CHAPTER

TEN

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

The Nth-derivative test developed in the preceding chapter equips us for the task of locating the extreme values of any objective function, as long as it involves only one choice variable, possesses derivatives to the desired order, and sooner or later yields a nonzero derivative value at the critical value x_0. In the examples cited in Chap. 9, however, we made use only of polynomial and rational functions, for which we know how to obtain the necessary derivatives. Suppose that our objective function happened to be an *exponential* one, such as

$$y = 8^{x - \sqrt{x}}$$

Then we are still helpless in applying the derivative criterion, because we have yet to learn how to differentiate such a function. This is what we shall do in the present chapter.

Exponential functions, as well as the closely related logarithmic functions, have important applications in economics, especially in connection with growth problems, and in economic dynamics in general. The particular application relevant to the present part of the book, however, involves a class of optimization problems in which the choice variable is *time*. For example, a certain wine dealer may have a stock of wine, the market value of which, owing to its vintage year, is known to increase with time in some prescribed fashion. The problem is to determine the best time to sell that stock on the basis of the wine-value function, after taking into account the interest cost involved in having the money capital tied up in that stock. Exponential functions may enter into such a problem in two ways. In the first place, the value of the wine may increase with time according to some *exponential law of growth*. In that event, we would have an exponential wine-value function. This is only a possibility, of course, and not a certainty. When we give consideration to the interest cost, however, a *sure* entry is provided for an exponential function because of the fact of interest compounding, which will be explained presently. Thus we must study the nature of exponential functions before we can discuss this type of optimization problem.

Since our primary purpose is to deal with time as a choice variable, let us now switch to the symbol t—in lieu of x—to indicate the independent variable in the subsequent discussion. (However, this same symbol t can very well represent variables other than time also.)

10.1 THE NATURE OF EXPONENTIAL FUNCTIONS

As introduced in connection with polynomial functions, the term *exponent* means an indicator of the power to which a variable is to be raised. In power expressions such as x^3 or x^5, the exponents are *constants*; but there is no reason why we cannot also have a *variable* exponent, such as in 3^x or 3^t, where the number 3 is to be raised to varying powers (various values of x). A function whose *independent* variable appears in the role of an exponent is called an *exponential function*.

Simple Exponential Function

In its simple version, the exponential function may be represented in the form

$$(10.1) \qquad y = f(t) = b^t \qquad (b > 1)$$

where y and t are the dependent and independent variables, respectively, and b denotes a fixed *base* of the exponent. The domain of such a function is the set of all real numbers. Thus, unlike the exponents in a polynomial function, the variable exponent t in (10.1) is not limited to positive integers—unless we wish to impose such a restriction.

But why the restriction of $b > 1$? The explanation is as follows. In view of the fact that the domain of the function in (10.1) consists of the set of all real numbers, it is possible for t to take a value such as $\frac{1}{2}$. If b is allowed to be negative, the half power of b will involve taking the square root of a negative number. While this is not an impossible task, we would certainly prefer to take the easy way out by restricting b to be positive. Once we adopt the restriction $b > 0$, however, we might as well go all the way to the restriction $b > 1$: The restriction $b > 1$ differs from $b > 0$ only in the further exclusion of the cases of (1) $0 < b < 1$ and (2) $b = 1$; but as will be shown, the first case can be subsumed under the restriction $b > 1$, whereas the second case can be dismissed outright. Consider the first case. If $b = \frac{1}{5}$, then we have

$$y = \left(\frac{1}{5}\right)^t = \frac{1}{5^t} = 5^{-t}$$

This shows that a function with a fractional base can easily be rewritten into one with a base greater than 1. As for the second case, the fact that $b = 1$ will give us the function $y = 1^t = 1$, so that the exponential function actually degenerates into a constant function; it may therefore be disqualified as a member of the exponential family.

Graphical Form

The graph of the exponential function in (10.1) takes the general shape of the curve in Fig. 10.1. The curve drawn is based on the value $b = 2$; but even for other values of b, the same general configuration will prevail.

Several salient features of this type of exponential curve may be noted. First, it is continuous and smooth everywhere; thus the function should be everywhere differentiable. As a matter of fact, it is continuously differentiable any number of times. Second, it is monotonically increasing, and in fact y increases at an increasing rate throughout. Consequently, both the first and second derivatives of the function $y = b^t$ should be positive—a fact we should be able to confirm after we have developed the relevant differentiation formulas. Third, we note that, even though the domain of the function contains negative as well as positive numbers, the range of the function is limited to the open interval $(0, \infty)$. That is, the dependent variable y is invariably *positive*, regardless of the sign of the independent variable t.

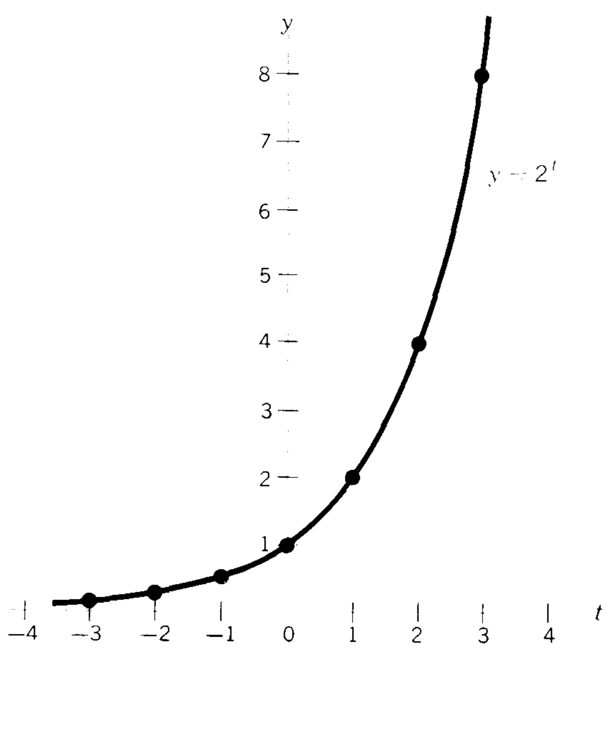


Figure 10.1

The monotonicity of the exponential function entails at least two interesting and significant implications. First, we may infer that the exponential function must have an inverse function, which is itself monotonic. This inverse function, we shall find, turns out to be a *logarithmic* function. Second, since monotonicity means that there is a unique value of t for a given value of y and since the range of the exponential function is the interval $(0, \infty)$, it follows that we should be able to express *any positive number* as a unique power of a base $b > 1$. This can be seen from Fig. 10.1, where the curve of $y = 2^t$ covers all the positive values of y in its range; therefore any positive value of y must be expressible as some unique power of the number 2. Actually, even if the base is changed to some other real number greater than 1, the same range holds, so that it is possible to express any positive number y as a power of any base $b > 1$.

Generalized Exponential Function

This last point deserves closer scrutiny. If a positive y can indeed be expressed as powers of various alternative bases, then there must exist a general procedure of *base conversion*. In the case of the function $y = 9^t$, for instance, we can readily transform it into $y = (3^2)^t = 3^{2t}$, thereby converting the base from 9 to 3, provided the exponent is duly altered from t to $2t$. This change in exponent, necessitated by the base conversion, does not create any new type of function, for, if we let $w = 2t$, then $y = 3^{2t} = 3^w$ is still in the form of (10.1). From the point of view of the base 3, however, the exponent is now $2t$ rather than t. What is the effect of adding a numerical coefficient (here, 2) to the exponent t?

The answer is to be found in Fig. 10.2a, where two curves are drawn—one for the function $y = f(t) = b^t$ and one for another function $y = g(t) = b^{2t}$. Since the exponent in the latter is exactly twice that of the former, and since the identical base is adopted for the two functions, the assignment of an arbitrary value $t = t_0$ in the function g and $t = 2t_0$ in the function f must yield the same value:

$$f(2t_0) = g(t_0) = b^{2t_0} = y_0$$

Thus the distance y_0J will be half of y_0K. By similar reasoning, for any value of y, the function g should be exactly halfway between the function f and the vertical axis. It may be concluded, therefore, that the *doubling* of the exponent has the effect of compressing the exponential curve *halfway* toward the y axis, whereas *halving* the exponent will extend the curve away from the y axis to *twice* the horizontal distance.

It is of interest that both functions share the same vertical intercept

$$f(0) = g(0) = b^0 = 1$$

The change of the exponent t to $2t$, or to any other multiple of t, will leave the vertical intercept unaffected. In terms of *compressing*, this is because compressing a zero horizontal distance will still yield a zero distance.

The change of exponent is one way of modifying—and generalizing—the exponential function of (10.1); another is to attach a coefficient to b^t, such as $2b^t$. [*Warning*: $2b^t \neq (2b)^t$.] The effect of such a coefficient is also to compress or extend the curve, except that this time the direction is vertical. In Fig. 10.2*b*, the higher curve represents $y = 2b^t$, and the lower one is $y = b^t$. For every value of t, the former must obviously be twice as high, because it has a y value twice as large as the latter. Thus we have $t_0J' = J'K'$. Note that the vertical intercept, too, is changed in the present case. We may conclude that *doubling* the coefficient (here, from 1 to 2) serves to extend the curve away from the horizontal axis to *twice* the vertical distance, whereas *halving* the coefficient will compress the curve *halfway* toward the t axis.

With the knowledge of the two modifications discussed above, the exponential function $y = b^t$ can now be generalized to the form

$$(10.2) \qquad y = ab^{ct}$$

where a and c are "compressing" or "extending" agents. When assigned various values, they will alter the position of the exponential curve, thus generating a whole family of exponential curves (functions). If a and c are positive, the general configuration shown in Fig. 10.2 will prevail; if a or c or both are *negative*, however, then fundamental modifications will occur in the configuration of the curve (see Exercise 10.1-5 below).

A Preferred Base

What prompted the discussion of the change of exponent from t to ct was the question of base conversion. But, granting the feasibility of base conversion, why

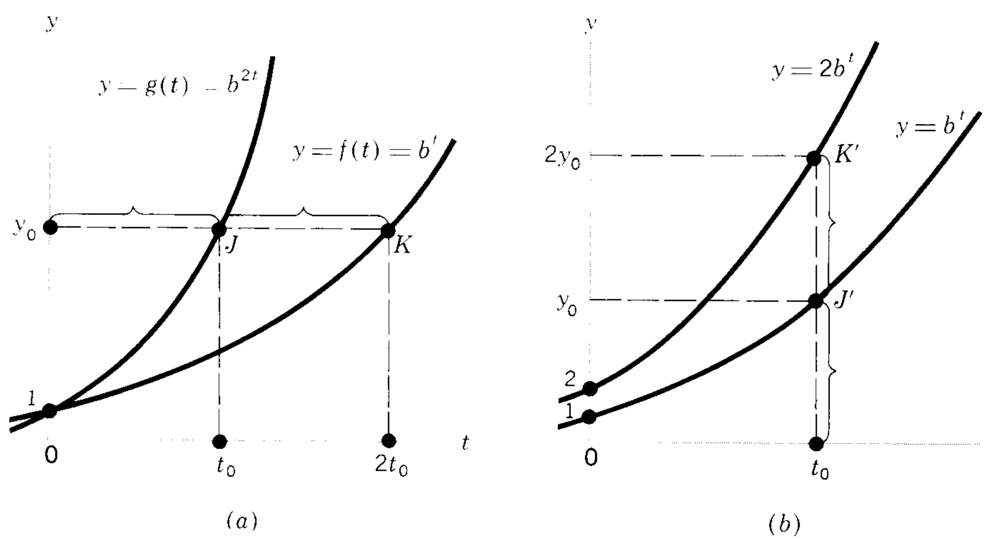


Figure 10.2

would one want to do it anyhow? One answer is that some bases are more convenient than others as far as mathematical manipulations are concerned.

Curiously enough, in calculus, the preferred base happens to be a certain irrational number denoted by the symbol e:

$$e = 2.71828\ldots$$

When this base e is used in an exponential function, it is referred to as a *natural exponential function*, examples of which are

$$y = e^t \qquad y = e^{3t} \qquad y = Ae^{rt}$$

These illustrative functions can also be expressed by the alternative notations

$$y = \exp(t) \qquad y = \exp(3t) \qquad y = A\exp(rt)$$

where the abbreviation exp (for exponential) indicates that e is to have as its exponent the expression in parentheses.

The choice of such an outlandish number as $e = 2.71828\ldots$ as the preferred base will no doubt seem bewildering. But there is an excellent reason for this choice, for the function e^t possesses the remarkable property of being its own derivative! That is,

$$\frac{d}{dt}e^t = e^t$$

a fact which will reduce the work of differentiation to practically no work at all. Moreover, armed with this differentiation rule—to be proved later in this chapter—it will also be easy to find the derivative of a more complicated natural exponential function such as $y = Ae^{rt}$. To do this, first let $w = rt$, so that the function becomes

$$y = Ae^w \qquad \text{where } w = rt\text{, and } A, r \text{ are constants}$$

Then, by the chain rule, we can write

$$\frac{dy}{dt} = \frac{dy}{dw}\frac{dw}{dt} = Ae^w(r) = rAe^{rt}$$

That is,

$$\frac{d}{dt}Ae^{rt} = rAe^{rt} \tag{10.3}$$

The mathematical convenience of the base e should thus be amply clear.

EXERCISE 10.1

1 Plot in a single diagram the graphs of the exponential functions $y = 3^t$ and $y = 3^{2t}$.

(*a*) Do the two graphs display the same general positional relationship as shown in Fig. 10.2*a*?

(*b*) Do these two curves share the same y intercept? Why?

(*c*) Sketch the graph of the function $y = 3^{3t}$ in the same diagram.

2 Plot in a single diagram the graphs of the exponential functions $y = 4^t$ and $y = 3(4^t)$.
(*a*) Do the two graphs display the general positional relationship suggested in Fig. 10.2*b*?
(*b*) Do the two curves have the same y intercept? Why?
(*c*) Sketch the graph of the function $y = \frac{3}{2}(4^t)$ in the same diagram.

3 Taking for granted that e^t is its own derivative, use the chain rule to find dy/dt for the following:
(*a*) $y = e^{5t}$ (*b*) $y = 4e^{3t}$ (*c*) $y = 6e^{-2t}$

4 In view of our discussion about (10.1), do you expect the function $y = e^t$ to be monotonically increasing at an increasing rate? Verify your answer by determining the signs of the first and second derivatives of this function. In doing so, remember that the domain of this function is the set of all real numbers, i.e., the interval $(-\infty, \infty)$.

5 In (10.2), if negative values are assigned to a and c, the general shape of the curves in Fig. 10.2 will no longer prevail. Examine the change in curve configuration by contrasting (*a*) the case of $a = -1$ against the case of $a = 1$, and (*b*) the case of $c = -1$ against the case of $c = 1$.

10.2 NATURAL EXPONENTIAL FUNCTIONS AND THE PROBLEM OF GROWTH

The pertinent questions still unanswered are: How is the number e defined? Does it have any economic meaning in addition to its mathematical significance as a convenient base? And, in what ways do natural exponential functions apply to economic analysis?

The Number e

Let us consider the following function:

$$(10.4) \qquad f(m) = \left(1 + \frac{1}{m}\right)^m$$

If larger and larger values are assigned to m, then $f(m)$ will also assume larger values; specifically, we find that

$$f(1) = \left(1 + \tfrac{1}{1}\right)^1 = 2$$
$$f(2) = \left(1 + \tfrac{1}{2}\right)^2 = 2.25$$
$$f(3) = \left(1 + \tfrac{1}{3}\right)^3 = 2.37037\ldots$$
$$f(4) = \left(1 + \tfrac{1}{4}\right)^4 = 2.44141\ldots$$
$$\vdots$$

Moreover, if m is increased indefinitely, then $f(m)$ will converge to the number

$2.71828\ldots \equiv e$; thus e may be defined as the limit of (10.4) as $m \to \infty$:

$$e \equiv \lim_{m \to \infty} f(m) = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m \tag{10.5}$$

That the approximate value of e is 2.71828 can be verified by finding the Maclaurin series of the function $\phi(x) = e^x$—with x used here to facilitate the application of the expansion formula (9.14). Such a series will give us a polynomial approximation to e^x, and thus the value of e $(= e^1)$ may be approximated by setting $x = 1$ in that polynomial. If the remainder term R_n approaches zero as the number of terms in the series is increased indefinitely, i.e., if the series is convergent to $\phi(x)$, then we can indeed approximate the value of e to any desired degree of accuracy by making the number of included terms sufficiently large.

To this end, we need to have derivatives of various orders for the function. Accepting the fact that the first derivative of e^x is e^x itself, we can see that the derivative of $\phi(x)$ is simply e^x and, similarly, that the second, third, or any higher-order derivatives must be e^x as well. Hence, when we evaluate all the derivatives at the expansion point $(x_0 = 0)$, we have the gratifyingly neat result

$$\phi'(0) = \phi''(0) = \cdots = \phi^{(n)}(0) = e^0 = 1$$

Consequently, by setting $x_0 = 0$ in (9.14), the Maclaurin series of e^x is

$$e^x = \phi(x) = \phi(0) + \phi'(0)x + \frac{\phi''(0)}{2!}x^2 + \frac{\phi'''(0)}{3!}x^3 + \cdots + \frac{\phi^{(n)}(0)}{n!}x^n + R_n$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + R_n$$

The remainder term R_n, according to (9.15), can be written as

$$R_n = \frac{\phi^{(n+1)}(p)}{(n+1)!}x^{n+1} = \frac{e^p}{(n+1)!}x^{n+1}$$

$$\left[\phi^{(n+1)}(x) = e^x; \therefore \phi^{(n+1)}(p) = e^p\right]$$

Inasmuch as the factorial expression $(n + 1)!$ will increase in value more rapidly than the power expression x^{n+1} (for a finite x) as n increases, it follows that $R_n \to 0$ as $n \to \infty$. Thus the Maclaurin series converges, and the value of e^x may, as a result, be expressed as an *infinite series*—an expression involving an infinite number $(n \to \infty)$ of additive terms which follow a consistent, recognizable pattern of formation, and in which the remainder term R_n disappears $(R_n \to 0)$:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots \tag{10.6}$$

As a special case, for $x = 1$, we find that

$$
\begin{aligned}
e &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots \\
&= 2 + 0.5 + 0.1666667 + 0.0416667 + 0.0083333 + 0.0013889 \\
&\qquad + 0.0001984 + 0.0000248 + 0.0000028 + 0.0000003 + \cdots \\
&= 2.7182819
\end{aligned}
$$

Thus, if we want a figure accurate to five decimal places, we can write $e = 2.71828$. Note that we need not worry about the subsequent terms in the infinite series, because they will be of negligible magnitude if we are concerned only with five decimal places.

An Economic Interpretation of e

Mathematically, the number e is the limit expression in (10.5). But does it also possess some economic meaning? The answer is that it can be interpreted as the result of a special process of interest compounding.

Suppose that, starting out with a principal (or capital) of \$1, we find a hypothetical banker to offer us the unusual interest rate of 100 percent per annum (\$1 interest per year). If interest is to be compounded once a year, the value of our asset at the end of the year will be \$2; we shall denote this value by $V(1)$, where the number in parentheses indicates the frequency of compounding within 1 year:

$$
\begin{aligned}
V(1) &= \text{initial principal } (1 + \text{interest rate}) \\
&= 1(1 + 100\%) = \left(1 + \tfrac{1}{1}\right)^1 = 2
\end{aligned}
$$

If interest is compounded semiannually, however, an interest amounting to 50 percent (half of 100 percent) of principal will accrue at the end of 6 months. We shall therefore have \$1.50 as the new principal during the second 6-month period, in which interest will be calculated at 50 percent of \$1.50. Thus our year-end asset value will be 1.50(1 + 50%); that is,

$$V(2) = (1 + 50\%)(1 + 50\%) = \left(1 + \tfrac{1}{2}\right)^2$$

By analogous reasoning, we can write $V(3) = (1 + \frac{1}{3})^3$, $V(4) = (1 + \frac{1}{4})^4$, etc.; or, in general,

$$V(m) = \left(1 + \frac{1}{m}\right)^m \tag{10.7}$$

where m represents the frequency of compounding in 1 year.

In the limiting case, when interest is compounded *continuously* during the year, i.e., when m becomes infinite, the value of the asset will grow in a "snowballing" fashion, becoming at the end of 1 year

$$\lim_{m \to \infty} V(m) = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m = e(\text{dollars}) \qquad [\text{by } (10.5)]$$

Thus, the number $e = 2.71828$ can be interpreted as the year-end value to which a principal of \$1 will grow if interest at the rate of 100 percent per annum is compounded continuously.

Note that the interest rate of 100 percent is only a *nominal interest rate*, for if \$1 becomes $\$e = \$2.718$ after 1 year, the *effective interest rate* is in this case approximately 172 percent per annum.

Interest Compounding and the Function Ae^{rt}

The continuous interest-compounding process just discussed can be generalized in three directions, to allow for: (1) more years of compounding, (2) a principal other than \$1, and (3) a nominal interest rate other than 100 percent.

If a principal of \$1 becomes $\$e$ after 1 year of continuous compounding and if we let $\$e$ be the new principal in the second year (during which every dollar will again grow into $\$e$), our asset value at the end of 2 years will obviously become $\$e(e) = \$\,e^2$. By the same token, it will become $\$e^3$ at the end of 3 years or, more generally, will become $\$e^t$ after t years.

Next, let us change the principal from \$1 to an unspecified amount, $\$A$. This change is easily taken care of: if \$1 will grow into $\$e^t$ after t years of continuous compounding at the nominal rate of 100 percent per annum, it stands to reason that $\$A$ will grow into $\$Ae^t$.

How about a nominal interest rate of other than 100 percent, for instance, $r = 0.05$ (= 5 percent)? The effect of this rate change is to alter the expression Ae^t to Ae^{rt}, as can be verified from the following. With an initial principal of $\$A$, to be invested for t years at a nominal interest rate r, the compound-interest formula (10.7) must be modified to the form

$$(10.8) \qquad V(m) = A\left(1 + \frac{r}{m}\right)^{mt}$$

The insertion of the coefficient A reflects the change of principal from the previous level of \$1. The quotient expression r/m means that, in each of the m compounding periods in a year, only $1/m$ of the nominal rate r will actually be applicable. Finally, the exponent mt tells us that, since interest is to be compounded m times a year, there should be a total of mt compoundings in t years.

The formula (10.8) can be transformed into an alternative form

$$(10.8') \qquad V(m) = A\left[\left(1 + \frac{r}{m}\right)^{m/r}\right]^{rt}$$

$$= A\left[\left(1 + \frac{1}{w}\right)^{w}\right]^{rt} \qquad \text{where } w \equiv \frac{m}{r}$$

As the frequency of compounding m is increased, the newly created variable w must increase pari passu; thus, as $m \to \infty$, we have $w \to \infty$, and the bracketed expression in (10.8′), by virtue of (10.5), tends to the number e. Consequently, we find the asset value in the generalized continuous-compounding process to be

$$(10.8'') \qquad V \equiv \lim_{m \to \infty} V(m) = Ae^{rt}$$

as anticipated above.

Note that, in (10.8), t is a *discrete* (as against a *continuous*) variable: it can only take values that are integral multiplies of $1/m$. For example, if $m = 4$ (compounding on a quarterly basis), then t can only take the values of $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$, etc., indicating that $V(m)$ will assume a new value only at the end of each new quarter. When $m \to \infty$, as in (10.8″), however, $1/m$ will become infinitesimal, and accordingly the variable t will become continuous. In that case, it becomes legitimate to speak of fractions of a year and to let t be, say, 1.2 or 2.35.

The upshot is that the expressions e, e^t, Ae^t, and Ae^{rt} can all be interpreted economically in connection with continuous interest compounding, as summarized in Table 10.1.

Instantaneous Rate of Growth

It should be pointed out, however, that interest compounding is an illustrative, but not exclusive, interpretation of the natural exponential function Ae^{rt}. Interest compounding merely exemplifies the general process of *exponential growth* (here, the growth of a sum of money capital over time), and we can apply the function equally well to the growth of population, wealth, or real capital.

Applied to some context other than interest compounding, the coefficient r in Ae^{rt} no longer denotes the nominal interest rate. What economic meaning does it then take? The answer is that r can be reinterpreted as the *instantaneous rate of growth* of the function Ae^{rt}. (In fact, this is why we have adopted the symbol r, for rate of growth, in the first place.) Given the function $V = Ae^{rt}$, which gives the value of V at each point of time t, the rate of change of V is to be found in the derivative

$$\frac{dV}{dt} = rAe^{rt} = rV \qquad [\text{see } (10.3)]$$

But the *rate of growth* of V is simply the *rate of change* in V expressed in relative (percentage) terms, i.e., expressed as a ratio to the value of V itself. Thus, for any

Table 10.1 Continuous interest compounding

Principal, \$	Nominal interest rate	Years of continuous compounding	Asset value, at the end of compounding process, \$
1	100% (= 1)	1	e
1	100%	t	e^t
A	100%	t	Ae^t
A	r	t	Ae^{rt}

given point of time, we have

$$\text{(10.9)} \qquad \text{Rate of growth of } V \equiv \frac{dV/dt}{V} = \frac{rV}{V} = r$$

as was stated above.

Several observations should be made about this rate of growth. But, first, let us clarify a fundamental point regarding the concept of time, namely, the distinction between a *point* of time and a *period* of time. The variable V (denoting a sum of money, or the size of population, etc.) is a *stock* concept, which is concerned with the question: How much of it *exists* at a given moment? As such, V is related to the *point* concept of time; at each point of time, V takes a unique value. The change in V, on the other hand, represents a *flow*, which involves the question: How much of it *takes place* during a given time span? Hence a change in V and, by the same token, the rate of change of V must have reference to some specified period of time, say, per year.

