pi_2 &= 150 - 4q_2 - 2((q_1 + q_2) = 150 - 2q_1 - 6q_2 = 0 \end{aligned}$$

$$\begin{aligned} q_1 &= \frac{\begin{vmatrix} 100 & 2 \\ 150 & 6 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ 2 & 6 \end{vmatrix}} = \frac{600 - 300}{20} = 15 \\ q_2 &= \frac{\begin{vmatrix} 4 & 100 \\ 2 & 150 \end{vmatrix}}{20} = \frac{600 - 200}{20} = 20 \end{aligned}$$

$$P_1^* = 85 \qquad P_2^* = 110$$

SOC's

$$|H| = \begin{vmatrix} -4 & -2 \\ -2 & -6 \end{vmatrix} \qquad \mathrm{H}_1 = -4 < 0 \qquad \mathrm{H}_2 = 20 > 0$$

Therefore a Max

**At home, verify that the Inverse Elasticity Rule holds here!

2.1 Concavity and Convexity

Let

$$y = f(\bar{x})$$

where

$$\bar{x} = [\bar{x}_1, ...\bar{x}_n] \quad \text{and let} \quad \hat{x} = [\hat{x}_1, ...\hat{x}_n]$$

such that $\bar{\mathrm{x}} \neq \hat{\mathrm{x}}$

Definition 1:
y=f($\bar{x}$) is a concave function if

$$\underbrace{f\left(k \cdot \bar{x} + (1-k) \cdot \hat{x}\right)}_{\text{Point on Dome}} \quad \geq \quad \underbrace{kf(\bar{x}) + (1-k)f(\hat{x})}_{\text{Line Segment}}$$

Definition 2:
y=f($\bar{x}$) is convex if

$$f(k\bar{x} + (1-k)\hat{x}) \quad \leq \quad kf(\bar{x}) + (1-k)f(\hat{x})$$

for strict concavity/convexity replace the weak inequalities with strict inequalities.

If the function y=f($\bar{x}$) is twice differentiable, then the following holds:

Theorem 1: y=f($\bar{x}$) is concave/convex if and only the Hessian, $|H|$ is negative/positive semidefinite

Theorem 2: If the Hessian is negative definite/positive definiate for all x, then y=f(x) is concave/convex

NOTE: Theorem 2 is a sufficient condition for strict concavity/convexity but it is not a necessary condition

2.2 Limit Output Model

Suppose a monopolist faces the following demand curve

$$p = a - q \qquad \text{a is a constant} > 0$$

His cost function is

$$TC = k + cq \qquad \text{where K = set up costs, cq = variable costs}$$

Therefore

$$ATC = \frac{k}{q} + c \qquad \{= AFC + AVC\}$$

The profit function is

$$\pi = pq - (K + cq)$$

Maximize

$$\frac{\partial \pi}{\partial q} = a - 2q - c = 0 \quad \longrightarrow \quad q = \frac{a-c}{2}$$

$$p = a - 1 = a - \left(\frac{a-c}{2}\right) = \frac{a+c}{2}$$

Set MR=MC

$$\begin{aligned} a - 2q &= c \\ q &= \frac{a-c}{2} \end{aligned}$$

Monopolists profit max graphically
[1]

Now consider a potential entrant to the monopolist‘s market
Assumption: Entrant takes monopolist‘s output as given
Let

$$\begin{aligned} q_e &= Entrant's\ Output \\ q_m &= Monopolist's\ Output \end{aligned}$$

[1] Graph - page 5 Cha. 11 part 2

If entrant does enter, market price will be:

$$p = a - (q_m - q_e)$$

Entrant's profits

$$\begin{aligned}
\pi &= pq_e - k - cq_e \\
\pi_e &= (a - q_e - q_m)q_e - k - cq_e \\
\frac{\partial \pi_e}{\partial q_e} &= a - q_m - 2q_e - c = 0 \\
q_e &= \frac{a - c - q_m}{2} \qquad \text{Entrant's output is a function of the monopolist's output.}
\end{aligned}$$

Entrant's output

$$q_e = \frac{a - c - q_m}{2}$$

Sub into profit function

$$\begin{aligned}
\pi_e &= (a - q_e - q_m)q_e - k - cq_e \\
\pi_e &= (a - q_m)\left(\frac{a - c - q_m}{2}\right) - \left(\frac{a - c - q_m}{2}\right)^2 - k - c\left(\frac{a - c - q_m}{2}\right)
\end{aligned}$$

Entrant's profit function is a function of a, c, k, and q_m

He will enter if: $\pi_e > 0$ OR if: $(a - q_m - q_e)q_e - cq_e > k$

Which says: If an entrant's profits (gross) can cover fixed costs (k) then he will enter the market of the monopolist.

Graphically:

· Entrant takes monopolist's q_m as given

· Entrant maximizes profits off the residual demand curve

MONOPOLIST'S DEMAND CURVE

[2]

[2] insert first graph on page 8 chap 11 part 2

· B=Entrants profit above variable costs
· if B>k then the entrant will enter
· if B<k then there will be no entry
RESIDUAL DEMAND CURVE
[3]

The monopolist knows that

$$q_e^* = \frac{a-c-q_m}{2}$$

or generally$q_e^* = f(q_m)$Therefore the monopolist can effect the entrant's choice q_e^*

The monopolist can choose q$_m$ such that when the entrant chooses the optimal q_e^* he will not earn any profits

Therefore the monopolists maximization problem is:

MAX:

$$\pi_m = (a - q_m) - q_m - k - cq_m$$

Subject to:

$$\pi_e = (a - q_m - q_e)q_e - cq_e \leq k$$

Substitute

$$q_e = \frac{a-c-q_m}{2}$$

into the monopolist's max problem, Max

$$aq_m - q_m^2 - cq_m - k$$

subject to

$$(a-q_m)\left[\frac{a-c-q_m}{2}\right] - \left[\frac{a-c-q_m}{2}\right]^2 - c\left[\frac{a-c-q_m}{2}\right] = K$$

Notice that there is now only one choice variable, q_m.