With this understanding, let us return to (10.9) for some comments:

1. The rate of growth defined in (10.9) is an *instantaneous* rate of growth. Since the derivative $dV/dt = rAe^{rt}$ takes a different value at a different point of t, as will $V = Ae^{rt}$, their ratio must also have reference to a specific point (or *instant*) of t. In this sense, the rate of growth is instantaneous.
2. In the present case, however, the instantaneous rate of growth happens to be a constant r, with the rate of growth thus remaining uniform at all points of time. This many not, of course, be true of all growth situations actually encountered.
3. Even though the rate of growth r is measured instantaneously, as of a particular point of time, its magnitude nevertheless has the connotation of so many percent *per unit of time*, say, per year (if t is measured in year units). Growth, by its very nature, can occur only over a time interval. This is why a single still picture (recording the situation at one instant) could never portray, say, the growth of a child, whereas two still pictures taken at different times —say, a year apart—can accomplish this. To say that V has a rate of growth of r at the instant $t = t_0$, therefore, really means that, if the rate r prevailing at $t = t_0$ is allowed to continue undisturbed for one whole unit of time (1 year), then V will have grown by the amount rV at the end of the year.
4. For the exponential function $V = Ae^{rt}$, the *percentage rate* of growth is constant at all points of t, but the *absolute amount* of increment of V increases as time goes on, because the percentage rate will be calculated on larger and larger bases.

Upon interpreting r as the instantaneous rate of growth, it is clear that little effort will henceforth be required to find the rate of growth of a natural exponential function of the form $y = Ae^{rt}$, provided r is a constant. Given a function $y = 75e^{0.02t}$, for instance, we can immediately read off the rate of growth of y as 0.02 or 2 percent per period.

Continuous versus Discrete Growth

The above discussion, though analytically interesting, is still open to question insofar as economic relevance is concerned, because in actuality growth does not always take place on a *continuous* basis—not even in interest compounding. Fortunately, however, even for cases of *discrete* growth, where changes occur only once per period rather than from instant to instant, the continuous exponential growth function can be justifiably used.

For one thing, in cases where the frequency of compounding is relatively high, though not infinite, the continuous pattern of growth may be regarded as an approximation to the true growth pattern. But, more importantly, we can show that a problem of discrete or discontinuous growth can always be transformed into an equivalent continuous version.

Suppose that we have a geometric pattern of growth (say, the *discrete* compounding of interest) as shown by the following sequence:

$$A,\ A(1+i),\ A(1+i)^2,\ A(1+i)^3,\ldots$$

where the effective interest rate per period is denoted by i and where the exponent of the expression $(1+i)$ denotes the number of periods covered in the compounding. If we consider $(1+i)$ to be the base b in an exponential expression, then the above sequence may be summarized by the exponential function Ab^t—except that, because of the discrete nature of the problem, t is restricted to integer values only. Moreover, $b = 1 + i$ is a positive number (positive even if i is a *negative* interest rate, say, -0.04), so that it can always be expressed as a power of any real number greater than 1, including e. This means that there must exist a number r such that*

$$1 + i = b = e^r$$

Thus we can transform Ab^t into a natural exponential function:

$$A(1+i)^t = Ab^t = Ae^{rt}$$

For any given value of t—in this context, integer values of t—the function Ae^{rt} will, of course, yield exactly the same value as $A(1+i)^t$, such as $A(1+i) = Ae^r$ and $A(1+i)^2 = Ae^{2r}$. Consequently, even though a *discrete* case $A(1+i)^t$ is being considered, we may still work with the *continuous* natural exponential function Ae^{rt}. This explains why natural exponential functions are extensively applied in economic analysis despite the fact that not all growth patterns may actually be continuous.

Discounting and Negative Growth

Let us now turn briefly from interest compounding to the closely related concept of *discounting*. In a compound-interest problem, we seek to compute the *future value* V (principal plus interest) from a given *present value* A (initial principal).

* The method of finding the number r, given a specific value of b, will be discussed in Sec. 10.4.

The problem of *discounting* is the opposite one of finding the present value A of a given sum V which is to be available t years from now.

Let us take the discrete case first. If the amount of principal A will grow into the future value of $A(1+i)^t$ after t years of annual compounding at the interest rate i per annum, i.e., if

$$V = A(1+i)^t$$

then, by dividing both sides of the equation by the nonzero expression $(1+i)^t$, we can get the discounting formula:

$$A = \frac{V}{(1+i)^t} = V(1+i)^{-t} \tag{10.10}$$

which involves a negative exponent. It should be realized that in this formula the roles of V and A have been reversed: V is now a given, whereas A is the unknown, to be computed from i (the rate of discount) and t (the number of years), as well as V.

Similarly, for the continuous case, if the principal A will grow into Ae^{rt} after t years of continuous compounding at the rate r in accordance with the formula

$$V = Ae^{rt}$$

then we can derive the corresponding continuous-discounting formula simply by dividing both sides of the last equation by e^{rt}:

$$A = \frac{V}{e^{rt}} = Ve^{-rt} \tag{10.11}$$

Here again, we have A (rather than V) as the unknown, to be computed from the given future value V, the nominal rate of discount r, and the number of years t.

Taking (10.11) as an exponential growth function, we can immediately read $-r$ as the instantaneous rate of growth of A. Being negative, this rate is sometimes referred to as a *rate of decay*. Just as interest compounding exemplifies the process of growth, discounting illustrates *negative* growth.

EXERCISE 10.2

1 Use the infinite-series form of e^x in (10.6) to find the approximate value of:
(*a*) e^2 (*b*) $\sqrt{e}$ $(= e^{1/2})$
(Round off your calculation of each term to 3 decimal places, and continue with the series till you get a term 0.000.)

2 Given the function $\phi(x) = e^{2x}$:
(*a*) Write the polynomial part P_n of its Maclaurin series.
(*b*) Write the Lagrange form of the remainder R_n. Determine whether $R_n \to 0$ as $n \to \infty$, that is, whether the series is convergent to $\phi(x)$.
(*c*) If convergent, so that $\phi(x)$ may be expressed as an infinite series, write out this series.

3 Write an exponential expression for the value:

(*a*) \$10, compounded continuously at the interest rate of 5% for 3 years

(*b*) \$690, compounded continuously at the interest rate of 4% for 2 years

(These interest rates are nominal rates per annum.)

4 What is the instantaneous rate of growth of y in each of the following?

(*a*) $y = e^{0.07t}$ (*c*) $y = Ae^{0.2t}$

(*b*) $y = 12e^{0.03t}$ (*d*) $y = 0.03e^{t}$

5 Show that the two functions $y_1 = Ae^{rt}$ (interest compounding) and $y_2 = Ae^{-rt}$ (discounting) are mirror images of each other with reference to the y axis [cf. Exercise 10.1-5, part (*b*)].

10.3 LOGARITHMS

Exponential functions are closely related to *logarithmic functions* (*log functions*, for short). Before we can discuss log functions, we must first understand the meaning of the term *logarithm*.

The Meaning of Logarithm

When we have two numbers such as 4 and 16, which can be related to each other by the equation $4^2 = 16$, we define the *exponent* 2 to be the *logarithm* of 16 to the base of 4, and write

$$\log_4 16 = 2$$

It should be clear from this example that the logarithm is nothing but the *power* to which a base (4) must be raised to attain a particular number (16). In general, we may state that

$$y = b^t \quad \Leftrightarrow \quad t = \log_b y \tag{10.12}$$

which indicates that the log of y to the base b (denoted by $\log_b y$) is the power to which the base b must be raised in order to attain the value y. For this reason, it is correct, though tautological, to write

$$b^{\log_b y} = y$$

In the discussion of exponential functions, we emphasized that the function $y = b^t$ (with $b > 1$) is monotonically increasing. This means that, for any positive value of y, there is a *unique* exponent t (not necessarily positive) such that $y = b^t$; moreover, the larger the value of y, the larger must be t, as can be seen from Fig. 10.2. Translated into logarithms, the monotonicity of the exponential function implies that any positive number y must possess a *unique* logarithm t to a base $b > 1$ such that the larger the y, the larger its logarithm. As Figs. 10.1 and 10.2

show, y is necessarily positive in the exponential function $y = b^t$; consequently, a negative number or zero cannot possess a logarithm.

Common Log and Natural Log

The base of the logarithm, $b > 1$, does not have to be restricted to any particular number, but in actual log applications two numbers are widely chosen as bases—the number 10 and the number e. When 10 is the base, the logarithm is known as *common logarithm*, symbolized by $\log_{10}$ (or if the context is clear, simply by log). With e as the base, on the other hand, the logarithm is referred to as *natural logarithm* and is denoted either by $\log_e$ or by ln (for natural log). We may also use the symbol log (without subscript e) if it is not ambiguous in the particular context.

Common logarithms, used frequently in *computational* work, are exemplified by the following:

$$\begin{aligned}
\log_{10} 1000 &= 3 && [\text{because } 10^3 = 1000] \\
\log_{10} 100 &= 2 && [\text{because } 10^2 = 100] \\
\log_{10} 10 &= 1 && [\text{because } 10^1 = 10] \\
\log_{10} 1 &= 0 && [\text{because } 10^0 = 1] \\
\log_{10} 0.1 &= -1 && [\text{because } 10^{-1} = 0.1] \\
\log_{10} 0.01 &= -2 && [\text{because } 10^{-2} = 0.01]
\end{aligned}$$

Observe the close relation between the set of numbers immediately to the left of the equals signs and the set of numbers immediately to the right. From these, it should be apparent that the common logarithm of a number between 10 and 100 must be between 1 and 2 and that the common logarithm of a number between 1 and 10 must be a positive fraction, etc. The exact logarithms can easily be obtained from a table of common logarithms or electronic calculators with log capabilities.*

In *analytical* work, however, natural logarithms prove vastly more convenient to use than common logarithms. Since, by the definition of logarithm, we have the relationship

$$(10.13) \qquad y = e^t \quad \Leftrightarrow \quad t = \log_e y \ (\text{or } t = \ln y)$$

it is easy to see that the analytical convenience of e in exponential functions will automatically extend into the realm of logarithms with e as the base.

* More fundamentally, the value of a logarithm, like the value of e, can be calculated (or approximated) by resorting to a Maclaurin-series expansion of a log function, in a manner similar to that outlined in (10.6). However, we shall not venture into this matter here.

The following examples will serve to illustrate natural logarithms:

$$\ln e^3 = \log_e e^3 = 3$$

$$\ln e^2 = \log_e e^2 = 2$$

$$\ln e^1 = \log_e e^1 = 1$$

$$\ln 1 = \log_e e^0 = 0$$

$$\ln \frac{1}{e} = \log_e e^{-1} = -1$$

The general principle emerging from these examples is that, given an expression e^n, where n is any real number, we can automatically read the exponent n as the natural log of e^n. In general, therefore, we have the result that $\ln e^n = n$.*

Common log and natural log are convertible into each other; i.e., the base of a logarithm can be changed, just as the base of an exponential expression can. A pair of conversion formulas will be developed after we have studied the basic rules of logarithms.

Rules of Logarithms

Logarithms are in the nature of exponents; therefore, they obey certain rules closely related to the rules of exponents introduced in Sec. 2.5. These can be of great help in simplifying mathematical operations. The first three rules are stated only in terms of natural log, but they are also valid when the symbol ln is replaced by $\log_b$.

Rule I (log of a product) $\quad \ln(uv) = \ln u + \ln v \qquad (u, v > 0)$

Example 1 $\ln(e^6 e^4) = \ln e^6 + \ln e^4 = 6 + 4 = 10$

Example 2 $\ln(Ae^7) = \ln A + \ln e^7 = \ln A + 7$

PROOF By definition, $\ln u$ is the power to which e must be raised to attain the value of u; thus $e^{\ln u} = u$.† Similarly, we have $e^{\ln v} = v$ and $e^{\ln(uv)} = uv$. The latter is an exponential expression for uv. However, another expression of uv is obtainable by direct multiplication of u and v:

$$uv = e^{\ln u} e^{\ln v} = e^{\ln u + \ln v}$$

Thus, by equating the two expressions for uv, we find

$$e^{\ln(uv)} = e^{\ln u + \ln v} \qquad \text{or} \qquad \ln(uv) = \ln u + \ln v$$

* As a mnemonic device, observe that when the symbol ln (or $\log_e$) is placed at the left of the expression e^n, the symbol ln seems to cancel out the symbol e, leaving n as the answer.

† Note that when e is raised to the power $\ln u$, the symbol e and the symbol ln again seem to cancel out, leaving u as the answer.

Rule II (log of a quotient) $\ln(u/v) = \ln u - \ln v \qquad (u, v > 0)$

Example 3 $\ln(e^2/c) = \ln e^2 - \ln c = 2 - \ln c$

Example 4 $\ln(e^2/e^5) = \ln e^2 - \ln e^5 = 2 - 5 = -3$

The proof of this rule is very similar to that of Rule I and is therefore left to you as an exercise.

Rule III (log of a power) $\ln u^a = a \ln u \qquad (u > 0)$

Example 5 $\ln e^{15} = 15 \ln e = 15$

Example 6 $\ln A^3 = 3 \ln A$

PROOF By definition, $e^{\ln u} = u$; and similarly, $e^{\ln u^a} = u^a$. However, another expression for u^a can be formed as follows:

$$u^a = (e^{\ln u})^a = e^{a \ln u}$$

By equating the exponents in the two expressions for u^a, we obtain the desired result, $\ln u^a = a \ln u$.

These three rules are useful devices for simplifying the mathematical operations in certain types of problems. Rule I serves to convert, via logarithms, a multiplicative operation (uv) into an additive one ($\ln u + \ln v$); Rule II turns a division (u/v) into a subtraction ($\ln u - \ln v$); and Rule III enables us to reduce a power to a multiplicative constant. Moreover, these rules can be used in combination.

Example 7 $\ln(uv^a) = \ln u + \ln v^a = \ln u + a \ln v$

You are warned, however, that when we have *additive* expressions to begin with, logarithms may be of no help at all. In particular, it should be remembered that

$$\ln(u \pm v) \neq \ln u \pm \ln v$$

Let us now introduce two additional rules concerned with changes in the base of a logarithm.

Rule IV (conversion of log base) $\log_b u = (\log_b e)(\log_e u) \qquad (u > 0)$

This rule, which resembles the chain rule in spirit (witness the "chain" $_b \nearrow^e \searrow_e \nearrow^u$), enables us to derive a logarithm $\log_e u$ (to base e) from the logarithm $\log_b u$ (to base b), or vice versa.

PROOF Let $u = e^p$, so that $p = \log_e u$. Then it follows that

$$\log_b u = \log_b e^p = p \log_b e = (\log_e u)(\log_b e)$$

Rule IV can readily be generalized to

$$\log_b u = (\log_b c)(\log_c u)$$

where c is some base other than b.

Rule V (inversion of log base) $\qquad \log_b e = \dfrac{1}{\log_e b}$

This rule, which resembles the inverse-function rule of differentiation, enables us to obtain the log of b to the base e immediately upon being given the log of e to the base b, and vice versa. (This rule can also be generalized to the form $\log_b c = 1/\log_c b$).

PROOF As an application of Rule IV, let $u = b$; then we have

$$\log_b b = (\log_b e)(\log_e b)$$

But the left-side expression is $\log_b b = 1$; therefore $\log_b e$ and $\log_e b$ must be reciprocal to each other, as Rule V asserts.

From the last two rules, it is easy to derive the following pair of conversion formulas between common log and natural log:

$$\begin{aligned} \log_{10} N &= (\log_{10} e)(\log_e N) = 0.4343 \log_e N \\ \log_e N &= (\log_e 10)(\log_{10} N) = 2.3026 \log_{10} N \end{aligned} \tag{10.14}$$

for N a positive real number. The first equals sign in each formula is easily justified by Rule IV. In the first formula, the value 0.4343 (the common log of 2.71828) can be found from a table of common logarithms or an electronic calculator; in the second, the value 2.3026 (the natural log of 10) is merely the reciprocal of 0.4343, so calculated because of Rule V.

Example 8 $\log_e 100 = 2.3026(\log_{10} 100) = 2.3026(2) = 4.6052$. Conversely, we have $\log_{10} 100 = 0.4343(\log_e 100) = 0.4343(4.6052) = 2$.

An Application

The above rules of logarithms enable us to solve with ease certain simple *exponential equations* (exponential *functions* set equal to zero). For instance, if we seek to find the value of x that satisfies the equation

$$ab^x - c = 0 \qquad (a, b, c > 0)$$

we can first try to transform this exponential equation, by the use of logarithms, into a *linear* equation and then solve it as such. For this purpose, the c term should first be transposed to the right side:

$$ab^x = c$$

Whereas we do not have a simple log expression for the additive expression $(ab^x - c)$, we do have convenient log expressions for the multiplicative term ab^x and for c individually. Thus, after the transposition of c and upon taking the log (say, to base 10) of both sides, we have

$$\log a + x \log b = \log c$$

which is a linear equation in the variable x, with the solution

$$x = \frac{\log c - \log a}{\log b}$$

EXERCISE 10.3

1 What are the values of the following logarithms?
  (a) $\log_{10} 10{,}000$ (c) $\log_3 81$
  (b) $\log_{10} 0.0001$ (d) $\log_5 3125$

2 Evaluate the following:
  (a) $\ln e^2$ (c) $\ln(1/e^3)$ (e) $(e^{\ln 3})!$
  (b) $\log_e e^{-4}$ (d) $\log_e(1/e^2)$ (f) $\ln e^x - e^{\ln x}$

3 Evaluate the following by application of the rules of logarithms:
  (a) $\log_{10}(100)^{14}$ (d) $\ln Ae^2$
  (b) $\log_{10} \frac{1}{100}$ (e) $\ln ABe^{-4}$
  (c) $\ln(3/B)$ (f) $(\log_4 e)(\log_e 64)$

4 Which of the following are valid?
  (a) $\ln u - 2 = \ln \dfrac{u}{e^2}$ (c) $\ln u + \ln v - \ln w = \ln \dfrac{uv}{w}$
  (b) $3 + \ln v = \ln \dfrac{e^3}{v}$ (d) $\ln 3 + \ln 5 = \ln 8$

5 Prove that $\ln(u/v) = \ln u - \ln v$.

10.4 LOGARITHMIC FUNCTIONS

When a variable is expressed as a function of the logarithm of another variable, the function is referred to as a *logarithmic function*. We have already seen two versions of this type of function in (10.12) and (10.13), namely,

$$t = \log_b y \qquad \text{and} \qquad t = \log_e y \; (= \ln y)$$

which differ from each other only in regard to the base of the logarithm.

Log Functions and Exponential Functions

As we stated earlier, log functions are inverse functions of certain exponential functions. An examination of the above two log functions will confirm that they are indeed the respective inverse functions of the exponential functions

$$y = b^t \qquad \text{and} \qquad y = e^t$$

because the log functions cited are the results of reversing the roles of the dependent and independent variables of the corresponding exponential functions. You should realize, of course, that the symbol t is being used here as a general symbol, and it does not necessarily stand for *time*. Even when it does, its appearance as a *dependent* variable does not mean that time is determined by some variable y; it means only that a given value of y is associated with a unique point of time.

As inverse functions of monotonically increasing (exponential) functions, logarithmic functions must also be monotonically increasing, which is consistent with our earlier statement that the larger a number, the larger is its logarithm to any given base. This property may be expressed symbolically in terms of the following two propositions: For two positive values of y (y_1 and y_2),

$$\begin{aligned} \ln y_1 &= \ln y_2 \quad &\Leftrightarrow \quad y_1 &= y_2 \\ \ln y_1 &> \ln y_2 \quad &\Leftrightarrow \quad y_1 &> y_2 \end{aligned} \tag{10.15}$$

These propositions are also valid, of course, if we replace ln by $\log_b$.

The Graphical Form

The monotonicity and other general properties of logarithmic functions can be clearly observed from their graphs. Given the graph of the exponential function $y = e^t$, we can obtain the graph of the corresponding log function by replotting the original graph with the two axes transposed. The result of such replotting is illustrated in Fig. 10.3. Note that if diagram b were laid over diagram a, with y axis on y axis and t axis on t axis, the two curves should coincide exactly. As they actually appear in Fig. 10.3—with interchanged axes—on the other hand, the two curves are seen to be mirror images of each other (as the graphs of any pair of inverse functions must be) with reference to the 45° line drawn through the origin.

This mirror-image relationship has several noteworthy implications. For one, although both are monotonically increasing, the log curve increases at a *decreasing rate* (second derivative negative), in contradistinction to the exponential curve, which increases at an increasing rate. Another interesting contrast is that, while the exponential function has a positive *range*, the log function has a positive *domain* instead. (This latter restriction on the domain of the log function is, of course, merely another way of stating that only positive numbers possess logarithms.) A third consequence of the mirror-image relationship is that, just as $y = e^t$ has a vertical intercept at 1, the log function $t = \log_e y$ must cross the

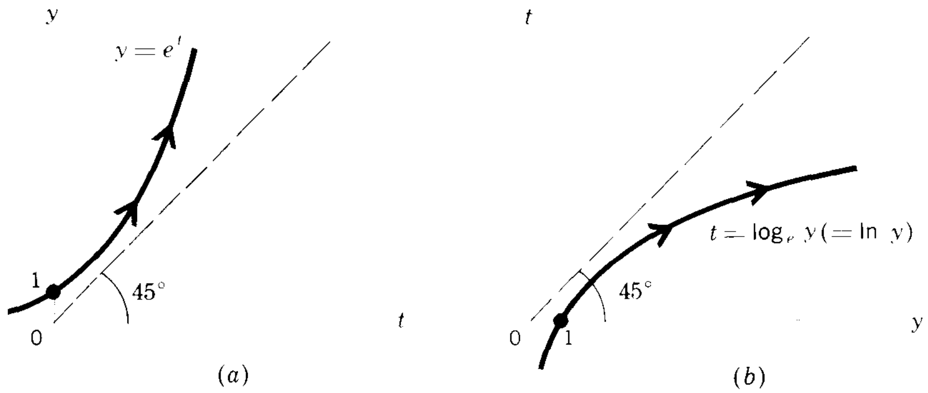


Figure 10.3

horizontal axis at $y = 1$, indicating that $\log_e 1 = 0$. Inasmuch as this horizontal intercept is unaffected by the base of the logarithm—for instance, $\log_{10} 1 = 0$ also—we may infer from the general shape of the log curve in Fig. 10.3*b* that, for *any* base,

$$(10.16) \qquad \left.\begin{matrix} 0 < y < 1 \\ y = 1 \\ y > 1 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} \log y < 0 \\ \log y = 0 \\ \log y > 0 \end{matrix}\right.$$

For verification, we can check the two sets of examples of common and natural logarithms given in Sec. 10.3. Furthermore, we may note that

$$(10.16') \qquad \log y \to \begin{Bmatrix} \infty \\ -\infty \end{Bmatrix} \qquad \text{as } y \to \begin{Bmatrix} \infty \\ 0^+ \end{Bmatrix}$$

The graphical comparison of the logarithmic function and the exponential function in Fig. 10.3 is based on the simple functions $y = e^t$ and $t = \ln y$. The same general result will prevail if we compare the generalized exponential function $y = Ae^{rt}$ with its corresponding log function. With the (positive) constants A and r to *compress* or *extend* the exponential curve, it will nevertheless resemble the general shape of Fig. 10.3*a*, except that its vertical intercept will be at $y = A$ rather than at $y = 1$ (when $t = 0$, we have $y = Ae^0 = A$). Its inverse function, accordingly, must have a *horizontal* intercept at $y = A$. In general, with reference to the 45° line, the corresponding log curve will be a mirror image of the exponential curve.

If the specific algebraic expression of the inverse of $y = Ae^{rt}$ is desired, it can be obtained by taking the natural log of both sides of this exponential function [which, according to the first proposition in (10.15), will leave the equation undisturbed] and then solving for t:

$$\ln y = \ln(Ae^{rt}) = \ln A + rt \ln e = \ln A + rt$$

hence

$$(10.17) \qquad t = \frac{\ln y - \ln A}{r} \qquad (r \neq 0)$$

This result, a log function, constitutes the inverse of the exponential function $y = Ae^{rt}$. As claimed earlier, the function in (10.17) has a horizontal intercept at $y = A$, because when $y = A$, we have $\ln y = \ln A$, and therefore $t = 0$.

Base Conversion

In Sec. 10.2, it was stated that the exponential function $y = Ab^t$ can always be converted into a *natural* exponential function $y = Ae^{rt}$. We are now ready to derive a conversion formula. Instead of Ab^t, however, let us consider the conversion of the more general expression Ab^{ct} into Ae^{rt}. Since the essence of the problem is to find an r from given values of b and c such that

$$e^r = b^c$$

all that is necessary is to express r as a function of b and c. Such a task is easily accomplished by taking the natural log of both sides of the last equation:

$$\ln e^r = \ln b^c$$

The left side can immediately be read as equal to r, so that the desired function (conversion formula) emerges as

$$r = \ln b^c = c \ln b \tag{10.18}$$

This indicates that the function $y = Ab^{ct}$ can always be rewritten in the natural-base form, $y = Ae^{(c \ln b)t}$

Example 1 Convert $y = 2^t$ to a natural exponential function. Here, we have $A = 1$, $b = 2$, and $c = 1$. Hence $r = c \ln b = \ln 2$, and the desired exponential function is

$$y = Ae^{rt} = e^{(\ln 2)t}$$

If we like, we can also calculate the numerical value of ($\ln 2$) by use of (10.14) and a table of common logarithms as follows:

$$\ln 2 = 2.3026 \log_{10} 2 = 2.3026(0.3010) = 0.6931 \tag{10.19}$$

Then we may express the earlier result alternatively as $y = e^{0.6931t}$.

Example 2 Convert $y = 3(5)^{2t}$ to a natural exponential function. In this example, $A = 3$, $b = 5$, and $c = 2$, and formula (10.18) gives us $r = 2 \ln 5$. Therefore the desired function is

$$y = Ae^{rt} = 3e^{(2 \ln 5)t}$$

Again, if we like, we can calculate that

$$2 \ln 5 = \ln 25 = 2.3026 \log_{10} 25 = 2.3026(1.3979) = 3.2188$$

so the earlier result can be alternatively expressed as $y = 3e^{3.2188t}$

It is also possible, of course, to convert log functions of the form $t = \log_b y$ into equivalent natural log functions. To that end, it is sufficient to apply Rule IV of logarithms, which may be expressed as

$$\log_b y = (\log_b e)(\log_e y)$$

The direct substitution of this result into the given log function will immediately give us the desired natural log function:

$$t = \log_b y = (\log_b e)(\log_e y)$$
$$= \frac{1}{\log_e b}\log_e y \qquad \text{[by Rule V of logarithms]}$$
$$= \frac{\ln y}{\ln b}$$

By the same procedure, we can transform the more general log function $t = a\log_b(cy)$ into the equivalent form

$$t = a(\log_b e)(\log_e cy) = \frac{a}{\log_e b}\log_e(cy) = \frac{a}{\ln b}\ln(cy)$$

Example 3 Convert the function $t = \log_2 y$ into the natural log form. Since in this example we have $b = 2$ and $a = c = 1$, the desired function is

$$t = \frac{1}{\ln 2}\ln y$$

By (10.19), however, we may also express it as $t = (1/0.6931)\ln y$.

Example 4 Convert the function $t = 7\log_{10} 2y$ into a natural logarithmic function. The values of the constants are in this case $a = 7$, $b = 10$, and $c = 2$; consequently, the desired function is

$$t = \frac{7}{\ln 10}\ln 2y$$

But since $\ln 10 = 2.3026$, as (10.14) indicates, the above function can be rewritten as $t = (7/2.3026)\ln 2y = 3.0400\ln 2y$.

In the above discussion, we have followed the practice of expressing t as a function of y when the function is logarithmic. The only reason for doing so is our desire to stress the inverse-function relationship between the exponential and logarithmic functions. When a log function is studied *by itself*, we shall write $y = \ln t$ (rather than $t = \ln y$), as is customary. Naturally, nothing in the analytical aspect of the discussion will be affected by such an interchange of symbols.

EXERCISE 10.4

1 The form of the inverse function of $y = Ae^{rt}$ in (10.17) requires r to be nonzero. What is the meaning of this requirement when viewed in reference to the original exponential function $y = Ae^{rt}$?

2 (*a*) Sketch a graph of the exponential function $y = Ae^{rt}$; indicate the value of the vertical intercept.

(*b*) Then sketch the graph of the log function $t = \dfrac{\ln y - \ln A}{r}$, and indicate the value of the horizontal intercept.

3 Find the inverse function of $y = ab^{ct}$.

4 Transform the following functions to their natural exponential forms:

(*a*) $y = 8^{3t}$ (*c*) $y = 5(5)^t$

(*b*) $y = 2(7)^{2t}$ (*d*) $y = 2(15)^{4t}$

5 Transform the following functions to their natural logarithmic forms:

(*a*) $t = \log_7 y$ (*c*) $t = 3\log_{15} 9y$

(*b*) $t = \log_8 3y$ (*d*) $t = 2\log_{10} y$

6 Find the continuous-compounding nominal interest rate per annum (r) that is equivalent to a discrete-compounding interest rate (i) of

(*a*) 5 percent per annum, compounded annually

(*b*) 5 percent per annum, compounded semiannually

(*c*) 6 percent per annum, compounded semiannually

(*d*) 6 percent per annum, compounded quarterly

10.5 DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Earlier it was claimed that the function e^t is its own derivative. As it turns out, the natural log function, $\ln t$, possesses a rather convenient derivative also, namely, $d(\ln t)/dt = 1/t$. This fact reinforces our preference for the base e. Let us now prove the validity of these two derivative formulas, and then we shall deduce the derivative formulas for certain variants of the exponential and log expressions e^t and $\ln t$.

Log-Function Rule

The derivative of the log function $y = \ln t$ is

$$\frac{d}{dt}\ln t = \frac{1}{t}$$

To prove this, we recall that, by definition, the derivative of $y = f(t) = \ln t$ has the following value at $t = N$:

$$f'(N) = \lim_{t \to N} \frac{f(t) - f(N)}{t - N} = \lim_{t \to N} \frac{\ln t - \ln N}{t - N} = \lim_{t \to N} \frac{\ln(t/N)}{t - N}$$

[by Rule II of logarithms]

Now let us introduce a shorthand symbol $m \equiv \dfrac{N}{t - N}$. Then we can write $\dfrac{1}{t - N} =$

$\dfrac{m}{N}$, and also $\dfrac{t}{N} = 1 + \dfrac{t-N}{N} = 1 + \dfrac{1}{m}$. Thus the expression to the right of the limit sign above can be converted to the form

$$\frac{1}{t-N}\ln\frac{t}{N} = \frac{m}{N}\ln\left(1+\frac{1}{m}\right) = \frac{1}{N}\ln\left(1+\frac{1}{m}\right)^m$$

[by Rule III of logarithms]

Note that, when t tends to N, m will tend to infinity. Thus, to find the desired derivative value, we may take the limit of the last expression above as $m \to \infty$:

$$f'(N) = \lim_{m\to\infty}\frac{1}{N}\ln\left(1+\frac{1}{m}\right)^m = \frac{1}{N}\ln e = \frac{1}{N} \qquad \text{[by (10.5)]}$$

Since N can be any number for which a logarithm is defined, however, we can generalize this result, and write $f'(t) = d(\ln t)/dt = 1/t$. This proves the log-function rule.

Exponential-Function Rule

The derivative of the function $y = e^t$ is

$$\frac{d}{dt}e^t = e^t$$

This result follows easily from the log-function rule. We know that the inverse function of the function $y = e^t$ is $t = \ln y$, with derivative $dt/dy = 1/y$. Thus, by the inverse-function rule, we may write immediately

$$\frac{d}{dt}e^t = \frac{dy}{dt} = \frac{1}{dt/dy} = \frac{1}{1/y} = y = e^t$$

The Rules Generalized

The above two rules can be generalized to cases where the variable t in the expression e^t and $\ln t$ is replaced by some *function* of t, say, $f(t)$. The generalized versions of the two rules are

$$\text{(10.20)} \qquad \begin{aligned} \frac{d}{dt}e^{f(t)} &= f'(t)e^{f(t)} &\qquad \left[\text{or } \frac{d}{dt}e^u = e^u\frac{du}{dt}\right] \\ \frac{d}{dt}\ln f(t) &= \frac{f'(t)}{f(t)} &\qquad \left[\text{or } \frac{d}{dt}\ln v = \frac{1}{v}\frac{dv}{dt}\right] \end{aligned}$$

The proofs for (10.20) involve nothing more than the straightforward application of the chain rule. Given a function $y = e^{f(t)}$, we can first let $u = f(t)$, so that $y = e^u$. Then, by the chain rule, the derivative emerges as

$$\frac{d}{dt}e^{f(t)} = \frac{d}{dt}e^u = \frac{d}{du}e^u\frac{du}{dt} = e^u\frac{du}{dt} = e^{f(t)}f'(t)$$

Similarly, given a function $y = \ln f(t)$, we can first let $v = f(t)$, so as to form a chain: $y = \ln v$, where $v = f(t)$. Then, by the chain rule, we have

$$\frac{d}{dt}\ln f(t) = \frac{d}{dt}\ln v = \frac{d}{dv}\ln v\frac{dv}{dt} = \frac{1}{v}\frac{dv}{dt} = \frac{1}{f(t)}f'(t)$$

Note that the only real modification introduced in (10.20) beyond the simpler rules $de^t/dt = e^t$ and $d(\ln t)/dt = 1/t$ is the multiplicative factor $f'(t)$.

Example 1 Find the derivative of the function $y = e^{rt}$. Here, the exponent is $rt = f(t)$, with $f'(t) = r$; thus

$$\frac{dy}{dt} = \frac{d}{dt}e^{rt} = re^{rt}$$

Example 2 Find dy/dt from the function $y = e^{-t}$. In this case, $f(t) = -t$, so that $f'(t) = -1$. As a result,

$$\frac{dy}{dt} = \frac{d}{dt}e^{-t} = -e^{-t}$$

Example 3 Find dy/dt from the function $y = \ln at$. Since in this case $f(t) = at$, with $f'(t) = a$, the derivative is

$$\frac{d}{dt}\ln at = \frac{a}{at} = \frac{1}{t}$$

which is, interestingly enough, identical with the derivative of $y = \ln t$.

This example illustrates the fact that a multiplicative constant for t *within* a log expression drops out in the process of derivation. But note that, for a constant k, we have

$$\frac{d}{dt}k\ln t = k\frac{d}{dt}\ln t = \frac{k}{t}$$

thus a multiplicative constant *without* the log expression is still retained in derivation.

Example 4 Find the derivative of the function $y = \ln t^c$. With $f(t) = t^c$ and $f'(t) = ct^{c-1}$, the formula in (10.20) yields

$$\frac{d}{dt}\ln t^c = \frac{ct^{c-1}}{t^c} = \frac{c}{t}$$

Example 5 Find dy/dt from $y = t^3\ln t^2$. Because this function is a product of two terms t^3 and $\ln t^2$, the product rule should be used:

$$\begin{aligned}\frac{dy}{dt} &= t^3\frac{d}{dt}\ln t^2 + \ln t^2\frac{d}{dt}t^3\\ &= t^3\left(\frac{2t}{t^2}\right) + (\ln t^2)(3t^2)\\ &= 2t^2 + 3t^2(2\ln t) \qquad \text{[Rule III of logarithms]}\\ &= 2t^2(1 + 3\ln t)\end{aligned}$$

The Case of Base b

For exponential and log functions with base b, the derivatives are

$$(10.21)\qquad \begin{aligned} \frac{d}{dt}b^t &= b^t \ln b \qquad \left[\textit{Warning}: \frac{d}{dt}b^t \neq tb^{t-1}\right] \\ \frac{d}{dt}\log_b t &= \frac{1}{t\ln b} \end{aligned}$$

Note that in the special case of base e (when $b = e$), we have $\ln b = \ln e = 1$, so that these two derivatives will reduce to $(d/dt)e^t = e^t$ and $(d/dt)\ln t = 1/t$, respectively.

The proofs for (10.21) are not difficult. For the case of b^t, the proof is based on the identity $b \equiv e^{\ln b}$, which enables us to write

$$b^t = e^{(\ln b)t} = e^{t\ln b}$$

(We write $t \ln b$, instead of $\ln b\ t$, in order to emphasize that t is not a part of the log expression.) Hence

$$\begin{aligned} \frac{d}{dt}b^t &= \frac{d}{dt}e^{t\ln b} = (\ln b)(e^{t\ln b}) \qquad [\text{by } (10.20)] \\ &= (\ln b)(b^t) = b^t \ln b \end{aligned}$$

To prove the second part of (10.21), on the other hand, we rely on the basic log property that

$$\log_b t = (\log_b e)(\log_e t) = \frac{1}{\ln b}\ln t$$

which leads us to the derivative

$$\frac{d}{dt}\log_b t = \frac{d}{dt}\left(\frac{1}{\ln b}\ln t\right) = \frac{1}{\ln b}\frac{d}{dt}\ln t = \frac{1}{\ln b}\left(\frac{1}{t}\right)$$

The more general versions of these two formulas are

$$(10.21')\qquad \begin{aligned} \frac{d}{dt}b^{f(t)} &= f'(t)b^{f(t)}\ln b \\ \frac{d}{dt}\log_b f(t) &= \frac{f'(t)}{f(t)}\frac{1}{\ln b} \end{aligned}$$

Again, it is seen that if $b = e$, then $\ln b = 1$, and these formulas will reduce to (10.20).

Example 6 Find the derivative of the function $y = 12^{1-t}$. Here, $b = 12$, $f(t) = 1 - t$, and $f'(t) = -1$; thus

$$\frac{dy}{dt} = -(12)^{1-t}\ln 12$$

Higher Derivatives

Higher derivatives of exponential and log functions, like those of other types of functions, are merely the results of repeated differentiation.

Example 7 Find the *second* derivative of $y = b^t$ (with $b > 1$). The first derivative, by (10.21), is $y'(t) = b^t \ln b$ (where $\ln b$ is, of course, a constant); thus, by differentiating once more with respect to t, we have

$$y''(t) = \frac{d}{dt}y'(t) = \left(\frac{d}{dt}b^t\right)\ln b = (b^t \ln b)\ln b = b^t(\ln b)^2$$

Note that $y = b^t$ is always positive and $\ln b$ (for $b > 1$) is also positive [by (10.16)]; thus $y'(t) = b^t \ln b$ must be positive. And $y''(t)$, being a product of b^t and a squared number, is also positive. These facts confirm our previous statement that the exponential function $y = b^t$ increases monotonically at an increasing rate.

Example 8 Find the *second* derivative of $y = \ln t$. The first derivative is $y' = 1/t = t^{-1}$; hence, the second derivative is

$$y'' = -t^{-2} = \frac{-1}{t^2}$$

Inasmuch as the domain of this function consists of the open interval $(0, \infty)$, $y' = 1/t$ must be a positive number. On the other hand, y'' is always negative. Together, these conclusions serve to confirm our earlier allegation that the log function $y = \ln t$ increases monotonically at a decreasing rate.

An Application

One of the prime virtues of the logarithm is its ability to convert a multiplication into an addition, and a division into a subtraction. This property can be exploited when we are differentiating a complicated product or quotient of any type of functions (not necessarily exponential or logarithmic).

Example 9 Find dy/dx from

$$y = \frac{x^2}{(x+3)(2x+1)}$$

Instead of applying the product and quotient rules, we may first take the natural log of both sides of the equation to reduce the function to the form

$$\ln y = \ln x^2 - \ln(x+3) - \ln(2x+1)$$

According to (10.20), the derivative of the left side with respect to x is

$$\frac{d}{dx}\text{(left side)} = \frac{1}{y}\frac{dy}{dx}$$

whereas the right side gives

$$\frac{d}{dx}\text{(right side)} = \frac{2x}{x^2} - \frac{1}{x+3} - \frac{2}{2x+1} = \frac{7x+6}{x(x+3)(2x+1)}$$

When the two results are equated and both sides are multiplied by y, we get the desired derivative as follows:

$$\begin{aligned}\frac{dy}{dx} &= \frac{7x+6}{x(x+3)(2x+1)}y \\ &= \frac{7x+6}{x(x+3)(2x+1)}\,\frac{x^2}{(x+3)(2x+1)} = \frac{x(7x+6)}{(x+3)^2(2x+1)^2}\end{aligned}$$

Example 10 Find dy/dx from $y = x^a e^{kx-c}$. Taking the natural log of both sides, we have

$$\ln y = a\ln x + \ln e^{kx-c} = a\ln x + kx - c$$

Differentiating both sides with respect to x, and using (10.20), we then get

$$\frac{1}{y}\frac{dy}{dx} = \frac{a}{x} + k$$

and $$\frac{dy}{dx} = \left(\frac{a}{x} + k\right)y = \left(\frac{a}{x} + k\right)x^a e^{kx-c}$$

EXERCISE 10.5

1 Find the derivatives of:

(a) $y = e^{2t+4}$	(e) $y = e^{ax^2+bx+c}$
(b) $y = e^{1-7t}$	(f) $y = xe^x$
(c) $y = e^{t^2+1}$	(g) $y = x^2e^{2x}$
(d) $y = 3e^{2-t^2}$	(h) $y = axe^{bx+c}$

2 (a) Verify the derivative in Example 3 by utilizing the equation $\ln at = \ln a + \ln t$.
(b) Verify the result in Example 4 by utilizing the equation $\ln t^c = c\ln t$.

3 Find the derivatives of:

(a) $y = \ln 8t^5$	(e) $y = \ln x - \ln(1+x)$
(b) $y = \ln at^c$	(f) $y = \ln[x(1-x)^8]$
(c) $y = \ln(t+9)$	(g) $y = \ln\left(\frac{3x}{1+x}\right)$
(d) $y = 5\ln(t+1)^2$	(h) $y = 5x^4\ln x^2$

4 Find the derivatives of:

(a) $y = 5^t$	(d) $y = \log_7 7x^2$
(b) $y = \log_2(t+1)$	(e) $y = \log_2(8x^2+3)$
(c) $y = 13^{2t+3}$	(f) $y = x^2\log_3 x$

5 Prove the two formulas in (10.21′).

6 Show that the function $V = Ae^{rt}$ (with $A, r > 0$) and the function $A = Ve^{-rt}$ (with $V, r > 0$) are both monotonic, but in opposite directions, and that they are both strictly convex in shape (cf. Exercise 10.2-5).

7 Find the derivatives of the following by first taking the natural log of both sides:

(*a*) $y = \dfrac{3x}{(x+2)(x+4)}$ (*b*) $y = (x^2 + 3)e^{x^2+1}$

10.6 OPTIMAL TIMING

What we have learned about exponential and log functions can now be applied to some simple problems of optimal timing.

A Problem of Wine Storage

Suppose that a certain wine dealer is in possession of a particular quantity (say, a case) of wine, which he can either sell at the present time ($t = 0$) for a sum of $K or else store for a variable length of time and then sell at a higher value. The growing value (V) of the wine is known to be the following function of time:

$$V = Ke^{\sqrt{t}} \qquad \left[= K\exp(t^{1/2})\right] \tag{10.22}$$

so that if $t = 0$ (sell now), then $V = K$. The problem is to ascertain when he should sell it in order to maximize profit, assuming the storage cost to be nil.*

Since the cost of wine is a "sunk" cost—the wine is already paid for by the dealer—and since storage cost is assumed to be nonexistent, to maximize profit is the same as maximizing the sales revenue, or the value of V. There is one catch, however. Each value of V corresponding to a specific point of t represents a dollar sum receivable at a different date and, because of the interest element involved, is not directly comparable with the V value of another date. The way out of this difficulty is to *discount* each V figure to its *present-value* equivalent (the value at time $t = 0$), for then all the V values will be on a comparable footing.

Let us assume that the interest rate on the continuous-compounding basis is at the level of r. Then, according to (10.11), the present value of V can be expressed as

$$A(t) = Ve^{-rt} = Ke^{\sqrt{t}}e^{-rt} = Ke^{\sqrt{t}-rt} \tag{10.22'}$$

where A, denoting the present value of V, is itself a function of t. Therefore our problem amounts to finding the value of t that maximizes A.

* The consideration of storage cost will entail a difficulty we are not yet equipped to handle. Later, in Chap. 13, we shall return to this problem.

Maximization Conditions

The first-order condition for maximizing A is to have $dA/dt = 0$. To find this derivative, we can either differentiate (10.22′) directly with respect to t, or do it indirectly by first taking the natural log of both sides of (10.22′) and then differentiating with respect to t. Let us illustrate the latter procedure.

First, we obtain from (10.22′) the equation

$$\ln A(t) = \ln K + \ln e^{\sqrt{t} - rt} = \ln K + (t^{1/2} - rt)$$

Upon differentiating both sides with respect to t, we then get

$$\frac{1}{A}\frac{dA}{dt} = \frac{1}{2}t^{-1/2} - r$$

$$\text{or} \qquad \frac{dA}{dt} = A\left(\frac{1}{2}t^{-1/2} - r\right)$$

Since $A \neq 0$, the condition $dA/dt = 0$ can be satisfied if and only if

$$\frac{1}{2}t^{-1/2} = r \qquad \text{or} \qquad \frac{1}{2\sqrt{t}} = r \qquad \text{or} \qquad \frac{1}{2r} = \sqrt{t}$$

This implies that the optimum length of storage time is

$$\bar{t} = \left(\frac{1}{2r}\right)^2 = \frac{1}{4r^2}$$

If $r = 0.10$, for instance, then $\bar{t} = 25$, and the dealer should store the case of wine for 25 years. Note that the higher the rate of interest (rate of discount) is, the shorter the optimum storage period will be.

The first-order condition, $1/(2\sqrt{t}) = r$, admits of an easy economic interpretation. The left-hand expression merely represents the rate of growth of wine value V, because from (10.22)

$$\begin{aligned} \frac{dV}{dt} &= \frac{d}{dt}K\exp(t^{1/2}) = K\frac{d}{dt}\exp(t^{1/2}) && [K \text{ constant}] \\ &= K\left(\frac{1}{2}t^{-1/2}\right)\exp(t^{1/2}) && [\text{by (10.20)}] \\ &= \left(\frac{1}{2}t^{-1/2}\right)V && [\text{by (10.22)}] \end{aligned}$$

so that the rate of growth of V is indeed the left-hand expression in the first-order condition:

$$r_V \equiv \frac{dV/dt}{V} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}}$$

The right-hand expression r is, in contrast, the rate of interest or the rate of compound-interest growth of the cash fund receivable *if* the wine is sold right away—an *opportunity-cost* aspect of storing the wine. Thus, the equating of the two instantaneous rates, as illustrated in Fig. 10.4, is an attempt to hold onto the

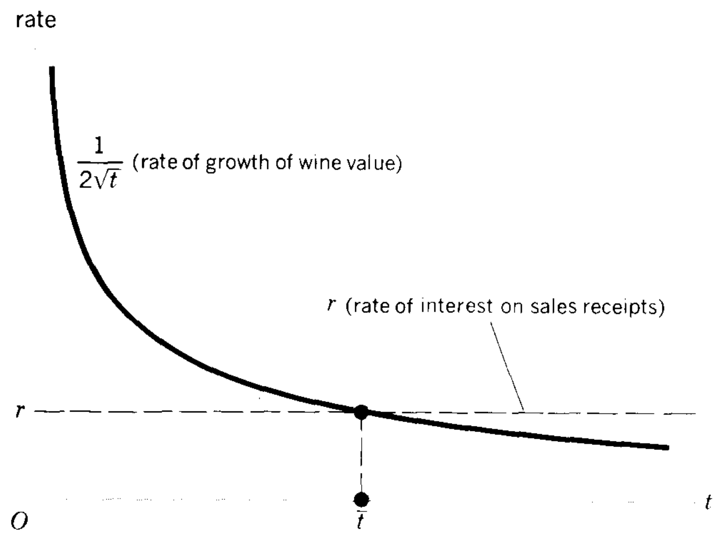


Figure 10.4

wine until the advantage of storage is completely wiped out, i.e., to wait till the moment when the (declining) rate of growth of wine value is just matched by the (constant) interest rate on cash sales receipts.

The next order of business is to check whether the value of $\bar{t}$ satisfies the second-order condition for maximization of A. The second derivative of A is

$$\frac{d^2A}{dt^2} = \frac{d}{dt}A\left(\frac{1}{2}t^{-1/2} - r\right) = A\frac{d}{dt}\left(\frac{1}{2}t^{-1/2} - r\right) + \left(\frac{1}{2}t^{-1/2} - r\right)\frac{dA}{dt}$$

But, since the final term drops out when we evaluate it at the equilibrium (optimum) point, where $dA/dt = 0$, we are left with

$$\frac{d^2A}{dt^2} = A\frac{d}{dt}\left(\frac{1}{2}t^{-1/2} - r\right) = A\left(-\frac{1}{4}t^{-3/2}\right) = \frac{-A}{4\sqrt{t^3}}$$

In view that $A > 0$, this second derivative is negative when evaluated at $\bar{t} > 0$, thereby ensuring that the solution value $\bar{t}$ is indeed profit-maximizing.

A Problem of Timber Cutting

A similar problem, which involves a choice of the best time to take action, is that of timber cutting.

Suppose the value of timber (already planted on some given land) is the following increasing function of time:

$$V = 2^{\sqrt{t}}$$

expressed in units of \$1000. Assuming a discount rate of r (on the continuous basis) and also assuming zero upkeep cost during the period of timber growth, what is the optimal time to cut the timber for sale?

As in the wine problem, we should first convert V into its present value:

$$A(t) = Ve^{-rt} = 2^{\sqrt{t}}e^{-rt}$$

thus $$\ln A = \ln 2^{\sqrt{t}} + \ln e^{-rt} = \sqrt{t}\ln 2 - rt = t^{1/2}\ln 2 - rt$$

To maximize A, we must set $dA/dt = 0$. The first derivative is obtainable by differentiating $\ln A$ with respect to t and then multiplying by A:

$$\frac{1}{A}\frac{dA}{dt} = \frac{1}{2}t^{-1/2}\ln 2 - r$$

thus $$\frac{dA}{dt} = A\left(\frac{\ln 2}{2\sqrt{t}} - r\right)$$

Since $A \neq 0$, the condition $dA/dt = 0$ can be met if and only if

$$\frac{\ln 2}{2\sqrt{t}} = r \qquad \text{or} \qquad \sqrt{t} = \frac{\ln 2}{2r}$$

Consequently, the optimum number of years of growth is

$$\bar{t} = \left(\frac{\ln 2}{2r}\right)^2$$

It is evident from this solution that, the higher the rate of discount, the earlier the timber should be cut.

To make sure that $\bar{t}$ is a maximizing (instead of minimizing) solution, the second-order condition should be checked. But this will be left to you as an exercise.