[3] insert second graph page 8 ch. 11 part 2

There q_m^* is determined by the constant
Without differentiating solve the constraint for q_m^*

Answer:

$$q_m^* = a - c - \sqrt[2]{k}$$

[4]

2.3 Cournot Duopoly

Suppose the monopolist decides to allow entry. The result: Duopoly

Assumption: Each firm takes the other firms output as exongenous and chooses the output to maximize its own profits

Market Demand:

$$\begin{aligned} P &= a - bq \\ \text{or } P &= a - b(q_1 + q_2) \qquad \{q_1 + q_2 = q\} \end{aligned}$$

where q_i is firm i's output $\{i = 1, 2\}$
Each firm faces the same cost function

$$TC = K + cq_i \qquad \{i = 1, 2\}$$

Each firm's profit function is:

$$\pi_i = pq_i - cq_i - K$$

Firm 1:

$$\begin{aligned} \pi_1 &= pq_1 - cq_1 - K \\ \pi_1 &= (a - bq_1 - bq_2)q_1 - cq_1 - K \end{aligned}$$

[4] GRAPH page 11

Max π_1, treating q$_2$ as a constant

$$\begin{aligned} \frac{\partial \pi_1}{\partial q_1} &= a - bq_2 - 2bq_1 - c = 0 \\ 2bq_1 &= a - c - bq_2 \\ q_1 &= \frac{a-c}{2b} - \frac{q_2}{2} \qquad \longrightarrow \quad \text{"Best Response Function"} \end{aligned}$$

Best Response Function: Tells firm 1 the profit maximizing q_1 for any level of q_2

For Firm 2:

$$\pi_2 = (a - bq_1 - bq_2)q_2 - cq_2 - K$$

Max π_2 (treating q_1 as a constant) gives

$$q_2 = \frac{a-c}{2b} - \frac{q_1}{2} \quad \text{Firm 2's Best Response Function}$$

The two "Best Response" Functions

$$\begin{array}{ll} (1) & q_1 = \frac{a-c}{2b} - \frac{q_2}{2} \\ (2) & q_2 = \frac{a-c}{2b} - \frac{q_1}{2} \end{array}$$

gives us two equations and two unknowns.

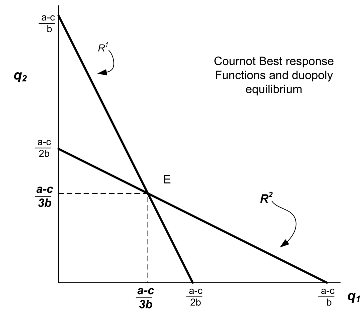
The solution to this system of equations is the equilibrium to the "Cournot Duopoly" game

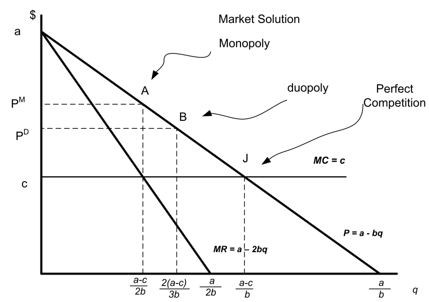
Using Cramer's Rule:

$$\begin{array}{ll} (1) & q_1^* = \frac{a-c}{3b} \\ (2) & q_2^* = \frac{a-c}{3b} \end{array}$$

$$\text{Market Output} \; : \; q_1^* + q_2^* = \frac{2(a-c)}{3b}$$

Best Response Functions Graphically





2.4 Stackelberg Duopoly

In the Cournot Duopoly , 2 firms picked output simultaneously. Suppose firm 1 was able to choose output first, knowing how firm 2's output would vary with firm 1's output.

2.4.1 Firm 1's Max Problem

$$Max\ q_1 : (a - bq_1 - bq_2)q_1 - cq_1 - K$$

Subject to:

$$q_2 = \frac{a-c}{2b} - \frac{q_1}{2} \quad \{\text{2's Response Function}\}$$

Sub in for q_2

$$\begin{aligned} Max\ q_1 &: aq_1 - bq_1^2 - bq_1\left(\frac{a-c}{2b} - \frac{q_1}{2}\right) - cq_1 - K \\ \frac{\partial \pi_1}{\partial q_1} &= a - 2bq_1 - \left(\frac{a-c}{2b}\right) + bq_1 - c = 0 \\ q_1^* &= \frac{a-c}{2b} \end{aligned}$$

Firm 2:

$$q_2 = \frac{a-c}{2b} - \frac{q_1}{2}$$

Sub in

$$\begin{aligned} q_1 &= \frac{a-c}{2b} \\ q_2^* &= \frac{a-c}{2b} - \frac{1}{2}\left(\frac{a-c}{2b}\right) = \frac{a-c}{4b} \end{aligned}$$

Graphically: Stackelberg and Cournot Equilibrium