In this example, we have abstracted from planting cost by assuming that the trees are already planted, in which case the (sunk) planting cost is legitimately excludable from consideration in the optimization decision. If the decision is not one of when to harvest but one of whether or not to plant at all, then the planting cost (incurred at the *present*) must be duly compared with the *present* value of the timber output, computed with t set at the optimum value $\bar{t}$. For instance, if $r = 0.05$, then we have

$$\bar{t} = \left(\frac{0.6931}{0.10}\right)^2 = (6.931)^2 = 48.0 \text{ years}$$

and $$\bar{A} = 2^{6.931}e^{-0.05(48.0)} = (122.0222)e^{-2.40}$$
$$= 122.0222(0.0907) = \$11.0674 \text{ (in thousands)}$$

So only a planting cost lower than $\bar{A}$ will make the venture worthwhile—again, provided that upkeep cost is nil.

EXERCISE 10.6

1 If the value of wine grows according to the function $V = Ke^{2\sqrt{t}}$, instead of as in (10.22), how long should the dealer store the wine?

2 Check the second-order condition for the timber-cutting problem.

3 As a generalization of the optimization problem illustrated in the present section, show that:

(*a*) With any value function $V = f(t)$ and a given continuous rate of discount r, the first-order condition for the present value A of V to reach a maximum is that the rate of growth of V be equal to r.

(*b*) The second-order sufficient condition for maximum really amounts to the stipulation that the rate of growth of V be decreasing with time.

10.7 FURTHER APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC DERIVATIVES

Aside from their use in optimization problems, the derivative formulas of Sec. 10.5 have further useful economic applications.

Finding the Rate of Growth

When a variable y is a function of time, $y = f(t)$, its instantaneous rate of growth is defined as*

$$(10.23) \qquad r_y \equiv \frac{dy/dt}{y} = \frac{f'(t)}{f(t)} = \frac{\text{marginal function}}{\text{total function}}$$

But, from (10.20), we see that this ratio is precisely the derivative of $\ln f(t) = \ln y$. Thus, to find the instantaneous rate of growth of a function of time $f(t)$, we can —instead of differentiating it with respect to t, and then dividing by $f(t)$—simply take its natural log and then differentiate $\ln f(t)$ with respect to time.† This alternative method may turn out to be the simpler approach, *if* $f(t)$ is a multiplicative or divisional expression which, upon logarithm-taking, will reduce to a sum or difference of additive terms.

Example 1 Find the rate of growth of $V = Ae^{rt}$, where t denotes time. It is already known to us that the rate of growth of V is r, but let us check it by finding the derivative of $\ln V$:

$$\ln V = \ln A + rt \ln e = \ln A + rt \qquad [A \text{ constant}]$$

Therefore,

$$r_V = \frac{d}{dt}\ln V = 0 + \frac{d}{dt}rt = r$$

as was to be demonstrated.

* If the variable t does *not* denote time, the expression $(dy/dt)/y$ is referred to as the *proportional rate of change* of y with respect to t.

† If we plot the natural log of a function $f(t)$ against t in a two-dimensional diagram, the slope of the curve, accordingly, will tell us the rate of growth of $f(t)$. This provides the rationale for the so-called "semilog scale" charts.

Example 2 Find the rate of growth of $y = 4^t$. In this case, we have

$$\ln y = \ln 4^t = t \ln 4$$

Hence $\quad r_y = \frac{d}{dt}\ln y = \ln 4$

This is as it should be, because $e^{\ln 4} \equiv 4$, and consequently, $y = 4^t$ can be rewritten as $y = e^{(\ln 4)t}$, which would immediately enable us to read $(\ln 4)$ as the rate of growth of y.

Rate of Growth of a Combination of Functions

To carry this discussion a step further, let us examine the instantaneous rate of growth of a *product* of two functions of time:

$$y = uv \qquad \text{where} \begin{cases} u = f(t) \\ v = g(t) \end{cases}$$

Taking the natural log of y, we obtain

$$\ln y = \ln u + \ln v$$

Thus the desired rate of growth is

$$r_y = \frac{d}{dt}\ln y = \frac{d}{dt}\ln u + \frac{d}{dt}\ln v$$

But the two terms on the right side are the rates of growth of u and v, respectively. Thus we have the rule

$$(10.24) \qquad r_{(uv)} = r_u + r_v$$

Expressed in words, the instantaneous rate of growth of a *product* is the *sum* of the instantaneous rates of growth of the components.

By a similar procedure, the rate of growth of a *quotient* can be shown to be the *difference* between the rates of growth of the components (see Exercise 10.7-4):

$$(10.25) \qquad r_{(u/v)} = r_u - r_v$$

Example 3 If consumption C is growing at the rate α, and if population H (for "heads") is growing at the rate β, what is the rate of growth of per capita consumption? Since per capita consumption is equal to C/H, its rate of growth should be

$$r_{(C/H)} = r_C - r_H = \alpha - \beta$$

Now consider the instantaneous rate of growth of a *sum* of two functions of time:

$$z = u + v \qquad \text{where} \begin{cases} u = f(t) \\ v = g(t) \end{cases}$$

This time, the natural log will be

$$\ln z = \ln(u+v) \qquad [\neq \ln u + \ln v]$$

Thus

$$\begin{aligned} r_z &= \frac{d}{dt}\ln z = \frac{d}{dt}\ln(u+v) \\ &= \frac{1}{u+v}\frac{d}{dt}(u+v) \qquad [\text{by } (10.20)] \\ &= \frac{1}{u+v}[f'(t)+g'(t)] \end{aligned}$$

But from (10.23) we have $r_u = f'(t)/f(t)$, so that $f'(t) = f(t)r_u = ur_u$. Similarly, we have $g'(t) = vr_v$. As a result, we can write the rule

$$(10.26) \qquad r_{(u+v)} = \frac{u}{u+v}r_u + \frac{v}{u+v}r_v$$

which states that the rate of growth of a *sum* is a *weighted average* of the rates of growth of the components.

By the same token, we have (see Exercise 10.7.5)

$$(10.27) \qquad r_{(u-v)} = \frac{u}{u-v}r_u - \frac{v}{u-v}r_v$$

Example 4 The exports of goods of a country, $G = G(t)$, has a growth rate of a/t, and its exports of services, $S = S(t)$, has a growth rate of b/t. What is the growth rate of its total exports? Since total exports is $X(t) = G(t) + S(t)$, a sum, its rate of growth should be

$$\begin{aligned} r_X &= \frac{G}{X}r_G + \frac{S}{X}r_S \\ &= \frac{G}{X}\left(\frac{a}{t}\right) + \frac{S}{X}\left(\frac{b}{t}\right) = \frac{Ga+Sb}{Xt} \end{aligned}$$

Finding the Point Elasticity

We have seen that, given $y = f(t)$, the derivative of $\ln y$ measures the instantaneous rate of growth of y. Now let us see what happens when, given a function $y = f(x)$, we differentiate ($\ln y$) with respect to ($\ln x$), rather than to x.

To begin with, let us define $u \equiv \ln y$ and $v \equiv \ln x$. Then we can observe a chain of relationship linking u to y, and thence to x and v as follows:

$$u \equiv \ln y \qquad y = f(x) \qquad x \equiv e^{\ln x} \equiv e^v$$

Accordingly, the derivative of ($\ln y$) with respect to ($\ln x$) is

$$\begin{aligned} \frac{d(\ln y)}{d(\ln x)} &= \frac{du}{dv} = \frac{du}{dy}\frac{dy}{dx}\frac{dx}{dv} \\ &= \left(\frac{d}{dy}\ln y\right)\left(\frac{dy}{dx}\right)\left(\frac{d}{dv}e^v\right) = \frac{1}{y}\frac{dy}{dx}e^v = \frac{1}{y}\frac{dy}{dx}x = \frac{dy}{dx}\frac{x}{y} \end{aligned}$$

But this expression is precisely that of the point elasticity of the function. Hence we have established the general principle that, for a function $y = f(x)$, the point elasticity of y with respect to x is

$$(10.28) \qquad \varepsilon_{yx} = \frac{d(\ln y)}{d(\ln x)}$$

It should be noted that the subscript yx in this symbol is an indicator that y and x are the two variables involved and does not imply the multiplication of y and x. This is unlike the case of $r_{(uv)}$, where the subscript *does* denote a product. Again, we now have an alternative way of finding the point elasticity of a function by use of logarithms, which may often prove to be an easier approach, *if* the given function comes in the form of a multiplicative or divisional expression.

Example 5 Find the point elasticity of demand, given that $Q = k/P$, where k is a positive constant. This is the equation of a rectangular hyperbola (see Fig. 2.8d); and, as is well known, a demand function of this form has a unitary point elasticity at all points. To show this, we shall apply (10.28). Since the natural log of the demand function is

$$\ln Q = \ln k - \ln P$$

the elasticity of demand (Q with respect to P) is indeed

$$\varepsilon_d = \frac{d(\ln Q)}{d(\ln P)} = -1 \qquad \text{or} \qquad |\varepsilon_d| = 1$$

The result in (10.28) was derived by use of the chain rule of derivatives. It is of interest that a similar chain rule holds for elasticities; i.e., given a function $y = g(w)$, where $w = h(x)$, we have

$$(10.29) \qquad \varepsilon_{yx} = \varepsilon_{yw}\varepsilon_{wx}$$

The proof is as follows:

$$\varepsilon_{yw}\varepsilon_{wx} = \left(\frac{dy}{dw}\frac{w}{y}\right)\left(\frac{dw}{dx}\frac{x}{w}\right) = \frac{dy}{dw}\frac{dw}{dx}\frac{w}{y}\frac{x}{w} = \frac{dy}{dx}\frac{x}{y} = \varepsilon_{yx}$$

EXERCISE 10.7

1 Find the instantaneous rate of growth:
(a) $y = 3t^2$ (c) $y = ab^t$ (e) $y = t/3^t$
(b) $y = at^c$ (d) $y = 2^t(t^2)$

2 If population grows according to the function $H = H_0(2)^{bt}$ and consumption by the function $C = C_0e^{at}$, find the rates of growth of population, of consumption, and of per capita consumption by using natural log.

3 If y is related to x by $y = x^k$, how will the rates of growth r_y and r_x be related?

4 Prove that if $y = u/v$, where $u = f(t)$ and $v = g(t)$, then the rate of growth of y will be $r = r_u - r_v$, as shown in (10.25).

5 Prove the rate-of-growth rule (10.27).

6 Given the demand function $Q_d = k/P^n$, where k and n are positive constants, find the point elasticity of demand ε_d by using (10.28) (cf. Exercise 8.1-5).

7 (*a*) Given $y = wz$, where $w = g(x)$ and $z = h(x)$, establish that

$$\varepsilon_{yx} = \varepsilon_{wx} + \varepsilon_{zx}$$

(*b*) Given $y = u/v$, where $u = G(x)$ and $v = H(x)$, establish that

$$\varepsilon_{yx} = \varepsilon_{ux} - \varepsilon_{vx}$$

8 Given $y = f(x)$, show that the derivative $d(\log_b y)/d(\log_b x)$—log to base b rather than e—also measures the point elasticity ε_{yx}.

9 Show that, if the demand for money M_d is a function of the national income $Y = Y(t)$ and the interest rate $i = i(t)$, the rate of growth of M_d can be expressed as a weighted sum of r_Y and r_i,

$$r_{M_d} = \varepsilon_{M_dY} r_Y + \varepsilon_{M_di} r_i$$

where the weights are the elasticities of M_d with respect to Y and i, respectively.

10 Given the production function $Q = F(K, L)$, find a general expression for the rate of growth of Q in terms of the rates of growth of K and L.

CHAPTER
ELEVEN
THE CASE OF MORE THAN ONE CHOICE VARIABLE

The problem of optimization was discussed in Chap. 9 within the framework of an objective function with a single choice variable. In the last chapter, the discussion was extended to exponential objective functions, but we still dealt with one choice variable only. Now we must develop a way of finding the extreme values of an objective function that involves two or more choice variables. Only then will we be able to tackle the type of problem confronting, say, a multiproduct firm, where the profit-maximizing decision consists of the choice of optimal output levels for several commodities and the optimal combination of several different inputs.

We shall discuss first the case of an objective function of two choice variables, $z = f(x, y)$, in order to take advantage of its graphability. Later the analytical results can be generalized to the nongraphable n-variable case. Regardless of the number of variables, however, we shall assume in general that, when written in a general form, our objective function possesses continuous partial derivatives to any desired order. This will ensure the smoothness and differentiability of the objective function as well as its partial derivatives.

For functions of several variables, extreme values are again of two kinds: (1) absolute or global and (2) relative or local. As before, our attention will be focused heavily on relative extrema, and for this reason we shall often drop the adjective "relative," with the understanding that, unless otherwise specified, the extrema referred to are *relative*. However, in Sec. 11.5, conditions for *absolute* extrema will be given due consideration.

11.1 THE DIFFERENTIAL VERSION OF OPTIMIZATION CONDITIONS

The discussion in Chap. 9 of optimization conditions for problems with a single choice variable was couched entirely in terms of *derivatives*, as against differentials. To prepare for the discussion of problems with two or more choice variables, it would be helpful also to know how those conditions can equivalently be expressed in terms of *differentials*.

First-Order Condition

Consider the function $z = f(x)$, as depicted in Fig. 11.1. At the maximum point A as well as the minimum point B, the value of z must be stationary. In other words, it is a necessary condition for an extremum of z that $dz = 0$ instantaneously as x varies. This condition constitutes the differential version of the first-order condition for an extremum. While the condition $dz = 0$ is necessary, it is clearly *not* sufficient for either a maximum or a minimum, for the inflection point C in Fig. 11.1 also shares the property that $dz = 0$.

To see that the above condition is equivalent to the derivative version of the first-order condition $dz/dx = 0$ or $f'(x) = 0$, recall that the differential of $z = f(x)$ is

$$(11.1) \qquad dz = f'(x)\, dx$$

We note that when there is no change in x ($dx = 0$), dz will automatically be zero. But this, of course, is not what the first-order condition is all about. What the first-order condition requires is that dz be zero as x is varied, that is, as arbitrary (positive or negative, but not zero) infinitesimal changes of x occur. In such a context, with $dx \neq 0$, dz can be zero if and only if $f'(x) = 0$. Thus the derivative condition $f'(x) = 0$ and the differential condition "$dz = 0$ for arbitrary nonzero values of dx" are indeed equivalent.

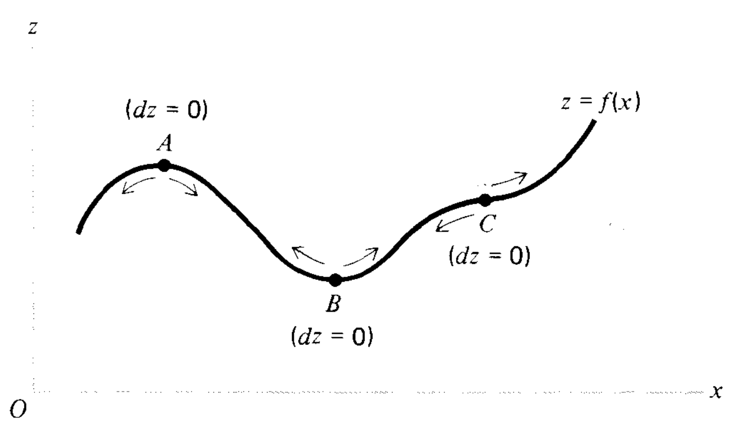


Figure 11.1

Second-Order Condition

A maximum point, such as point A in Fig. 11.1, has the graphical property that as we slide along the curve infinitesimally toward the left ($dx < 0$) and the right ($dx > 0$) of A, we are *descending* in *both* directions. A sufficient condition for achieving this is that $dz < 0$ on both sides of A in the immediate neighborhood of that point.* The fact that $dz = 0$ at point A, but $dz < 0$ at points on the two sides of A, means that dz is invariably *decreasing* as we move away from A in either direction. In other words, the condition amounts to $d(dz) < 0$—or, in a simpler notation, $d^2z < 0$—for arbitrary nonzero values of dx. The symbol $d^2z \equiv d(dz)$, denoting the differential of a differential, is known as the *second-order* differential of z. And the above condition on d^2z constitutes the differential version of the second-order sufficient condition for a maximum.

Note that the negativity of d^2z is *sufficient*, but *not necessary*, for a maximum of z. The reason is that, in certain cases, d^2z may happen to be *zero* (rather than negative) at a maximum of z. This possibility is, of course, strongly reminiscent of the cases under the Nth-derivative test where a maximum may be characterized by a *zero* second-derivative value. Indeed, in the case of a function of a single variable, there exists a very close relationship between the sign of the second-order differential d^2z and that of the second-order derivative d^2z/dx^2 or $f''(x)$, as we shall presently show.

Given that $dz = f'(x)\,dx$, we can obtain d^2z merely by further differentiation of dz. In so doing, however, we should bear in mind that dx, representing in this context an arbitrary or given nonzero change in x, is to be treated as a constant during differentiation. Consequently, dz can vary only with $f'(x)$, but since $f'(x)$ is in turn a function of x, dz can in the final analysis vary only with x. In view of this, we have

$$\begin{aligned}
(11.2) \qquad d^2z \equiv d(dz) &= d[f'(x)\,dx] && [\text{by } (11.1)]\\
&= [df'(x)]\,dx && [dx \text{ is constant}]\\
&= [f''(x)\,dx]\,dx = f''(x)\,dx^2
\end{aligned}$$

Note that the exponent 2 appears in (11.2) in two fundamentally different ways. In the symbol d^2z, the exponent 2 indicates the *second-order* differential of z; but in the symbol $dx^2 \equiv (dx)^2$, the exponent 2 denotes the *squaring* of the first-order differential dx. The result in (11.2) provides a direct link between d^2z and $f''(x)$. Inasmuch as we are considering nonzero values of dx only, the dx^2 term is always positive; thus d^2z and $f''(x)$ must take the same algebraic sign.

This fact serves to confirm our earlier claim that the differential condition "$d^2z < 0$ for arbitrary nonzero values of dx" is equivalent to the derivative condition $f''(x) < 0$ as a sufficient condition for a maximum of z. But, turning to

* This can be clarified by referring to (11.1). Let $dz < 0$ on both sides of point A. Then $f'(x)$ and dx must be opposite in sign. This means that to the left of point A (letting $dx < 0$), $f'(x)$ must be positive, so the f curve must be upward-sloping. Similarly, to the right of A (letting $dx > 0$), $f'(x)$ must be negative, so the f curve must be downward-sloping. Hence, point A is the peak of a hill.

the case of a *minimum* of z, we can also see from (11.2) that the sufficient derivative condition $f''(x) > 0$ can be equivalently stated as "$d^2z > 0$ for arbitrary nonzero values of dx." Finally, we may infer from (11.2) that the second-order *necessary* conditions

$$\text{For maximum of } z: \quad f''(x) \leq 0$$
$$\text{For minimum of } z: \quad f''(x) \geq 0$$

can be translated, respectively, into

$$\left.\begin{array}{ll}\text{For maximum of } z: & d^2z \leq 0 \\ \text{For minimum of } z: & d^2z \geq 0\end{array}\right\} \text{ for arbitrary nonzero values of } dx$$

Differential Conditions versus Derivative Conditions

Now that we have demonstrated the possibility of expressing the derivative version of first- and second-order conditions in terms of dz and d^2z, you may very well ask why we bothered to develop a new set of differential conditions when derivative conditions were already available. The answer is that differential conditions—but not derivative conditions—are stated in forms that can be directly generalized from the one-variable case to cases with two or more choice variables. To be more specific, the first-order condition (zero value for dz) and the second-order condition (negativity or positivity for d^2z) are applicable with equal validity to all cases, provided the phrase "for arbitrary nonzero values of dx" is duly modified to reflect the change in the number of choice variables.

This does not mean, however, that derivative conditions will have no further role to play. To the contrary, since derivative conditions are operationally more convenient to apply, we shall—after the generalization process is carried out by means of the differential conditions to cases with more choice variables—still attempt to develop and make use of derivative conditions appropriate to those cases.

11.2 EXTREME VALUES OF A FUNCTION OF TWO VARIABLES

For a function of one choice variable, an extreme value is represented graphically by the peak of a hill or the bottom of a valley in a two-dimensional graph. With *two* choice variables, the graph of the function—$z = f(x, y)$—becomes a surface in a 3-space, and while the extreme values are still to be associated with peaks and bottoms, these "hills" and "valleys" themselves now take on a three-dimensional character. They will, in this new context, be shaped like domes and bowls, respectively. The two diagrams in Fig. 11.2 serve to illustrate. Point A in diagram a, the peak of a dome, constitutes a maximum; the value of z at this point is larger than at any other point in its immediate neighborhood. Similarly, point B in diagram b, the bottom of a bowl, represents a minimum; everywhere in its immediate neighborhood the value of the function exceeds that at point B.

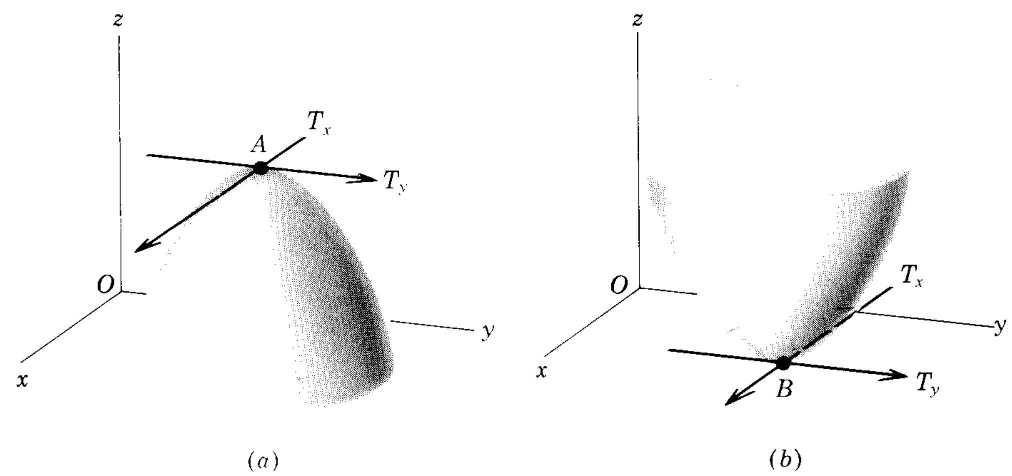


Figure 11.2

First-Order Condition

For the function

$$z = f(x, y)$$

the first-order necessary condition for an extremum (either maximum or minimum) again involves $dz = 0$. But since there are two independent variables here, dz is now a *total* differential; thus the first-order condition should be modified to the form

(11.3) $\quad dz = 0$ for arbitrary values of dx and dy, not both zero

The rationale behind (11.3) is similar to the explanation of the condition $dz = 0$ for the one-variable case: an extremum point must be a stationary point, and at a stationary point, z must be constant for arbitrary infinitesimal changes of the two variables x and y.

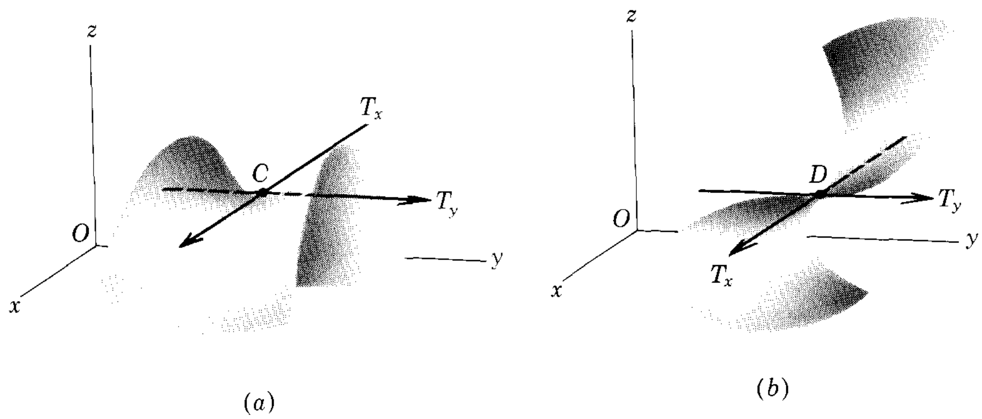
In the present two-variable case, the total differential is

$$dz = f_x\,dx + f_y\,dy \tag{11.4}$$

In order to satisfy condition (11.3), it is necessary-and-sufficient that the two partial derivatives f_x and f_y be simultaneously equal to zero. Thus the equivalent derivative version of the first-order condition (11.3) is

$$f_x = f_y = 0 \qquad \left[\text{or } \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0\right] \tag{11.5}$$

There is a simple graphical interpretation of this condition. With reference to point A in Fig. 11.2a, to have $f_x = 0$ at that point means that the tangent line T_x, drawn through A and parallel to the xz plane (holding y constant), must have a zero slope. By the same token, to have $f_y = 0$ at point A means that the tangent line T_y, drawn through A and parallel to the yz plane (holding x constant), must also have a zero slope. You can readily verify that these tangent-line requirements



(a) (b)

Figure 11.3

actually also apply to the minimum point B in Fig. 11.2b. This is because condition (11.5), like condition (11.3), is a necessary condition for *both* a maximum and a minimum.

As in the earlier discussion, the first-order condition is *necessary*, but *not sufficient*. That it is not sufficient to establish an extremum can be seen from the two diagrams in Fig. 11.3. At point C in diagram a, both T_x and T_y have zero slopes, but this point does not qualify as an extremum: Whereas it is a *minimum* when viewed against the background of the yz plane, it turns out to be a *maximum* when looked at against the xz plane! A point with such a "dual personality" is referred to, for graphical reasons, as a *saddle point*. Similarly, point D in Fig. 11.3b, while characterized by flat T_x and T_y, is no extremum, either; its location on the twisted surface makes it an *inflection point*, whether viewed against the xz or the yz plane. These counterexamples decidedly rule out the first-order condition as a sufficient condition for an extremum.

To develop a sufficient condition, we must look to the second-order total differential, which is related to second-order partial derivatives.

Second-Order Partial Derivatives

The function $z = f(x, y)$ can give rise to *two* first-order partial derivatives,

$$f_x \equiv \frac{\partial z}{\partial x} \qquad \text{and} \qquad f_y \equiv \frac{\partial z}{\partial y}$$

Since f_x is itself a function of x (as well as of y), we can measure the rate of change of f_x with respect to x, while y remains fixed, by a particular second-order (or second) partial derivative denoted by either f_{xx} or $\partial^2 z/\partial x^2$:

$$f_{xx} \equiv \frac{\partial}{\partial x}(f_x) \qquad \text{or} \qquad \frac{\partial^2 z}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

The notation f_{xx} has a double subscript signifying that the primitive function f has been differentiated partially with respect to x twice, whereas the notation $\partial^2 z/\partial x^2$ resembles that of $d^2 z/dx^2$ except for the use of the partial symbol. In a perfectly analogous manner, we can use the second partial derivative

$$f_{yy} \equiv \frac{\partial}{\partial y}(f_y) \qquad \text{or} \qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

to denote the rate of change of f_y with respect to y, while x is held constant.

Recall, however, that f_x is also a function of y and that f_y is also a function of x. Hence, there can be written two more second partial derivatives:

$$f_{xy} \equiv \frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \qquad \text{and} \qquad f_{yx} \equiv \frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

These are called *cross* (or *mixed*) *partial derivatives* because each measures the rate of change of one first-order partial derivative with respect to the "other" variable.

It bears repeating that the second-order partial derivatives of $z = f(x, y)$, like z and the first derivatives f_x and f_y, are also functions of the variables x and y. When that fact requires emphasis, we can write f_{xx} as $f_{xx}(x, y)$, and f_{xy} as $f_{xy}(x, y)$, etc. And, along the same line, we can use the notation $f_{yx}(1, 2)$ to denote the value of f_{yx} evaluated at $x = 1$ and $y = 2$, etc.

Even though f_{xy} and f_{yx} have been separately defined, they will—according to a proposition known as *Young's theorem*—be identical with each other, as long as the two cross partial derivatives are both continuous. In that case, the sequential order in which partial differentiation is undertaken becomes immaterial, because $f_{xy} = f_{yx}$. For the ordinary types of *specific* functions with which we work, this continuity condition is usually met; for *general* functions, as mentioned earlier, we always assume the continuity condition to hold. Hence, we may in general expect to find identical cross partial derivatives. In fact, the theorem applies also to functions of three or more variables. Given $z = g(u, v, w)$, for instance, the mixed partial derivatives will be characterized by $g_{uv} = g_{vu}$, $g_{vw} = g_{wv}$, etc., provided these partial derivatives are all continuous.

Example 1 Find the four second-order partial derivatives of

$$z = x^3 + 5xy - y^2$$

The first partial derivatives of this function are

$$f_x = 3x^2 + 5y \qquad \text{and} \qquad f_y = 5x - 2y$$

Therefore, upon further differentiation, we get

$$f_{xx} = 6x \qquad f_{yx} = 5 \qquad f_{xy} = 5 \qquad f_{yy} = -2$$

As expected, f_{yx} and f_{xy} are identical.

Example 2 Find all the second partial derivatives of $z = x^2 e^{-y}$. In this case, the first partial derivatives are

$$f_x = 2xe^{-y} \qquad \text{and} \qquad f_y = -x^2 e^{-y}$$

Thus we have

$$f_{xx} = 2e^{-y} \qquad f_{yx} = -2xe^{-y} \qquad f_{xy} = -2xe^{-y} \qquad f_{yy} = x^2e^{-y}$$

Again, we see that $f_{yx} = f_{xy}$.

Note that the second partial derivatives are all functions of the original variables x and y. This fact is clear enough in Example 2, but it is true even for Example 1, although some second partial derivatives happen to be *constant* functions in that case.

Second-Order Total Differential

Given the total differential dz in (11.4), and with the concept of second-order partial derivatives at our command, we can derive an expression for the second-order total differential d^2z by further differentiation of dz. In so doing, we should remember that in the equation $dz = f_x\,dx + f_y\,dy$, the symbols dx and dy represent arbitrary or given changes in x and y; so they must be treated as constants during differentiation. As a result, dz depends only on f_x and f_y, and since f_x and f_y are themselves functions of x and y, dz, like z itself, is a function of x and y.

To obtain d^2z, we merely apply the definition of a differential—as shown in (11.4)—to dz itself. Thus,

$$
\begin{aligned}
(11.6)\qquad d^2z \equiv d(dz) &= \frac{\partial(dz)}{\partial x}dx + \frac{\partial(dz)}{\partial y}dy \qquad [\text{cf. } (11.4)]\\
&= \frac{\partial}{\partial x}(f_x\,dx + f_y\,dy)\,dx + \frac{\partial}{\partial y}(f_x\,dx + f_y\,dy)\,dy\\
&= (f_{xx}\,dx + f_{xy}\,dy)\,dx + (f_{yx}\,dx + f_{yy}\,dy)\,dy\\
&= f_{xx}\,dx^2 + f_{xy}\,dy\,dx + f_{yx}\,dx\,dy + f_{yy}\,dy^2\\
&= f_{xx}\,dx^2 + 2f_{xy}\,dx\,dy + f_{yy}\,dy^2 \qquad [f_{xy} = f_{yx}]
\end{aligned}
$$

Note, again, that the exponent 2 appears in (11.6) in two different ways. In the symbol d^2z, the exponent 2 indicates the *second-order* total differential of z; but in the symbol $dx^2 \equiv (dx)^2$, the exponent denotes the *squaring* of the first-order differential dx.

The result in (11.6) shows the magnitude of d^2z (the change in dz) in terms of given values of dx and dy, measured from some point (x_0, y_0) in the domain. In order to calculate d^2z, however, we also need to know the second-order partial derivatives f_{xx}, f_{xy}, and f_{yy}, all evaluated at (x_0, y_0)—just as we need the first-order partial derivatives to calculate dz from (11.4).

Example 3 Given $z = x^3 + 5xy - y^2$, find dz and d^2z. This function is the same as the one in Example 1. Thus, substituting the various derivatives already obtained there into (11.4) and (11.6), we find*

$$dz = (3x^2 + 5y)\,dx + (5x - 2y)\,dy$$

and

$$d^2z = 6x\,dx^2 + 10dx\,dy - 2dy^2$$

At the point $x = 1$ and $y = 2$, for instance, we have

$$dz = 13dx + dy \qquad \text{and} \qquad d^2z = 6dx^2 + 10dx\,dy - 2dy^2$$

And for given dx and dy from the point $x = 1$ and $y = 2$ in the domain, the sign of dz tells the direction of change of z, whereas the sign of d^2z reveals whether dz is increasing ($d^2z > 0$) or decreasing ($d^2z < 0$).

Second-Order Condition

Using the concept of d^2z, we can state the second-order sufficient condition for a maximum of $z = f(x, y)$ as follows:

(11.7) $\quad d^2z < 0$ for arbitrary values of dx and dy, not both zero

The rationale behind (11.7) is very similar to that of the d^2z condition for the one-variable case, and it can be explained by means of Fig. 11.4, which depicts the bird's-eye view of a surface. Let point A on the surface—the point lying directly above the point (x_0, y_0) in the domain—satisfy the first-order condition (11.3). Then point A is a prospective candidate for a maximum. Whether it in fact qualifies depends on the surface configuration in the neighborhood of A. If an infinitesimal movement away from A in *any* direction along the surface (see the arrows in Fig. 11.4) invariably results in a decrease in z—that is, if $dz < 0$ for arbitrary values of dx and dy, not both zero—A is a peak of a dome. Given that $dz = 0$ at point A, however, the condition $dz < 0$ at other points in the neighborhood of A amounts to the stipulation that dz is decreasing, that is, $d^2z \equiv d(dz) < 0$, for arbitrary values of dx and dy, not both zero. Thus (11.7) constitutes a sufficient condition for identifying a stationary value as a maximum of z. Analogous reasoning would show that a counterpart second-order sufficient condition for identifying a stationary value as a *minimum* of $z = f(x, y)$ is

(11.8) $\quad d^2z > 0$ for arbitrary values of dx and dy, not both zero

* An alternative way of reaching these results is by direct differentiation of the function:

$$\begin{aligned} dz &= d(x^3) + d(5xy) - d(y^2) \\ &= 3x^2\,dx + 5y\,dx + 5x\,dy - 2y\,dy \end{aligned}$$

Further differentiation of dz (bearing in mind that dx and dy are constants) will then yield

$$\begin{aligned} d^2z &= d(3x^2)\,dx + d(5y)\,dx + d(5x)\,dy - d(2y)\,dy \\ &= (6x\,dx)\,dx + (5dy)\,dx + (5dx)\,dy - (2dy)\,dy \\ &= 6x\,dx^2 + 10dx\,dy - 2dy^2 \end{aligned}$$

The reason why (11.7) and (11.8) are only sufficient, but not necessary, conditions is that it is again possible for d^2z to take a zero value at a maximum or a minimum. For this reason, second-order *necessary* conditions must be stated with weak inequalities as follows:

$$\text{(11.9)} \qquad \left.\begin{array}{ll} \text{For maximum of } z: & d^2z \leq 0 \\ \text{For minimum of } z: & d^2z \geq 0 \end{array}\right\} \text{ for arbitrary values of } dx \text{ and } dy, \text{ not both zero}$$

In the following, however, we shall pay more attention to the second-order sufficient conditions.

For operational convenience, second-order differential conditions can be translated into equivalent conditions on second-order derivatives. In the two-variable case, (11.6) shows that this would entail restrictions on the signs of the second-order partial derivatives f_{xx}, f_{xy}, and f_{yy}. The actual translation would require a knowledge of quadratic forms, which will be discussed in the next section. But we may first introduce the main result here: For any values of dx and dy, not both zero,

$$d^2z \begin{cases} < 0 & \text{iff} \quad f_{xx} < 0; \quad f_{yy} < 0; \quad \text{and} \quad f_{xx}f_{yy} > f_{xy}^2 \\ > 0 & \text{iff} \quad f_{xx} > 0; \quad f_{yy} > 0; \quad \text{and} \quad f_{xx}f_{yy} > f_{xy}^2 \end{cases}$$

Note that the sign of d^2z hinges not only on f_{xx} and f_{yy}, which have to do with the surface configuration around point A (Fig. 11.4) in the two basic directions shown by T_x (east-west) and T_y (north-south), but also on the cross partial derivative f_{xy}. The role played by this latter partial derivative is to ensure that the surface in question will yield (two-dimensional) cross sections with the same type of

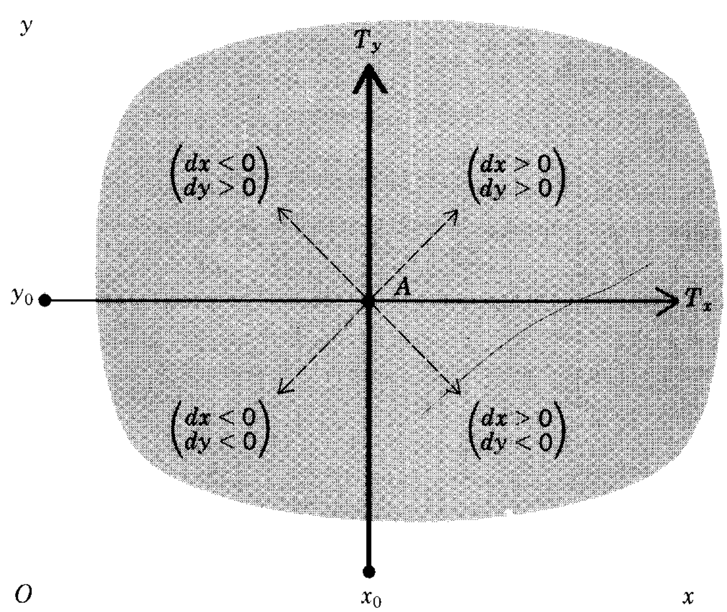


Figure 11.4

Table 11.1 Conditions for relative extremum: $z = f(x, y)$

Condition	Maximum	Minimum
First-order necessary condition	$f_x = f_y = 0$	$f_x = f_y = 0$
Second-order sufficient condition*	$f_{xx}, f_{yy} < 0$ and $f_{xx}f_{yy} > f_{xy}^2$	$f_{xx}, f_{yy} > 0$ and $f_{xx}f_{yy} > f_{xy}^2$

*Applicable only after the first-order necessary condition has been satisfied.

configuration (hill or valley, as the case may be) not only in the two basic directions (east-west and north-south), but in all other possible directions (such as northeast-southwest) as well.

The above result, together with the first-order condition (11.5), enables us to construct Table 11.1. It should be understood that all the second partial derivatives therein are to be evaluated at the stationary point where $f_x = f_y = 0$. It should also be stressed that the second-order sufficient condition is *not necessary* for an extremum. In particular, if a stationary value is characterized by $f_{xx}f_{yy} = f_{xy}^2$ in violation of that condition, that stationary value may nevertheless turn out to be an extremum. On the other hand, in the case of another type of violation, with a stationary point characterized by $f_{xx}f_{yy} < f_{xy}^2$, we can identify that point as a saddle point, because the sign of d^2z will in that case be indefinite (positive for some values of dx and dy, but negative for others).

Example 4 Find the extreme value(s) of $z = 8x^3 + 2xy - 3x^2 + y^2 + 1$. First let us find all the first and second partial derivatives:

$$f_x = 24x^2 + 2y - 6x \qquad f_y = 2x + 2y$$

$$f_{xx} = 48x - 6 \qquad f_{yy} = 2 \qquad f_{xy} = 2$$

The first-order condition calls for satisfaction of the simultaneous equations $f_x = 0$ and $f_y = 0$; that is,

$$\begin{aligned} 24x^2 + 2y - 6x &= 0 \\ 2y + 2x &= 0 \end{aligned}$$

The second equation implies that $y = -x$, and when this information is substituted into the first equation, we get $24x^2 - 8x = 0$, which yields the pair of solutions

$$\bar{x}_1 = 0 \qquad [\text{implying } \bar{y}_1 = -\bar{x}_1 = 0]$$

$$\bar{x}_2 = \tfrac{1}{3} \qquad [\text{implying } \bar{y}_2 = -\tfrac{1}{3}]$$

To apply the second-order condition, we note that, when

$$\bar{x}_1 = \bar{y}_1 = 0$$

f_{xx} turns out to be -6, while f_{yy} is 2, so that $f_{xx}f_{yy}$ is negative and is necessarily less than a squared value f_{xy}^2. This fails the second-order condition. The fact that f_{xx} and f_{yy} have opposite signs suggests, of course, that the surface in question will curl upward in one direction but downward in another, thereby giving rise to a saddle point.

What about the other solution? When evaluated at $\bar{x}_2 = \frac{1}{3}$, we find that $f_{xx} = 10$, which, together with the fact that $f_{yy} = f_{xy} = 2$, meets all three parts of the second-order sufficient condition for a minimum. Therefore, by setting $x = \frac{1}{3}$ and $y = -\frac{1}{3}$ in the given function, we can obtain as a minimum of z the value $\bar{z} = \frac{23}{27}$. In the present example, there thus exists only one relative extremum (a minimum), which can be represented by the ordered triple

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{1}{3}, \frac{-1}{3}, \frac{23}{27}\right)$$

Example 5 Find the extreme value(s) of $z = x + 2ey - e^x - e^{2y}$. The relevant derivatives of this function are

$$f_x = 1 - e^x \qquad f_y = 2e - 2e^{2y}$$

$$f_{xx} = -e^x \qquad f_{yy} = -4e^{2y} \qquad f_{xy} = 0$$

To satisfy the necessary condition, we must have

$$1 - e^x = 0$$
$$2e - 2e^{2y} = 0$$

which has only one solution, namely, $\bar{x} = 0$ and $\bar{y} = \frac{1}{2}$. To ascertain the status of the value of z corresponding to this solution (the stationary value), we evaluate the second-order derivatives at $x = 0$ and $y = \frac{1}{2}$, and find that $f_{xx} = -1$, $f_{yy} = -4e$, and $f_{xy} = 0$. Since f_{xx} and f_{yy} are both negative and since, in addition, $(-1)(-4e) > 0$, we may conclude that the z value in question, namely,

$$\bar{z} = 0 + e - e^0 - e^1 = -1$$

is a maximum value of the function. This maximum point on the given surface can be denoted by the ordered triple $(\bar{x}, \bar{y}, \bar{z}) = (0, \frac{1}{2}, -1)$.

Again, note that, to evaluate the second partial derivatives at $\bar{x}$ and $\bar{y}$, differentiation must be undertaken first, and then the specific values of $\bar{x}$ and $\bar{y}$ are to be substituted into the derivatives as the final step.

EXERCISE 11.2

Use Table 11.1 to find the extreme value(s) of each of the following four functions, and determine whether they are maxima or minima:

1 $z = x^2 + xy + 2y^2 + 3$

2 $z = -x^2 + xy - y^2 + 2x + y$

3 $z = ax^2 + by^2 + c$; consider each of the three subcases:
(*a*) $a > 0, b > 0$ (*b*) $a < 0, b < 0$ (*c*) a and b opposite in sign

4 $z = e^{2x} - 2x + 2y^2 + 3$

5 Consider the function $z = (x - 2)^4 + (y - 3)^4$.
(*a*) Establish by intuitive reasoning that z attains a minimum ($\bar{z} = 0$) at $\bar{x} = 2$ and $\bar{y} = 3$.
(*b*) Is the first-order necessary condition in Table 11.1 satisfied?
(*c*) Is the second-order sufficient condition in Table 11.1 satisfied?
(*d*) Find the value of d^2z. Does it satisfy the second-order necessary condition for a minimum in (11.9)?

11.3 QUADRATIC FORMS—AN EXCURSION

The expression for d^2z on the last line of (11.6) exemplifies what are known as *quadratic forms*, for which there exist established criteria for determining whether their signs are always positive, negative, nonpositive, or nonnegative, for arbitrary values of dx and dy, not both zero. Since the second-order condition for extremum hinges directly on the sign of d^2z, those criteria are of direct interest.

To begin with, we define a *form* as a polynomial expression in which each component term has a uniform degree. Our earlier encounter with polynomials was confined to the case of a single variable: $a_0 + a_1x + \cdots + a_nx^n$. When more variables are involved, each term of a polynomial may contain either one variable or several variables, each raised to a nonnegative integer power, such as $3x + 4x^2y^3 - 2yz$. In the special case where each term has a uniform degree—i.e., where the sum of exponents in each term is uniform—the polynomial is called a *form*. For example, $4x - 9y + z$ is a *linear form* in three variables, because each of its terms is of the first degree. On the other hand, the polynomial $4x^2 - xy + 3y^2$, in which each term is of the second degree (sum of integer exponents = 2), constitutes a *quadratic form* in two variables. We may also encounter quadratic forms in three variables, such as $x^2 + 2xy - yw + 7w^2$, or indeed in n variables.

Second-Order Total Differential as a Quadratic Form

If we consider the differentials dx and dy in (11.6) as variables and the partial derivatives as coefficients, i.e., if we let

$$(11.10)\qquad \begin{array}{lll} u \equiv dx & v \equiv dy & \\ a \equiv f_{xx} & b \equiv f_{yy} & h \equiv f_{xy}\left[= f_{yx}\right] \end{array}$$

then the second-order total differential

$$d^2z = f_{xx}\,dx^2 + 2f_{xy}\,dx\,dy + f_{yy}\,dy^2$$

can easily be identified as a quadratic form q in the two variables u and v:

$$(11.6')\qquad q = au^2 + 2huv + bv^2$$

Note that, in this quadratic form, $dx \equiv u$ and $dy \equiv v$ are cast in the role of variables, whereas the second partial derivatives are treated as constants—the exact opposite of the situation when we were differentiating dz to get d^2z. The reason for this reversal lies in the changed nature of the problem we are now dealing with. The second-order sufficient condition for extremum stipulates d^2z to be definitely positive (for a minimum) and definitely negative (for a maximum), regardless of the values that dx and dy may take (so long as they are not both zero). It is obvious, therefore, that in the present context dx and dy must be considered as *variables*. The second partial derivatives, on the other hand, will assume specific values at the points we are examining as possible extremum points, and thus may be regarded as *constants*.

The major question becomes, then: What restrictions must be placed upon a, b, and h in (11.6′), when u and v are allowed to take any values, in order to ensure a definite sign for q?

Positive and Negative Definiteness

As a matter of terminology, let us remark that a quadratic form q is said to be

$$\left.\begin{array}{l} \textit{positive definite} \\ \textit{positive semidefinite} \\ \textit{negative semidefinite} \\ \textit{negative definite} \end{array}\right\} \quad \text{if } q \text{ is invariably} \quad \left\{\begin{array}{ll} \text{positive} & (>0) \\ \text{nonnegative} & (\geq 0) \\ \text{nonpositive} & (\leq 0) \\ \text{negative} & (<0) \end{array}\right.$$

regardless of the values of the variables in the quadratic form, not all zero. If q changes signs when the variables assume different values, on the other hand, q is said to be *indefinite*. Clearly, the cases of positive and negative definiteness of $q = d^2z$ are related to the second-order *sufficient* conditions for a minimum and a maximum, respectively. The cases of *semi*definiteness, on the other hand, relate to second-order *necessary* conditions. When $q = d^2z$ is indefinite, we have the symptom of a saddle point.

Determinantal Test for Sign Definiteness

A widely used test for the sign definiteness of q calls for the examination of the signs of certain determinants. This test happens to be more easily applicable to positive and negative definiteness (as against semidefiniteness); that is, it applies more easily to second-order sufficient (as against necessary) conditions. We shall confine our discussion here to the sufficient conditions only.*

For the two-variable case, determinantal conditions for the sign definiteness of q are relatively easy to derive. In the first place, we see that the signs of the first and third terms in (11.6′) are independent of the values of the variables u and v,

* For a discussion of determinantal test for second-order necessary conditions, see Akira Takayama, *Mathematical Economics*, The Dryden Press, Hinsdale, IL, 1974, pp. 118–120.

because these variables appear in squares. Thus it is easy to specify the condition for the positive or negative definiteness of these terms alone, by restricting the signs of a and b. The trouble spot lies in the middle term. But if we can convert the entire polynomial into an expression such that the variables u and v appear only in some squares, the definiteness of the sign of q will again become tractable.

The device that will do the trick is that of completing the square. By adding h^2v^2/a to, and subtracting the same quantity from, the right side of (11.6′), we can rewrite the quadratic form as follows:

$$\begin{aligned} q &= au^2 + 2huv + \frac{h^2}{a}v^2 + bv^2 - \frac{h^2}{a}v^2 \\ &= a\left(u^2 + \frac{2h}{a}uv + \frac{h^2}{a^2}v^2\right) + \left(b - \frac{h^2}{a}\right)v^2 \\ &= a\left(u + \frac{h}{a}v\right)^2 + \frac{ab - h^2}{a}(v^2) \end{aligned}$$

Now that the variables u and v appear only in squares, we can predicate the sign of q entirely on the values of the coefficients a, b, and h as follows:

$$(11.11) \qquad q \text{ is } \begin{Bmatrix} \text{positive definite} \\ \text{negative definite} \end{Bmatrix} \qquad \text{iff} \qquad \begin{Bmatrix} a > 0 \\ a < 0 \end{Bmatrix} \quad \text{and} \quad ab - h^2 > 0$$

Note that (1) $ab - h^2$ should be *positive* in both cases and (2) as a prerequisite for the positivity of $ab - h^2$, the product ab must be positive (since it must exceed the squared term h^2); hence, the above condition automatically implies that a and b must take the identical algebraic sign.

The condition just derived may be stated more succinctly by the use of determinants. We observe first that the quadratic form in (11.6′) can be rearranged into the following square, symmetric format:

$$\begin{aligned} q = \quad & a(u^2) + h(uv) \\ & + h(vu) + b(v^2) \end{aligned}$$

with the squared terms placed on the diagonal and with the $2huv$ term split into two equal parts and placed off the diagonal. The coefficients now form a symmetric matrix, with a and b on the principal diagonal and h off the diagonal. Viewed in this light, the quadratic form is also easily seen to be the 1×1 matrix (a scalar) resulting from the following matrix multiplication:

$$q = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

The determinant of the 2×2 coefficient matrix, $\begin{vmatrix} a & h \\ h & b \end{vmatrix}$—which is referred to as the *discriminant* of the quadratic form q, and which we shall therefore denote by $|D|$—supplies the clue to the criterion in (11.11), for the latter can be alternatively expressed as:

$$(11.11') \qquad q \text{ is } \begin{Bmatrix} \text{positive definite} \\ \text{negative definite} \end{Bmatrix} \quad \text{iff} \quad \begin{Bmatrix} |a| > 0 \\ |a| < 0 \end{Bmatrix} \quad \text{and} \quad \begin{vmatrix} a & h \\ h & b \end{vmatrix} > 0$$

The determinant $|a| = a$ is a subdeterminant of $|D|$ that consists of the *first* element on the principal diagonal; thus it is called the *first principal minor* of $|D|$. The determinant $\begin{vmatrix} a & h \\ h & b \end{vmatrix}$ can also be considered a subdeterminant of $|D|$; since it involves the *first and second* elements on the principal diagonal, it is called the *second principal minor* of $|D|$. In the present case, there are only two principal minors available, and their signs will serve to determine the positive or negative definiteness of q.

When (11.11') is translated, via (11.10), into terms of the second-order total differential d^2z, we have

$$d^2z \text{ is } \begin{Bmatrix} \text{positive definite} \\ \text{negative definite} \end{Bmatrix}$$

$$\text{iff} \quad \begin{Bmatrix} f_{xx} > 0 \\ f_{xx} < 0 \end{Bmatrix} \quad \text{and} \quad \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2 > 0$$

Recalling that the last inequality above implies that f_{xx} and f_{yy} are required to take the *same* sign, we see that this is precisely the second-order sufficient condition presented in Table 11.1.

In general, the discriminant of a quadratic form

$$q = au^2 + 2huv + bv^2$$

is the symmetric determinant $\begin{vmatrix} a & h \\ h & b \end{vmatrix}$. In the particular case of the quadratic form

$$d^2z = f_{xx}\,dx^2 + 2f_{xy}\,dx\,dy + f_{yy}\,dy^2$$

the discriminant is a determinant with the second-order partial derivatives as its elements. Such a determinant is called a *Hessian determinant* (or simply a *Hessian*). In the two-variable case, the Hessian is

$$|H| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

which, in view of Young's theorem ($f_{xy} = f_{yx}$), is symmetric—as a discriminant should be. You should carefully distinguish the Hessian determinant from the Jacobian determinant discussed in Sec. 7.6.

Example 1 Is $q = 5u^2 + 3uv + 2v^2$ either positive or negative definite? The discriminant of q is $\begin{vmatrix} 5 & 1.5 \\ 1.5 & 2 \end{vmatrix}$, with principal minors

$$5 > 0 \qquad \text{and} \qquad \begin{vmatrix} 5 & 1.5 \\ 1.5 & 2 \end{vmatrix} = 7.75 > 0$$

Therefore q is positive definite.

Example 2 Given $f_{xx} = -2$, $f_{xy} = 1$, and $f_{yy} = -1$ at a certain point on a function $z = f(x, y)$, does d^2z have a definite sign at that point regardless of the values of dx and dy? The discriminant of the quadratic form d^2z is in this case $\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}$, with principal minors

$$-2 < 0 \qquad \text{and} \qquad \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = 1 > 0$$

Thus d^2z is negative definite.

Three-Variable Quadratic Forms

Can similar conditions be obtained for a quadratic form in *three* variables?

A quadratic form with three variables u_1, u_2, and u_3 may be generally represented as

$$\begin{aligned}(11.12) \qquad q(u_1, u_2, u_3) = {} & d_{11}(u_1^2) + d_{12}(u_1u_2) + d_{13}(u_1u_3) \\ & + d_{21}(u_2u_1) + d_{22}(u_2^2) + d_{23}(u_2u_3) \\ & + d_{31}(u_3u_1) + d_{32}(u_3u_2) + d_{33}(u_3^2) \\ = {} & \sum_{i=1}^{3} \sum_{j=1}^{3} d_{ij}u_iu_j \end{aligned}$$

where the double-Σ (double-sum) notation means that both the index i and the index j are allowed to take the values 1, 2, and 3; and thus the double-sum expression is equivalent to the 3×3 array shown above. Such a square array of the quadratic form is, incidentally, always to be considered a symmetric one, even though we have written the pair of coefficients (d_{12}, d_{21}) or (d_{23}, d_{32}) as if the two members of each pair were different. For if the term in the quadratic form involving the variables u_1 and u_2 happens to be, say, $12u_1u_2$, we always let $d_{12} = d_{21} = 6$, so that $d_{12}u_1u_2 = d_{21}u_2u_1$, and a similar procedure may be applied to make the other off-diagonal elements symmetrical.

Actually, this three-variable quadratic form is again expressible as a product of three matrices:

$$(11.12') \qquad q(u_1, u_2, u_3) = [u_1 \quad u_2 \quad u_3] \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \equiv u'Du$$

As in the two-variable case, the first matrix (a row vector) and the third matrix (a column vector) merely list the variables, and the middle one (D) is a symmetric coefficient matrix from the square-array version of the quadratic form in (11.12). This time, however, a total of *three* principal minors can be formed from its discriminant, namely,

$$|D_1| \equiv d_{11} \qquad |D_2| \equiv \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} \qquad |D_3| \equiv \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}$$

where $|D_i|$ denotes the ith principal minor of the discriminant $|D|$.* It turns out that the conditions for positive or negative definiteness can again be stated in terms of certain sign restrictions on these principal minors.

By the now-familiar device of completing the square, the quadratic form in (11.12) can be converted into an expression in which the three variables appear only as components of some squares. Specifically, recalling that $a_{12} = a_{21}$, etc., we have

$$\begin{aligned} q = {} & d_{11}\left(u_1 + \frac{d_{12}}{d_{11}}u_2 + \frac{d_{13}}{d_{11}}u_3\right)^2 \\ & + \frac{d_{11}d_{22} - d_{12}^2}{d_{11}}\left(u_2 + \frac{d_{11}d_{23} - d_{12}d_{13}}{d_{11}d_{22} - d_{12}^2}u_3\right)^2 \\ & + \frac{d_{11}d_{22}d_{33} - d_{11}d_{23}^2 - d_{22}d_{13}^2 - d_{33}d_{12}^2 + 2d_{12}d_{13}d_{23}}{d_{11}d_{22} - d_{12}^2}(u_3)^2 \end{aligned}$$

This sum of squares will be positive (negative) for any values of u_1, u_2, and u_3, not all zero, if and only if the coefficients of the three squared expressions are all positive (negative). But the three coefficients (in the order given) can be expressed in terms of the three principal minors as follows:

$$|D_1| \qquad \frac{|D_2|}{|D_1|} \qquad \frac{|D_3|}{|D_2|}$$

Hence, for *positive definiteness*, the necessary-and-sufficient condition is threefold:

$$\begin{aligned} & |D_1| > 0 \\ & |D_2| > 0 \qquad [\text{given that } |D_1| > 0 \text{ already}] \\ & |D_3| > 0 \qquad [\text{given that } |D_2| > 0 \text{ already}] \end{aligned}$$

In other words, the three principal minors must all be positive. For *negative definiteness*, on the other hand, the necessary-and-sufficient condition becomes:

$$\begin{aligned} & |D_1| < 0 \\ & |D_2| > 0 \qquad [\text{given that } |D_1| < 0 \text{ already}] \\ & |D_3| < 0 \qquad [\text{given that } |D_2| > 0 \text{ already}] \end{aligned}$$

That is, the three principal minors must alternate in sign in the specified manner.

* We have so far viewed the ith principal minor $|D_i|$ as a subdeterminant formed by retaining the first i principal-diagonal elements of $|D|$. Since the notion of a *minor* implies the *deletion* of something from the original determinant, however, you may prefer to view the ith principal minor alternatively as a subdeterminant formed by deleting the last $(n - i)$ rows and columns of $|D|$.

Example 3 Determine whether $q = u_1^2 + 6u_2^2 + 3u_3^2 - 2u_1u_2 - 4u_2u_3$ is either positive or negative definite. The discriminant of q is

$$\begin{vmatrix} 1 & -1 & 0 \\ -1 & 6 & -2 \\ 0 & -2 & 3 \end{vmatrix}$$

with principal minors as follows:

$$1 > 0 \qquad \begin{vmatrix} 1 & -1 \\ -1 & 6 \end{vmatrix} = 5 > 0 \qquad \text{and} \qquad \begin{vmatrix} 1 & -1 & 0 \\ -1 & 6 & -2 \\ 0 & -2 & 3 \end{vmatrix} = 11 > 0$$

Therefore, the quadratic form is positive definite.

Example 4 Determine whether $q = 2u^2 + 3v^2 - w^2 + 6uv - 8uw - 2vw$ is either positive or negative definite. The discriminant may be written as $\begin{vmatrix} 2 & 3 & -4 \\ 3 & 3 & -1 \\ -4 & -1 & -1 \end{vmatrix}$, and we find its first principal minor to be $2 > 0$, but the second principal minor is $\begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} = -3 < 0$. This violates the condition for both positive and negative definiteness; thus q is neither positive nor negative definite.

n-Variable Quadratic Forms

As an extension of the above result to the n-variable case, we shall state without proof that, for the quadratic form

$$\begin{aligned} q(u_1, u_2, \dots, u_n) &= \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} u_i u_j & & [\text{where } d_{ij} = d_{ji}] \\ &= \underset{(1\times n)}{u'} \ \underset{(n\times n)}{D} \ \underset{(n\times 1)}{u} & & [\text{cf. } (11.12')] \end{aligned}$$

the necessary-and-sufficient condition for *positive definiteness* is that the principal minors of $|D|$, namely,

$$|D_1| \equiv d_{11} \qquad |D_2| \equiv \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} \qquad \cdots \qquad |D_n| \equiv \begin{vmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{vmatrix}$$

all be positive. The corresponding necessary-and-sufficient condition for *negative definiteness* is that the principal minors alternate in sign as follows:

$$|D_1| < 0 \qquad |D_2| > 0 \qquad |D_3| < 0 \qquad \text{(etc.)}$$

so that all the *odd*-numbered principal minors are negative and all *even*-numbered ones are positive. The nth principal minor, $|D_n| = |D|$, should be positive if n is even, but negative if n is odd. This can be expressed succinctly by the inequality $(-1)^n|D_n| > 0$.

Characteristic-Root Test for Sign Definiteness

Aside from the above determinantal test for the sign definiteness of a quadratic form $u'Du$, there is an alternative test that utilizes the concept of the so-called "characteristic roots" of the matrix D. This concept arises in a problem of the following nature. Given an $n \times n$ matrix D, can we find a scalar r, and an $n \times 1$ vector $x \neq 0$, such that the matrix equation

$$\underset{(n\times n)}{D}\ \underset{(n\times 1)}{x} = r\ \underset{(n\times 1)}{x} \tag{11.13}$$

is satisfied? If so, the scalar r is referred to as a *characteristic root* of matrix D and x as a *characteristic vector* of that matrix.*

The matrix equation $Dx = rx$ can be rewritten as $Dx - rIx = 0$, or

$$(D - rI)x = 0 \qquad \text{where 0 is } n \times 1 \tag{11.13'}$$

This, of course, represents a system of n homogeneous linear equations. Since we want a nontrivial solution for x, the coefficient matrix $(D - rI)$—called the *characteristic matrix* of D—is required to be singular. In other words, its determinant must be made to vanish:

$$|D - rI| = \begin{vmatrix} d_{11} - r & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} - r & \cdots & d_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} - r \end{vmatrix} = 0 \tag{11.14}$$

Equation (11.14) is called the *characteristic equation* of matrix D. Since the determinant $|D - rI|$ will yield, upon Laplace expansion, an nth-degree polynomial in the variable r, (11.14) is in fact an nth-degree polynomial equation. There will thus be a total of n roots, $(r_1, \ldots, r_n)$, each of which qualifies as a characteristic root. If D is symmetric, as is the case in the quadratic-form context, the characteristic roots will always turn out to be real numbers, but they can take either algebraic sign, or be zero.

Inasmuch as these values of r will all make the determinant $|D - rI|$ vanish, the substitution of any of these (say, r_i) into the equation system (11.13') will produce a corresponding vector $x|_{r=r_i}$. More accurately, the system being homogeneous, it will yield an infinite number of vectors corresponding to the root r_i. We shall, however, apply a process of *normalization* (to be explained below) and

* Characteristic roots are also known by the alternative names of *latent roots*, or *eigenvalues*. Characteristic vectors are also called *eigenvectors*.

select a particular member of that infinite set as *the* characteristic vector corresponding to r_i; this vector will be denoted by v_i. With a total of n characteristic roots, there should be a total of n such corresponding characteristic vectors.

Example 5 Find the characteristic roots and vectors of the matrix $\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$. By substituting the given matrix for D in (11.14), we get the equation

$$\begin{vmatrix} 2-r & 2 \\ 2 & -1-r \end{vmatrix} = r^2 - r - 6 = 0$$

with roots $r_1 = 3$ and $r_2 = -2$. When the first root is used, the matrix equation (11.13′) takes the form of

$$\begin{bmatrix} 2-3 & 2 \\ 2 & -1-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The two rows of the coefficient matrix being linearly dependent, as we would expect in view of (11.14), there is an infinite number of solutions, which can be expressed by the equation $x_1 = 2x_2$. To force out a unique solution, we *normalize* the solution by imposing the restriction $x_1^2 + x_2^2 = 1$.* Then, since

$$x_1^2 + x_2^2 = (2x_2)^2 + x_2^2 = 5x_2^2 = 1$$

we can obtain (by taking the positive square root) $x_2 = 1/\sqrt{5}$, and also $x_1 = 2x_2 = 2/\sqrt{5}$. Thus the first characteristic vector is

$$v_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

Similarly, by using the second root $r_2 = -2$ in (11.13′), we get the equation

$$\begin{bmatrix} 2-(-2) & 2 \\ 2 & -1-(-2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which has the solution $x_1 = -\frac{1}{2}x_2$. Upon normalization, we find

$$x_1^2 + x_2^2 = \left(-\tfrac{1}{2}x_2\right)^2 + x_2^2 = \tfrac{5}{4}x_2^2 = 1$$

which yields $x_2 = 2/\sqrt{5}$ and $x_1 = -1/\sqrt{5}$. Thus the second characteristic vector is

$$v_2 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

The set of characteristic vectors obtained in this manner possesses two

* More generally, for the n-variable case, we require that $\sum_{i=1}^{n} x_i^2 = 1$.

important properties: First, the scalar product $v_i'v_i$ ($i = 1, 2, \ldots, n$) must be equal to unity, since

$$v_i'v_i = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i^2 = 1 \qquad \text{[by normalization]}$$

Second, the scalar product $v_i'v_j$ (where $i \neq j$) can always be taken to be zero.* In sum, therefore, we may write that

$$(11.15) \qquad v_i'v_i = 1 \qquad \text{and} \qquad v_i'v_j = 0 \qquad (i \neq j)$$

These properties will prove useful below. As a matter of terminology, when two vectors yield a zero-valued scalar product, the vectors are said to be *orthogonal* (perpendicular) to each other.† Hence each pair of characteristic vectors of matrix D must be orthogonal. The other property, $v_i'v_i = 1$, is indicative of normalization. Together, these two properties account for the fact that the characteristic vectors ($v_1, \ldots, v_n$) are said to be a set of *orthonormal* vectors. You should try to verify the orthonormality of the two characteristic vectors found in Example 5.

Now we are ready to explain how the characteristic roots and characteristic vectors of matrix D can be of service in determining the sign definiteness of the quadratic form $u'Du$. In essence, the idea is again to transform $u'Du$ (which involves not only squared terms $u_1^2, \ldots, u_n^2$, but also cross-product terms such as u_1u_2 and u_2u_3) into a form that contains only squared terms. Thus the approach is similar in intent to the completing-the-square process used in deriving the determinantal test above. However, in the present case, the transformation possesses the additional feature that each squared term has as its coefficient one of the characteristic roots, so that the signs of the n roots will provide sufficient information for determining the sign definiteness of the quadratic form.

* To demonstrate this, we note that, by (11.13), we may write $Dv_j = r_jv_j$, and $Dv_i = r_iv_i$. By premultiplying both sides of each of these equations by an appropriate row vector, we have

$$v_i'Dv_j = v_i'r_jv_j = r_jv_i'v_j \qquad [r_j \text{ is a scalar}]$$

$$v_j'Dv_i = v_j'r_iv_i = r_iv_j'v_i = r_iv_i'v_j \qquad [v_i'v_j = v_j'v_i]$$

Since $v_i'Dv_j$ and $v_j'Dv_i$ are both 1×1, and since they are transposes of each other (recall that $D' = D$ because D is symmetric), they must represent the same scalar. It follows that the extreme-right expressions in these two equations are equal; hence, by subtracting, we have

$$(r_j - r_i)v_i'v_j = 0$$

Now if $r_j \neq r_i$ (distinct roots), then $v_i'v_j$ has to be zero in order for the equation to hold, and this establishes our claim. If $r_j = r_i$ (repeated roots), moreover, it will always be possible, as it turns out, to find two linearly independent normalized vectors satisfying $v_i'v_j = 0$. Thus, we may state in general that $v_i'v_j = 0$, whenever $i \neq j$.

† As a simple illustration of this, think of the two unit vectors of a 2-space, $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. These vectors lie, respectively, on the two axes, and are thus perpendicular. At the same time, we do find that $e_1'e_2 = e_2'e_1 = 0$.

The transformation that will do the trick is as follows. Let the characteristic vectors $v_1, \ldots, v_n$ constitute the columns of a matrix T:

$$\underset{(n \times n)}{T} = [v_1 \quad v_2 \quad \cdots \quad v_n]$$

and then apply the transformation $\underset{(n\times 1)}{u} = \underset{(n\times n)}{T} \underset{(n\times 1)}{y}$ to the quadratic form $u'Du$:

$$\begin{aligned} u'Du &= (Ty)'D(Ty) = y'T'DTy \qquad [\text{by } (4.11)] \\ &= y'Ry \qquad \text{where} \qquad R \equiv T'DT \end{aligned}$$

As a result, the original quadratic form in the variables u_i is now turned into another quadratic form in the variables y_i. Since the u_i variables and the y_i variables take the same range of values, the transformation does not affect the sign definiteness of the quadratic form. Thus we may now just as well consider the sign of the quadratic form $y'Ry$ instead. What makes this latter quadratic form intriguing is that the matrix R will turn out to be a diagonal one, with the roots $r_1, \ldots, r_n$ of matrix D displayed along its diagonal, and with zeros everywhere else, so that we have in fact

$$(11.16) \qquad u'Du = y'Ry = [y_1 \quad y_2 \quad \cdots \quad y_n] \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & r_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= r_1 y_1^2 + r_2 y_2^2 + \cdots + r_n y_n^2$$

which is an expression involving squared terms only. The transformation $R \equiv T'DT$ provides us, therefore, with a procedure for *diagonalizing* the symmetric matrix D into the special diagonal matrix R.

Example 6 Verify that the matrix $\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$ given in Example 5 can be diagonalized into the matrix $\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$. On the basis of the characteristic vectors found in Example 5, the transformation matrix T should be

$$T = [v_1 \quad v_2] = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

Thus we may write

$$R \equiv T'DT = \begin{bmatrix} \dfrac{2}{\sqrt{5}} & \dfrac{1}{\sqrt{5}} \\ -\dfrac{1}{\sqrt{5}} & \dfrac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \dfrac{2}{\sqrt{5}} & -\dfrac{1}{\sqrt{5}} \\ \dfrac{1}{\sqrt{5}} & \dfrac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

which duly verifies the diagonalization process.

To prove the diagonalization result in (11.16), let us (partially) write out the matrix R as follows:

$$R \equiv T'DT = \begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix} D[v_1 \quad v_2 \quad \cdots \quad v_n]$$

We may easily verify that $D[v_1 \quad v_2 \quad \cdots \quad v_n]$ can be rewritten as $[Dv_1 \quad Dv_2 \quad \cdots \quad Dv_n]$. Besides, by (11.13), we can further rewrite this as $[r_1 v_1 \quad r_2 v_2 \quad \cdots \quad r_n v_n]$. Hence, we see that

$$R = \begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix} [r_1 v_1 \quad r_2 v_2 \quad \cdots \quad r_n v_n] = \begin{bmatrix} r_1 v_1' v_1 & r_2 v_1' v_2 & \cdots & r_n v_1' v_n \\ r_1 v_2' v_1 & r_2 v_2' v_2 & \cdots & r_n v_2' v_n \\ \cdots & \cdots & \cdots & \cdots \\ r_1 v_n' v_1 & r_2 v_n' v_2 & \cdots & r_n v_n' v_n \end{bmatrix}$$

$$= \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & r_n \end{bmatrix} \qquad [\text{by } (11.15)]$$

which is precisely what we intended to show.

In view of the result in (11.16), we may formally state the characteristic-root test for the sign definiteness of a quadratic form as follows:

a $q = u'Du$ is positive (negative) definite, if and only if *every* characteristic root of D is positive (negative)

b $q = u'Du$ is positive (negative) semidefinite, if and only if *all* characteristic roots of D are nonnegative (nonpositive)

c $q = u'Du$ is indefinite, if and only if some of the characteristic roots of D are positive and some are negative

Note that, in applying this test, all we need are the characteristic roots; the characteristic vectors are not required unless we wish to find the transformation matrix T. Note, also, that this test, unlike the determinantal test outlined above, permits us to check the second-order necessary conditions (part *b* above) simultaneously with the sufficient conditions (part *a*). However, it does have a drawback. When the matrix D is of a high dimension, the polynomial equation (11.14) may not be easily solvable for the characteristic roots needed for the test. In such cases, the determinantal test might yet be preferable.

EXERCISE 11.3

1 By direct matrix multiplication, express each matrix product below as a quadratic form:

(a) $[u \quad v]\begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$ (c) $[x \quad y]\begin{bmatrix} 5 & 2 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(b) $[u \quad v]\begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$ (d) $[dx \quad dy]\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\begin{bmatrix} dx \\ dy \end{bmatrix}$

2 In parts b and c of the preceding problem, the coefficient matrices are not symmetric with respect to the principal diagonal. Verify that by averaging the off-diagonal elements and thus converting them, respectively, into $\begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}$ and $\begin{bmatrix} 5 & 3 \\ 3 & 0 \end{bmatrix}$ we will get the same quadratic forms as before.

3 On the basis of their coefficient matrices (the *symmetric* versions), determine by the determinantal test whether the quadratic forms in Exercise 11.3-1a, b, and c are either positive definite or negative definite.

4 Express each quadratic form below as a matrix product involving a *symmetric* coefficient matrix:

(a) $q = 3u^2 - 4uv + 7v^2$
(b) $q = u^2 + 7uv + 3v^2$
(c) $q = 8uv - u^2 - 31v^2$
(d) $q = 6xy - 5y^2 - 2x^2$
(e) $q = 3u_1^2 - 2u_1u_2 + 4u_1u_3 + 5u_2^2 + 4u_3^2 - 2u_2u_3$
(f) $q = -u^2 + 4uv - 6uw - 4v^2 - 7w^2$

5 From the discriminants obtained from the symmetric coefficient matrices of the preceding problem, ascertain by the determinantal test which of the quadratic forms are positive definite and which are negative definite.

6 Find the characteristic roots of each of the following matrices:

(a) $D = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$ (b) $E = \begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}$ (c) $F = \begin{bmatrix} 5 & 3 \\ 3 & 0 \end{bmatrix}$

What can you conclude about the signs of the quadratic forms $u'Du$, $u'Eu$, and $u'Fu$? (Check your results against Exercise 11.3-3.)

7 Find the characteristic vectors of the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$.

8 Given a quadratic form $u'Du$, where D is 2×2, the characteristic equation of D can be written as

$$\begin{vmatrix} d_{11} - r & d_{12} \\ d_{21} & d_{22} - r \end{vmatrix} = 0 \qquad (d_{12} = d_{21})$$

Expand the determinant; express the roots of this equation by use of the quadratic formula; and deduce the following:

(a) No imaginary number (a number involving $\sqrt{-1}$) can occur in r_1 and r_2.
(b) To have repeated roots, matrix D must be in the form of $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$
(c) To have either positive or negative semidefiniteness, the discriminant of the quadratic form may vanish, that is, $|D| = 0$ is possible.

11.4 OBJECTIVE FUNCTIONS WITH MORE THAN TWO VARIABLES

When there appear in an objective function $n > 2$ choice variables, it is no longer possible to graph the function, although we can still speak of a *hypersurface* in an $(n + 1)$-dimensional space. On such a (nongraphable) hypersurface, there again may exist $(n + 1)$-dimensional analogs of peaks of domes and bottoms of bowls. How do we identify them?

First-Order Condition for Extremum

Let us specifically consider a function of three choice variables,

$$z = f(x_1, x_2, x_3)$$

with first partial derivatives f_1, f_2, and f_3 and second partial derivatives f_{ij} $(\equiv \partial^2 z/\partial x_i \partial x_j)$, with $i, j = 1, 2, 3$. By virtue of Young's theorem, we have $f_{ij} = f_{ji}$.

Our earlier discussion suggests that, to have a maximum or a minimum of z, it is necessary that $dz = 0$ for arbitrary values of dx_1, dx_2, and dx_3, not all zero. Since the value of dz is now

$$dz = f_1\, dx_1 + f_2\, dx_2 + f_3\, dx_3 \tag{11.17}$$

and since dx_1, dx_2, and dx_3 are arbitrary (infinitesimal) changes in the independent variables, not all zero, the only way to guarantee a zero dz is to have $f_1 = f_2 = f_3 = 0$. Thus, again, the necessary condition for extremum is that all the first-order partial derivatives be zero, the same as for the two-variable case.*

Second-Order Condition

The satisfaction of the first-order condition earmarks certain values of z as the stationary values of the objective function. If at a stationary value of z we find that d^2z is positive definite, this will suffice to establish that value of z as a minimum. Analogously, the negative definiteness of d^2z is a sufficient condition for the stationary value to be a maximum. This raises the questions of how to express d^2z when there are three variables in the function and how to determine its positive or negative definiteness.

* As a special case, note that if we happen to be working with a function $z = f(x_1, x_2, x_3)$ implicitly defined by an equation $F(z, x_1, x_2, x_3) = 0$, where

$$f_i \equiv \frac{\partial z}{\partial x_i} = \frac{-\partial F/\partial x_i}{\partial F/\partial z} \qquad (i = 1, 2, 3)$$

then the first-order condition $f_1 = f_2 = f_3 = 0$ will amount to the condition

$$\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = \frac{\partial F}{\partial x_3} = 0$$

since the value of the denominator $\partial F/\partial z \neq 0$ makes no difference.

The expression for d^2z can be obtained by differentiating dz in (11.17). In such a process, as in (11.6), we should treat the derivatives f_i as variables and the differentials dx_i as constants. Thus we have

$$(11.18)\qquad \begin{aligned} d^2z = d(dz) &= \frac{\partial(dz)}{\partial x_1}dx_1 + \frac{\partial(dz)}{\partial x_2}dx_2 + \frac{\partial(dz)}{\partial x_3}dx_3 \\ &= \frac{\partial}{\partial x_1}(f_1\,dx_1 + f_2\,dx_2 + f_3\,dx_3)\,dx_1 \\ &\quad + \frac{\partial}{\partial x_2}(f_1\,dx_1 + f_2\,dx_2 + f_3\,dx_3)\,dx_2 \\ &\quad + \frac{\partial}{\partial x_3}(f_1\,dx_1 + f_2\,dx_2 + f_3\,dx_3)\,dx_3 \\ &= f_{11}\,dx_1^2 + f_{12}\,dx_1\,dx_2 + f_{13}\,dx_1\,dx_3 \\ &\quad + f_{21}\,dx_2\,dx_1 + f_{22}\,dx_2^2 + f_{23}\,dx_2\,dx_3 \\ &\quad + f_{31}\,dx_3\,dx_1 + f_{32}\,dx_3\,dx_2 + f_{33}\,dx_3^2 \end{aligned}$$

which is a quadratic form similar to (11.12). Consequently, the criteria for positive and negative definiteness we learned earlier are directly applicable here.

In determining the positive or negative definiteness of d^2z, we must again, as we did in (11.6′), regard dx_i as variables that can take any values (though not all zero), while considering the derivatives f_{ij} as coefficients upon which to impose certain restrictions. The coefficients in (11.18) give rise to the symmetric Hessian determinant

$$|H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

whose successive principal minors may be denoted by

$$|H_1| = f_{11} \qquad |H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \qquad |H_3| = |H|$$

Thus, on the basis of the determinantal criteria for positive and negative definiteness, we may state the second-order sufficient condition for an extremum of z as follows:

$$(11.19)\qquad \bar{z} \text{ is a } \begin{Bmatrix} \text{maximum} \\ \text{minimum} \end{Bmatrix}$$

$$\text{if} \quad \begin{cases} |H_1| < 0; \quad |H_2| > 0; \quad |H_3| < 0 & (d^2z \text{ negative definite}) \\ |H_1| > 0; \quad |H_2| > 0; \quad |H_3| > 0 & (d^2z \text{ positive definite}) \end{cases}$$

In using this condition, we must evaluate all the principal minors at the stationary point where $f_1 = f_2 = f_3 = 0$.

We may, of course, also apply the characteristic-root test and associate the positive definiteness (negative definiteness) of d^2z with the positivity (negativity) of all the characteristic roots of the *Hessian matrix* $\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$. In fact, instead of saying that the second-order total differential d^2z is positive (negative) definite, it is also acceptable to state that the Hessian matrix H (to be distinguished from the Hessian determinant $|H|$) is positive (negative) definite. In this usage, however, note that the sign definiteness of H refers to the sign of the quadratic form d^2z with which H is associated, *not* to the signs of the elements of H per se.

Example 1 Find the extreme value(s) of

$$z = 2x_1^2 + x_1x_2 + 4x_2^2 + x_1x_3 + x_3^2 + 2$$

The first-order condition for extremum involves the simultaneous satisfaction of the following three equations:

$$\begin{aligned} (f_1 =)\ 4x_1 + x_2 + x_3 &= 0 \\ (f_2 =)\ x_1 + 8x_2 &= 0 \\ (f_3 =)\ x_1 + 2x_3 &= 0 \end{aligned}$$

Because this is a homogeneous linear-equation system, in which all the three equations are independent (the determinant of the coefficient matrix does not vanish), there exists only the single solution $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 0$. This means that there is only one stationary value, $\bar{z} = 2$.

The Hessian determinant of this function is

$$|H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

the principal minors of which are all positive:

$$|H_1| = 4 \qquad |H_2| = 31 \qquad |H_3| = 54$$

Thus we can conclude, by (11.9), that $\bar{z} = 2$ is a minimum.

Example 2 Find the extreme value(s) of

$$z = -x_1^3 + 3x_1x_3 + 2x_2 - x_2^2 - 3x_3^2$$

The first partial derivatives are found to be

$$f_1 = -3x_1^2 + 3x_3 \qquad f_2 = 2 - 2x_2 \qquad f_3 = 3x_1 - 6x_3$$

By setting all f_i equal to zero, we get three simultaneous equations, one nonlinear and two linear:

$$\begin{aligned} -3x_1^2 \qquad\quad &+3x_3 = 0 \\ -2x_2 \qquad &\quad\; = -2 \\ 3x_1 \qquad\quad &-6x_3 = 0 \end{aligned}$$

Since the second equation gives $\bar{x}_2 = 1$ and the third equation implies $\bar{x}_1 = 2\bar{x}_3$, substitution of these into the first equation yields two solutions:

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \begin{cases} (0, 1, 0), \text{ implying } \bar{z} = 1 \\ (\frac{1}{2}, 1, \frac{1}{4}), \text{ implying } \bar{z} = \frac{17}{16} \end{cases}$$

The second-order partial derivatives, properly arranged, give us the Hessian

$$|H| = \begin{vmatrix} -6x_1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{vmatrix}$$

in which the first element $(-6x_1)$ reduces to 0 under the first solution (with $\bar{x}_1 = 0$) and to -3 under the second (with $\bar{x}_1 = \frac{1}{2}$). It is immediately obvious that the first solution does not satisfy the second-order sufficient condition, since $|H_1| = 0$. We may, however, resort to the characteristic-root test for further information. For this purpose, we apply the characteristic equation (11.14). Since the quadratic form being tested is d^2z, whose discriminant is the Hessian determinant, we should, of course, substitute the elements of the Hessian for the d_{ij} elements in that equation. Hence the characteristic equation is (for the first solution)

$$\begin{vmatrix} -r & 0 & 3 \\ 0 & -2-r & 0 \\ 3 & 0 & -6-r \end{vmatrix} = 0$$

which, upon expansion, becomes the cubic equation

$$r^3 + 8r^2 + 3r - 18 = 0$$

By trial and error, we are able to factor the cubic function and rewrite the above equation as

$$(r+2)(r^2 + 6r - 9) = 0$$

It is clear from the $(r+2)$ term that one of the characteristic roots is $r_1 = -2$. The other two roots can be found by applying the quadratic formula to the other term; they are $r_2 = -3 + \frac{1}{2}\sqrt{72}$, and $r_3 = -3 - \frac{1}{2}\sqrt{72}$. Inasmuch as r_1 and r_3 are negative but r_2 is positive, the quadratic form d^2z is indefinite, thereby violating the second-order necessary conditions for both a maximum and a minimum z. Thus the first solution $(\bar{z} = 1)$ is not an extremum at all, but an inflection point.

As for the second solution, the situation is simpler. Since the successive principal minors

$$|H_1| = -3 \qquad |H_2| = 6 \qquad \text{and} \qquad |H_3| = -18$$

duly alternate in sign, the determinantal test is conclusive. According to (11.19), the solution $\bar{z} = \frac{17}{16}$ is a maximum.

n-Variable Case

When there are n choice variables, the objective function may be expressed as

$$z = f(x_1, x_2, \ldots, x_n)$$

The total differential will then be

$$dz = f_1\,dx_1 + f_2\,dx_2 + \cdots + f_n\,dx_n$$

so that the necessary condition for extremum ($dz = 0$ for arbitrary dx_i) means that all the n first-order partial derivatives are required to be zero.

The second-order differential d^2z will again be a quadratic form, derivable analogously to (11.18) and expressible by an $n \times n$ array. The coefficients of that array, properly arranged, will now give the (symmetric) Hessian

$$|H| = \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}$$

with principal minors $|H_1|, |H_2|, \ldots, |H_n|$, as defined before. The second-order sufficient condition for extremum is, as before, that all the n principal minors be positive (for a minimum in z) and that they duly alternate in sign (for a maximum in z), the first one being negative.

In summary, then—if we concentrate on the determinantal test—we have the criteria as listed in Table 11.2, which is valid for an objective function of any number of choice variables. As special cases, we can have $n = 1$ or $n = 2$. When $n = 1$, the objective function is $z = f(x)$, and the conditions for maximization, $f_1 = 0$ and $|H_1| < 0$, reduce to $f'(x) = 0$ and $f''(x) < 0$, exactly as we learned in Sec. 9.4. Similarly, when $n = 2$, the objective function is $z = f(x_1, x_2)$, so that the first-order condition for maximum is $f_1 = f_2 = 0$, whereas the second-order

Table 11.2 Determinantal test for relative extremum: $z = f(x_1, x_2, \ldots, x_n)$

Condition	Maximum	Minimum
First-order necessary condition	$f_1 = f_2 = \cdots = f_n = 0$	$f_1 = f_2 = \cdots = f_n = 0$
Second-order sufficient condition*	$\lvert H_1\rvert < 0$; $\lvert H_2\rvert > 0$; $\lvert H_3\rvert < 0$; $\cdots$; $(-1)^n\lvert H_n\rvert > 0$	$\lvert H_1\rvert, \lvert H_2\rvert, \ldots, \lvert H_n\rvert > 0$

*Applicable only after the first-order necessary condition has been satisfied.

sufficient condition becomes

$$f_{11} < 0 \qquad \text{and} \qquad \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = f_{11}f_{22} - f_{12}^2 > 0$$

which is merely a restatement of the information presented in Table 11.1.

EXERCISE 11.4

Find the extreme values, if any, of the following five functions. Check whether they are maxima or minima by the determinantal test.

1 $z = x_1^2 + 3x_2^2 - 3x_1x_2 + 4x_2x_3 + 6x_3^2$

2 $z = 29 - (x_1^2 + x_2^2 + x_3^2)$

3 $z = x_1x_3 + x_1^2 - x_2 + x_2x_3 + x_2^2 + 3x_3^2$

4 $z = e^x + e^y + e^{w^2} - 2e^w - (x + y)$

5 $z = e^{2x} + e^{-y} + e^{w^2} - (2x + 2e^w - y)$

Then answer the following questions regarding Hessian matrices and their characteristic roots:

6 (*a*) Which of the above five problems yield diagonal Hessian matrices? In each such case, do the diagonal elements possess a uniform sign?

(*b*) What can you conclude about the characteristic roots of each diagonal Hessian matrix found? About the sign definiteness of d^2z?

(*c*) Do the results of the characteristic-root test check with those of the determinantal test?

7 (*a*) Find the characteristic roots of the Hessian matrix for problem 3.

(*b*) What can you conclude from your results?

(*c*) Is your answer to (*b*) consistent with the result of the determinantal test for problem 3 above?

11.5 SECOND-ORDER CONDITIONS IN RELATION TO CONCAVITY AND CONVEXITY

Second-order conditions—whether stated in terms of the principal minors of the Hessian determinant or the characteristic roots of the Hessian matrix—are always concerned with the question of whether a stationary point is the peak of a hill or the bottom of a valley. In other words, they relate to how a curve, surface, or hypersurface (as the case may be) bends itself around a stationary point. In the single-choice-variable case, with $z = f(x)$, the hill (valley) configuration is manifest in an inverse U-shaped (U-shaped) curve. For the two-variable function $z = f(x, y)$, the hill (valley) configuration takes the form of a dome-shaped (bowl-shaped) surface, as illustrated in Fig. 11.2*a* (Fig. 11.2*b*). When three or more choice variables are present, the hills and valleys are no longer graphable, but we may nevertheless think of "hills" and "valleys" on hypersurfaces.

A function that gives rise to a hill (valley) over the entire domain is said to be a *concave* (*convex*) function.* For the present discussion, we shall take the domain to be the entire R^n, where n is the number of choice variables. Inasmuch as the hill and valley characterizations refer to the entire domain, concavity and convexity are, of course, global concepts. For a finer classification, we may also distinguish between concavity and convexity on the one hand, and *strict* concavity and *strict* convexity on the other hand. In the *non*strict case, the hill or valley is allowed to contain one or more flat (as against curved) portions, such as line segments (on a curve) or line segments and plane segments (on a surface). The presence of the word "strict," however, rules out such line or plane segments. The two surfaces shown in Fig. 11.2 illustrate strictly concave and strictly convex functions, respectively. The curve in Fig. 6.5, on the other hand, is convex (it shows a valley) but not strictly convex (it contains line segments). A strictly concave (strictly convex) function must be concave (convex), but the converse is not true.

In view of the association of concavity and strict concavity with a global hill configuration, an extremum of a concave function must be a peak—a maximum (as against minimum). Moreover, that maximum must be an absolute maximum (as against relative maximum), since the hill covers the entire domain. However, that absolute maximum may not be unique, because multiple maxima may occur if the hill contains a flat horizontal top. The latter possibility can be dismissed only when we specify strict concavity. For only then will the peak consist of a single point and the absolute maximum be *unique*. A unique (nonunique) absolute maximum is also referred to as a *strong* (*weak*) absolute maximum.

By analogous reasoning, an extremum of a *convex* function must be an absolute (or global) minimum, which may not be unique. But an extremum of a *strictly convex* function must be a unique absolute minimum.

In the preceding paragraphs, the properties of concavity and convexity are taken to be global in scope. If they are valid only for a portion of the curve or surface (only in a subset S of the domain), then the associated maximum and minimum are relative (or local) to that subset of the domain, since we cannot be certain of the situation outside of subset S. In our earlier discussion of the sign definiteness of d^2z (or of the Hessian matrix H), we evaluated the principal minors of the Hessian determinant only at the stationary point. By thus limiting the verification of the hill or valley configuration to a small neighborhood of the stationary point, we could discuss only *relative* maxima and minima. But it may happen that d^2z has a definite sign everywhere, regardless of where the principal minors are evaluated. In that event, the hill or valley would cover the entire domain, and the maximum or minimum found would be absolute in nature. More specifically, if d^2z is *everywhere* negative (positive) semidefinite, the function

* If the hill (valley) pertains only to a subset S of the domain, the function is said to be *concave* (*convex*) *on* S.

$z = f(x_1, x_2, \ldots, x_n)$ must be concave (convex), and if d^2z is *everywhere* negative (positive) definite, the function f must be strictly concave (strictly convex).

The preceding discussion is summarized in Fig. 11.5 for a twice continuously differentiable function $z = f(x_1, x_2, \ldots, x_n)$. For clarity, we concentrate exclusively on concavity and maximum; however, the relationships depicted will remain valid if the words "concave," "negative," and "maximum" are replaced, respectively, by "convex," "positive," and "minimum." To read Fig. 11.5, recall that the $\Rightarrow$ symbol (here elongated and even bent) means "implies." When that symbol extends from one enclosure (say, a rectangle) to another (say, an oval), it

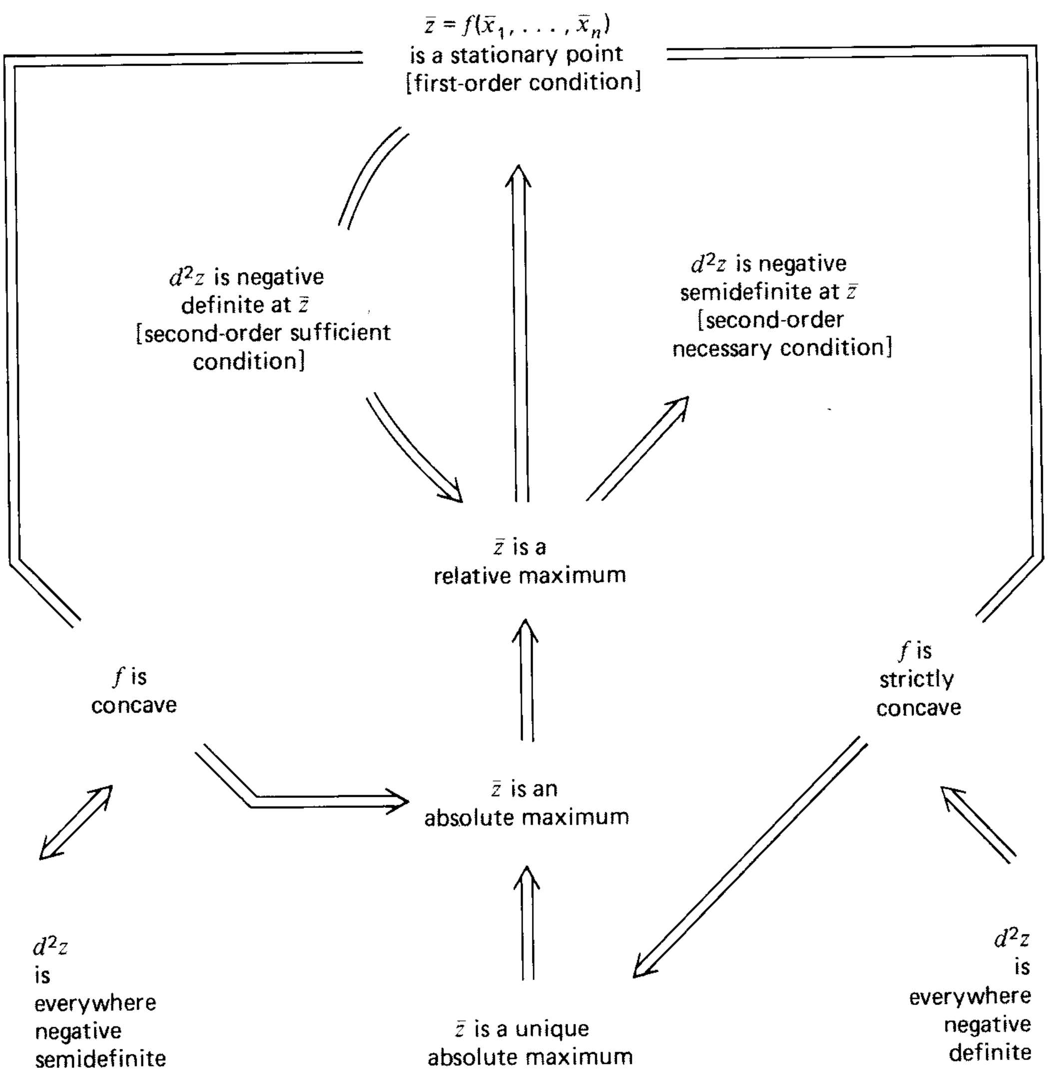


Figure 11.5

means that the former implies (is sufficient for) the latter; it also means that the latter is necessary for the former. And when the $\Rightarrow$ symbol extends from one enclosure through a second to a third, it means that the first enclosure, when accompanied by the second, implies the third.

In this light, the middle column in Fig. 11.5, read from top to bottom, states that the first-order condition is necessary for $\bar{z}$ to be a relative maximum, and the relative-maximum status of $\bar{z}$ is, in turn, necessary for $\bar{z}$ to be an absolute maximum, and so on. Alternatively, reading that column from bottom to top, we see that the fact that $\bar{z}$ is a unique absolute maximum is sufficient to establish $\bar{z}$ as an absolute maximum, and the absolute-maximum status of $\bar{z}$ is, in turn, sufficient for $\bar{z}$ to be a relative maximum, and so forth. The three ovals at the top have to do with the first- and second-order conditions at the stationary point $\bar{z}$. Hence they relate only to a relative maximum. The diamonds and triangles in the lower part, on the other hand, describe global properties that enable us to draw conclusions about an absolute maximum. Note that while our earlier discussion indicated only that the everywhere negative semidefiniteness of d^2z is *sufficient* for the concavity of function f, we have added in Fig. 11.5 the information that the condition is *necessary*, too. In contrast, the stronger property of everywhere negative definiteness of d^2z is *sufficient*, but *not necessary*, for the strict concavity of f—because strict concavity of f is compatible with a zero value of d^2z at a stationary point.

The most important message conveyed by Fig. 11.5, however, lies in the two extended $\Rightarrow$ symbols passing through the two diamonds. The one on the left states that, given a *concave* objective function, any stationary point can immediately be identified as an absolute maximum. Proceeding a step further, we see that the one on the right indicates that if the objective function is *strictly* concave, the stationary point must in fact be a unique absolute maximum. In either case, once the first-order condition is met, concavity or strict concavity effectively replaces the second-order condition as a sufficient condition for maximum—nay, for an absolute maximum. The powerfulness of this new sufficient condition becomes clear when we recall that d^2z can happen to be zero at a peak, causing the second-order sufficient condition to fail. Concavity or strict concavity, however, can take care of even such troublesome peaks, because it guarantees that a higher-order sufficient condition is satisfied even if the second-order one is not. It is for this reason that concavity is often assumed from the very outset when a maximization model is to be formulated with a *general* objective function (and, similarly, convexity is often assumed for a minimization model). For then all one needs to do is to apply the first-order condition. Note, however, that if a *specific* objective function is used, the property of concavity or convexity can no longer simply be assumed. Rather, it must be checked.

Checking Concavity and Convexity

Concavity and convexity, strict or nonstrict, can be defined (and checked) in several ways. We shall first introduce a geometric definition of concavity and convexity for a two-variable function $z = f(x_1, x_2)$, similar to the one-variable version discussed in Sec. 9.3:

> The function $z = f(x_1, x_2)$ is *concave* (*convex*) iff, for any pair of distinct points M and N on its graph—a surface—line segment MN lies either *on* or *below* (*above*) the surface. The function is *strictly concave* (*strictly convex*) iff line segment MN lies entirely *below* (*above*) the surface, except at M and N.

The case of a strictly concave function is illustrated in Fig. 11.6, where M and N, two arbitrary points on the surface, are joined together by a broken line segment as well as a solid arc, with the latter consisting of points on the surface that lie directly above the line segment. Since strict concavity requires line segment MN to lie entirely below arc MN (except at M and N) for *any* pair of points M and N, the surface must typically be dome-shaped. Analogously, the surface of a strictly convex function must typically be bowl-shaped. As for (nonstrictly) concave and convex functions, since line segment MN is allowed to lie on the surface itself, some portion of the surface, or even the entire surface, may be a plane—flat, rather than curved.

To facilitate generalization to the nongraphable n-dimensional case, the geometric definition needs to be translated into an equivalent algebraic version. Returning to Fig. 11.6, let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be any two distinct ordered pairs (2-vectors) in the domain of $z = f(x_1, x_2)$. Then the z values (height of surface) corresponding to these will be $f(u) = f(u_1, u_2)$ and $f(v) = f(v_1, v_2)$, respectively. We have assumed that the variables can take all real values, so if u and v are in the domain, then all the points on the line segment uv are also in the

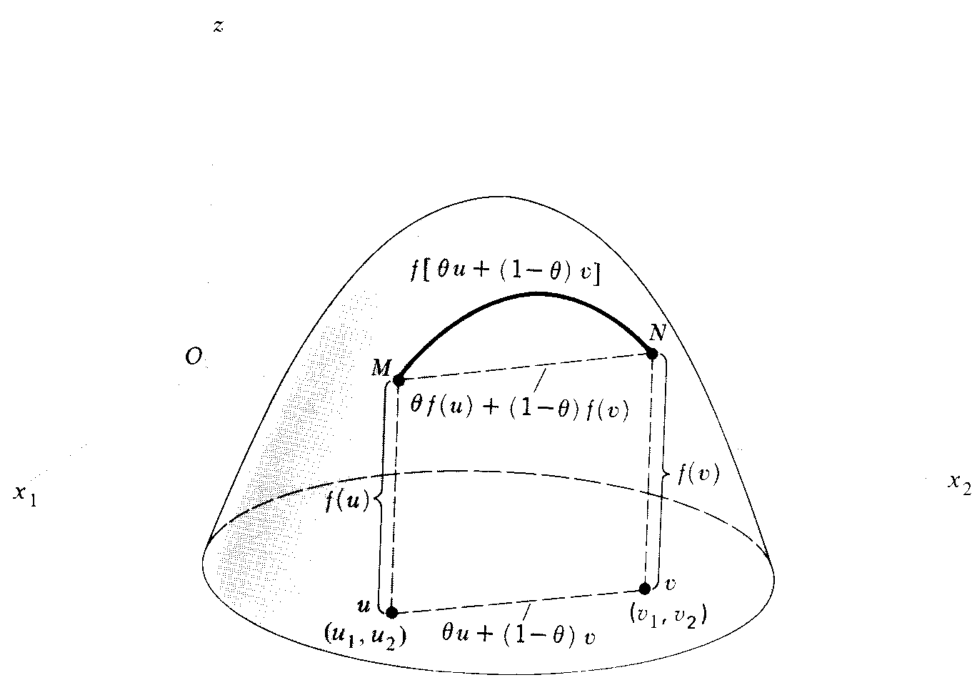


Figure 11.6

domain. Now each point on the said line segment is in the nature of a "weighted average" of u and v. Thus we can denote this line segment by $\theta u + (1 - \theta)v$, where θ (the Greek letter theta)—unlike u and v—is a (variable) scalar with the range of values $0 \leq \theta \leq 1$.* By the same token, line segment MN, representing the set of all weighted averages of $f(u)$ and $f(v)$, can be expressed by $\theta f(u) + (1 - \theta)f(v)$, with θ again varying from 0 to 1. What about arc MN along the surface? Since that arc shows the values of the function f evaluated at the various points on line segment uv, it can be written simply as $f[\theta u + (1 - \theta)v]$. Using these expressions, we may now state the following algebraic definition:

A function f is $\begin{Bmatrix} \text{concave} \\ \text{convex} \end{Bmatrix}$ iff, for any pair of distinct points u and v in the domain of f, and for $0 < \theta < 1$,

$$\underbrace{\theta f(u) + (1 - \theta)f(v)}_{\text{height of line segment}} \begin{Bmatrix} \leq \\ \geq \end{Bmatrix} \underbrace{f[\theta u + (1 - \theta)v]}_{\text{height of arc}} \tag{11.20}$$

Note that, in order to exclude the two end points M and N from the height comparison, we have restricted θ to the open interval (0, 1) only.

This definition is easily adaptable to *strict* concavity and convexity by changing the weak inequalities $\leq$ and $\geq$ to the strict inequalities $<$ and $>$, respectively. The advantage of the algebraic definition is that it can be applied to a function of any number of variables, for the vectors u and v in the definition can very well be interpreted as n-vectors instead of 2-vectors.

From (11.20), the following three theorems on concavity and convexity can be deduced fairly easily. These will be stated in terms of functions $f(x)$ and $g(x)$, but x can be interpreted as a vector of variables; that is, the theorems are valid for functions of any number of variables.

Theorem I (linear function) If $f(x)$ is a linear function, then it is a concave function as well as a convex function, but not strictly so.

Theorem II (negative of a function) If $f(x)$ is a concave function, then $-f(x)$ is a convex function, and vice versa. Similarly, if $f(x)$ is a strictly concave function, then $-f(x)$ is a strictly convex function, and vice versa.

Theorem III (sum of functions) If $f(x)$ and $g(x)$ are both concave (convex) functions, then $f(x) + g(x)$ is also a concave (convex) function. If $f(x)$ and $g(x)$

* The weighted-average expression $\theta u + (1 - \theta)v$, for any specific value of θ between 0 and 1, is technically known as a *convex combination* of the two vectors u and v. Leaving a more detailed explanation of this to a later point of this section, we may note here that when $\theta = 0$, the given expression reduces to vector v and similarly that when $\theta = 1$, the expression reduces to vector u. An intermediate value of θ, on the other hand, gives us an average of the two vectors u and v.

are both concave (convex) and, in addition, either one or both of them are strictly concave (strictly convex), then $f(x) + g(x)$ is strictly concave (strictly convex).

Theorem I follows from the fact that a linear function plots as a straight line, plane, or hyperplane, so that "line segment MN" always coincides with "arc MN." Consequently, the equality part of the two weak inequalities in (11.20) are simultaneously satisfied, making the function qualify as both concave and convex. However, since it fails the strict-inequality part of the definition, the linear function is neither strictly concave nor strictly convex.

Underlying Theorem II is the fact that the definitions of concavity and convexity differ only in the sense of inequality. Suppose that $f(x)$ is concave; then

$$\theta f(u) + (1-\theta)f(v) \le f[\theta u + (1-\theta)v]$$

Multiplying through by -1, and duly reversing the sense of the inequality, we get

$$\theta[-f(u)] + (1-\theta)[-f(v)] \ge -f[\theta u + (1-\theta)v]$$

This, however, is precisely the condition for $-f(x)$ to be convex. Thus the theorem is proved for the concave $f(x)$ case. The geometric interpretation of this result is very simple: the mirror image of a hill with reference to the base plane or hyperplane is a valley. The other cases can be proved similarly.

To see the reason behind Theorem III, suppose that $f(x)$ and $g(x)$ are both concave. Then the following two inequalities hold:

$$\theta f(u) + (1-\theta)f(v) \le f[\theta u + (1-\theta)v] \tag{11.21}$$

$$\theta g(u) + (1-\theta)g(v) \le g[\theta u + (1-\theta)v] \tag{11.22}$$

Adding these, we obtain a new inequality

$$\begin{aligned} \theta[f(u) + g(u)] &+ (1-\theta)[f(v) + g(v)] \\ &\le f[\theta u + (1-\theta)v] + g[\theta u + (1-\theta)v] \end{aligned} \tag{11.23}$$

But this is precisely the condition for $[f(x) + g(x)]$ to be concave. Thus the theorem is proved for the concave case. The proof for the convex case is similar.

Moving to the second part of Theorem III, let $f(x)$ be *strictly* concave. Then (11.21) becomes a *strict* inequality:

$$\theta f(u) + (1-\theta)f(v) < f[\theta u + (1-\theta)v] \tag{11.21'}$$

Adding this to (11.22), we find the sum of the left-side expressions in these two inequalities to be *strictly* less than the sum of the right-side expressions, regardless of whether the $<$ sign or the $=$ sign holds in (11.22). This means that (11.23) now becomes a *strict* inequality, too, thereby making $[f(x) + g(x)]$ *strictly* concave. Besides, the same conclusion emerges a fortiori, if $g(x)$ is made strictly concave along with $f(x)$, that is, if (11.22) is converted into a strict inequality along with (11.21). This proves the second part of the theorem for the concave case. The proof for the convex case is similar.

This theorem, which is also valid for a sum of more than two concave (convex) functions, may prove useful sometimes because it makes possible the compartmentalization of the task of checking concavity or convexity of a function that consists of additive terms. If the additive terms are found to be individually concave (convex), that would be sufficient for the sum function to be concave (convex).

Example 1 Check $z = x_1^2 + x_2^2$ for concavity or convexity. To apply (11.20), let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be any two distinct points in the domain. Then we have

$$f(u) = f(u_1, u_2) = u_1^2 + u_2^2$$

$$f(v) = f(v_1, v_2) = v_1^2 + v_2^2$$

and
$$f[\theta u + (1-\theta)v] = f\Big[\underbrace{\theta u_1 + (1-\theta)v_1}_{\text{value of } x_1}, \underbrace{\theta u_2 + (1-\theta)v_2}_{\text{value of } x_2}\Big]$$

$$= [\theta u_1 + (1-\theta)v_1]^2 + [\theta u_2 + (1-\theta)v_2]^2$$

Substituting these into (11.20), subtracting the right-side expression from the left-side one, and collecting terms, we find their difference to be

$$\theta(1-\theta)(u_1^2 + u_2^2) + \theta(1-\theta)(v_1^2 + v_2^2) - 2\theta(1-\theta)(u_1v_1 + u_2v_2)$$
$$= \theta(1-\theta)\left[(u_1 - v_1)^2 + (u_2 - v_2)^2\right]$$

Since θ is a positive fraction, $\theta(1-\theta)$ must be positive. Moreover, since (u_1, u_2) and (v_1, v_2) are distinct points, so that either $u_1 \neq v_1$ or $u_2 \neq v_2$ (or both), the bracketed expression must also be positive. Thus the strict $>$ inequality holds in (11.20), and $z = x_1^2 + x_2^2$ is strictly convex.

Alternatively, we may check the x_1^2 and x_2^2 terms separately. Since each of them is individually strictly convex, their sum is also strictly convex.

Because this function is strictly convex, it possesses a unique absolute minimum. It is easy to verify that the said minimum is $\bar{z} = 0$, attained at $\bar{x}_1 = \bar{x}_2 = 0$, and that it is indeed absolute and unique because any ordered pair $(x_1, x_2) \neq (0, 0)$ yields a z value greater than zero.

Example 2 Check $z = -x_1^2 - x_2^2$ for concavity or convexity. This function is the negative of the function in Example 1. Thus, by Theorem II, it is strictly concave.

Example 3 Check $z = (x + y)^2$ for concavity or convexity. Even though the variables are denoted by x and y instead of x_1 and x_2, we can still let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ denote two distinct points in the domain, with the subscript i

referring to the ith variable. Then we have

$$f(u) = f(u_1, u_2) = (u_1 + u_2)^2$$

$$f(v) = f(v_1, v_2) = (v_1 + v_2)^2$$

and
$$f[\theta u + (1-\theta)v] = [\theta u_1 + (1-\theta)v_1 + \theta u_2 + (1-\theta)v_2]^2$$
$$= [\theta(u_1 + u_2) + (1-\theta)(v_1 + v_2)]^2$$

Substituting these into (11.20), subtracting the right-side expression from the left-side one, and simplifying, we find their difference to be

$$\theta(1-\theta)(u_1+u_2)^2 - 2\theta(1-\theta)(u_1+u_2)(v_1+v_2) + \theta(1-\theta)(v_1+v_2)^2$$
$$= \theta(1-\theta)[(u_1+u_2) - (v_1+v_2)]^2$$

As in Example 1, $\theta(1-\theta)$ is positive. The square of the bracketed expression is nonnegative (zero cannot be ruled out this time). Thus the $\geq$ inequality holds in (11.20), and the function $(x+y)^2$ is convex, though not strictly so.

Accordingly, this function has an absolute minimum that may not be unique. It is easy to verify that the absolute minimum is $\bar{z} = 0$, attained whenever $\bar{x} + \bar{y} = 0$. That this is an absolute minimum is clear from the fact that whenever $x + y \neq 0$, z will be greater than $\bar{z} = 0$. That it is not unique follows from the fact that an infinite number of $(\bar{x}, \bar{y})$ pairs can satisfy the condition $\bar{x} + \bar{y} = 0$.

Differentiable Functions

As stated in (11.20), the definition of concavity and convexity uses no derivatives and thus does not require differentiability. If the function *is* differentiable, however, concavity and convexity can also be defined in terms of its first derivatives. In the one-variable case, the definition is:

> A differentiable function $f(x)$ is $\begin{Bmatrix}\text{concave}\\ \text{convex}\end{Bmatrix}$ iff, for any given point u and any other point v in the domain,

$$(11.24) \qquad f(v) \begin{Bmatrix}\leq\\ \geq\end{Bmatrix} f(u) + f'(u)(v-u)$$

Concavity and convexity will be *strict*, if the weak inequalities in (11.24) are replaced by the *strict* inequalities $<$ and $>$, respectively. Interpreted geometrically, this definition depicts a concave (convex) curve as one that lies on or below (above) all its tangent lines. To qualify as a strictly concave (strictly convex) curve, on the other hand, the curve must lie strictly below (above) all the tangent lines, except at the points of tangency.

In Fig. 11.7, let point A be any given point on the curve, with height $f(u)$ and with tangent line AB. Let x increase from the value u. Then a strictly concave curve (as drawn) must, in order to form a hill, curl progressively away from the

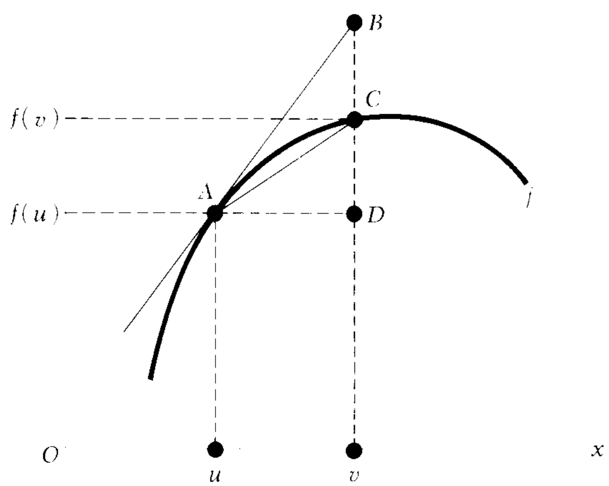


Figure 11.7

tangent line AB, so that point C, with height $f(v)$, has to lie below point B. In this case, the slope of line segment AC is less than that of tangent AB. If the curve is *non*strictly concave, on the other hand, it may contain a line segment, so that, for instance, arc AC may turn into a line segment and be coincident with line segment AB, as a linear portion of the curve. In the latter case the slope of AC is equal to that of AB. Together, these two situations imply that

$$\left(\text{Slope of line segment } AC = \frac{DC}{AD} = \right)\frac{f(v)-f(u)}{v-u} \leq (\text{slope of } AB =)f'(u)$$

When multiplied through by the positive quantity $(v - u)$, this inequality yields the result in (11.24) for the concave function. The same result can be obtained, if we consider instead x values less than u.

When there are two or more independent variables, the definition needs a slight modification:

> A differentiable function $f(x) = f(x_1, \ldots, x_n)$ is $\begin{Bmatrix} \text{concave} \\ \text{convex} \end{Bmatrix}$ iff, for any given point $u = (u_1, \ldots, u_n)$ and any other point $v = (v_1, \ldots, v_n)$ in the domain,
>
> $$(11.24') \qquad f(v) \begin{Bmatrix} \leq \\ \geq \end{Bmatrix} f(u) + \sum_{j=1}^{n} f_j(u)(v_j - u_j)$$
>
> where $f_j(u) \equiv \partial f/\partial x_j$ is evaluated at $u = (u_1, \ldots, u_n)$.

This definition requires the graph of a concave (convex) function $f(x)$ to lie on or below (above) all its tangent planes or hyperplanes. For *strict* concavity and convexity, the weak inequalities in (11.24′) should be changed to *strict* inequalities, which would require the graph of a strictly concave (strictly convex) function to lie strictly below (above) all its tangent planes or hyperplanes, except at the points of tangency.

Finally, consider a function $z = f(x_1, \ldots, x_n)$ which is twice continuously differentiable. For such a function, second-order partial derivatives exist, and thus d^2z is defined. Concavity and convexity can then be checked by the sign of d^2z:

(11.25)
A twice continuously differentiable function $z = f(x_1, \ldots, x_n)$ is $\begin{Bmatrix} \text{concave} \\ \text{convex} \end{Bmatrix}$ if, and only if, d^2z is everywhere $\begin{Bmatrix} \text{negative} \\ \text{positive} \end{Bmatrix}$ semidefinite. The said function is strictly $\begin{Bmatrix} \text{concave} \\ \text{convex} \end{Bmatrix}$ if (but *not* only if) d^2z is everywhere $\begin{Bmatrix} \text{negative} \\ \text{positive} \end{Bmatrix}$ definite.

You will recall that the concave and strictly concave aspects of (11.25) have already been incorporated into Fig. 11.5.

Example 4 Check $z = -x^4$ for concavity or convexity by the derivative conditions. We first apply (11.24). The left- and right-side expressions in that inequality are in the present case $-v^4$ and $-u^4 - 4u^3(v-u)$, respectively. Subtracting the latter from the former, we find their difference to be

$$-v^4 + u^4 + 4u^3(v-u) = (v-u)\left(-\frac{v^4 - u^4}{v-u} + 4u^3\right) \qquad \text{[factoring]}$$
$$= (v-u)\left[-(v^3 + v^2u + vu^2 + u^3) + 4u^3\right]$$
$$\text{[by (7.2)]}$$

It would be nice if the bracketed expression turned out to be divisible by $(v-u)$, for then we could again factor out $(v-u)$ and obtain a squared term $(v-u)^2$ to facilitate the evaluation of sign. As it turns out, this is indeed the case. Thus the difference cited above can be written as

$$-(v-u)^2[v^2 + 2vu + 3u^2] = -(v-u)^2\left[(v+u)^2 + 2u^2\right]$$

Given that $v \neq u$, the sign of this expression must be negative. With the strict $<$ inequality holding in (11.24), the function $z = -x^4$ is strictly concave. This means that it has a unique absolute maximum. As can be easily verified, that maximum is $\bar{z} = 0$, attained at $\bar{x} = 0$.

Because this function is twice continuously differentiable, we may also apply (11.25). Since there is only one variable, (11.25) gives us

$$d^2z = f''(x)\,dx^2 = -12x^2\,dx^2 \qquad \text{[by (11.2)]}$$

We know that dx^2 is positive (only nonzero changes in x are being considered); but $-12x^2$ can be either negative or zero. Thus the best we can do is to conclude that d^2z is everywhere negative *semi*definite, and that $z = -x^4$ is (nonstrictly) concave. This conclusion from (11.25) is obviously weaker than the one obtained earlier from (11.24); namely, $z = -x^4$ is strictly concave. What limits us to the weaker conclusion in this case is the same culprit that causes the second-derivative test to fail on occasions—the fact that d^2z may take a zero value at a stationary point of a function known to be strictly concave, or strictly convex. This is why, of course, the negative (positive) definiteness of d^2z is presented in (11.25) as only a sufficient, but not necessary, condition for strict concavity (strict convexity).

Example 5 Check $z = x_1^2 + x_2^2$ for concavity or convexity by the derivative conditions. This time we have to use (11.24′) instead of (11.24). With $u = (u_1, u_2)$ and $v = (v_1, v_2)$ as any two points in the domain, the two sides of (11.24′) are

$$\text{Left side} = v_1^2 + v_2^2$$

$$\text{Right side} = u_1^2 + u_2^2 + 2u_1(v_1 - u_1) + 2u_2(v_2 - u_2)$$

Subtracting the latter from the former, and simplifying, we can express their difference as

$$v_1^2 - 2v_1u_1 + u_1^2 + v_2^2 - 2v_2u_2 + u_2^2 = (v_1 - u_1)^2 + (v_2 - u_2)^2$$

Given that $(v_1, v_2) \neq (u_1, u_2)$, this difference is always positive. Thus the strict $>$ inequality holds in (11.24′), and $z = x_1^2 + x_2^2$ is strictly convex. Note that the present result merely reaffirms what we have previously found in Example 1.

As for the use of (11.25), since $f_1 = 2x_1$, and $f_2 = 2x_2$, we have

$$f_{11} = 2 > 0 \qquad \text{and} \qquad \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$$

regardless of where the second-order partial derivatives are evaluated. Thus d^2z is everywhere positive definite, which duly satisfies the sufficient condition for strict convexity. In the present example, therefore, (11.24′) and (11.25) do yield the same conclusion.

Convex Functions versus Convex Sets

Having clarified the meaning of the adjective "convex" as applied to a function, we must hasten to explain its meaning when used to describe a *set*. Although convex sets and convex functions are not unrelated, they are distinct concepts, and it is important not to confuse them.

For easier intuitive grasp, let us begin with the geometric characterization of a convex set. Let S be a set of points in a 2-space or 3-space. If, for any two points in set S, the line segment connecting these two points lies entirely in S, then S is said to be a *convex set*. It should be obvious that a straight line satisfies this definition and constitutes a convex set. By convention, a set consisting of a single point is also considered as a convex set, and so is the null set (with no point). For additional examples, let us look at Fig. 11.8. The disk—namely, the "solid" circle, a circle plus all the points within it—is a convex set, because a line joining any two points in the disk lies entirely in the disk, as exemplified by *ab* (linking two boundary points) and *cd* (linking two interior points). Note, however, that a (hollow) circle is *not* in itself a convex set. Similarly, a triangle, or a pentagon, is not in itself a convex set, but its solid version is. The remaining two solid figures in Fig. 11.8 are not convex sets. The palette-shaped figure is reentrant (indented); thus a line segment such as *gh* does not lie entirely in the set. In the key-shaped figure, moreover, we find not only the feature of reentrance, but also the presence of a hole, which is yet another cause of nonconvexity. Generally speaking, to qualify as a convex set, the set of points must contain no holes, and its boundary must not be indented anywhere.

The geometric definition of convexity also applies readily to point sets in a 3-space. For instance, a solid cube is a convex set, whereas a hollow cylinder is not. When a 4-space or a space of higher dimension is involved, however, the geometric interpretation becomes less obvious. We then need to turn to the algebraic definition of convex sets.

To this end, it is useful to introduce the concept of *convex combination* of vectors (points), which is a special type of linear combination. A linear combination of two vectors u and v can be written as

$$k_1 u + k_2 v$$

where k_1 and k_2 are two scalars. When these two scalars both lie in the closed interval $[0, 1]$ and add up to unity, the linear combination is said to be a convex combination, and can be expressed as

$$\theta u + (1 - \theta) v \qquad (0 \le \theta \le 1) \tag{11.26}$$

As an illustration, the combination $\frac{1}{3}\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 4 \\ 9 \end{bmatrix}$ is a convex combination. In view of the fact that these two scalar multipliers are positive fractions adding up

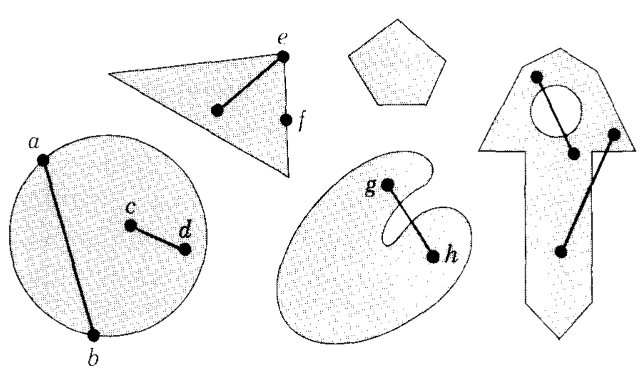


Figure 11.8

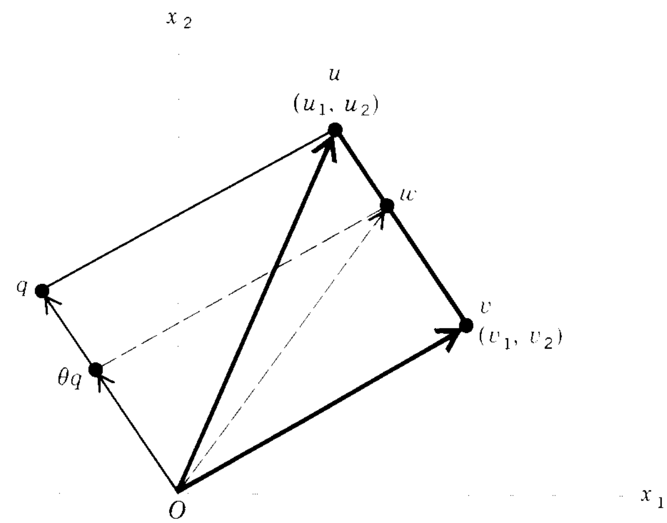


Figure 11.9

to 1, such a convex combination may be interpreted as a *weighted average* of the two vectors.*

The unique characteristic of the combination in (11.26) is that, for every acceptable value of θ, the resulting sum vector lies on the line segment connecting the points u and v. This can be demonstrated by means of Fig. 11.9, where we have plotted two vectors $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ as two points with coordinates (u_1, u_2) and (v_1, v_2), respectively. If we plot another vector q such that $Oquv$ forms a parallelogram, then we have (by virtue of the discussion in Fig. 4.3)

$$u = q + v \qquad \text{or} \qquad q = u - v$$

It follows that a convex combination of vectors u and v (let us call it w) can be expressed in terms of vector q, because

$$w = \theta u + (1 - \theta)v = \theta u + v - \theta v = \theta(u - v) + v = \theta q + v$$

Hence, to plot the vector w, we can simply add θq and v by the familiar parallelogram method. If the scalar θ is a positive fraction, the vector θq will merely be an abridged version of vector q; thus θq must lie on the line segment Oq. Adding θq and v, therefore, we must find vector w lying on the line segment uv, for the new, smaller parallelogram is nothing but the original parallelogram with the qu side shifted downward. The exact location of vector w will, of course, vary according to the value of the scalar θ; by varying θ from zero to unity, the location of w will shift from v to u. Thus the set of all points on the line segment uv, including u and v themselves, corresponds to the set of all convex combinations of vectors u and v.

* This interpretation has been made use of earlier in the discussion of concave and convex functions.

In view of the above, a convex set may now be redefined as follows: A set S is convex if and only if, for any two points $u \in S$ and $v \in S$, and for every scalar $\theta \in [0, 1]$, it is true that $w = \theta u + (1 - \theta)v \in S$. Because this definition is algebraic, it is applicable regardless of the dimension of the space in which the vectors u and v are located. Comparing this definition of a convex set with that of a convex function in (11.20), we see that even though the same adjective "convex" is used in both, the meaning of this word changes radically from one context to the other. In describing a *function*, the word "convex" specifies how a curve or surface bends itself—it must form a valley. But in describing a *set*, the word specifies how the points in the set are "packed" together—they must not allow any holes to arise, and the boundary must not be indented. Thus convex functions and convex sets are clearly distinct mathematical entities.

Yet convex functions and convex sets are not unrelated. For one thing, in defining a convex function, we need a convex set for the domain. This is because the definition (11.20) requires that, for any two points u and v in the domain, all the convex combinations of u and v—specifically, $\theta u + (1 - \theta)v$, $0 \leq \theta \leq 1$—must also be in the domain, which is, of course, just another way of saying that the domain must be a convex set. To satisfy this requirement, we adopted earlier the rather strong assumption that the domain consists of the entire n-space (where n is the number of choice variables), which is indeed a convex set. However, with the concept of convex sets at our disposal, we can now substantially weaken that assumption. For all we need to assume is that the domain is a convex subset of R^n, rather than R^n itself.

There is yet another way in which convex functions are related to convex sets. If $f(x)$ is a convex function, then for any constant k, it can give rise to a convex set

$$(11.27) \qquad S^{\leq} \equiv \{x \mid f(x) \leq k\} \qquad [f(x) \text{ convex}]$$

This is illustrated in Fig. 11.10a for the one-variable case. The set $S^{\leq}$ consists of all the x values associated with the segment of the $f(x)$ curve lying on or below the broken horizontal line. Hence it is the line segment on the horizontal axis marked by the heavy dots, which is a convex set. Note that if the k value is changed, the $S^{<}$ set will become a different line segment on the horizontal axis, but it will still be a convex set.

Going a step further, we may observe that even a *concave* function is related to convex sets in ways similar. First, the definition of a concave function in (11.20) is, like the convex-function case, predicated upon a domain that is a convex set. Moreover, even a concave function—say, $g(x)$—can generate an associated convex set, given some constant k. That convex set is

$$(11.28) \qquad S^{>} \equiv \{x \mid g(x) \geq k\} \qquad [g(x) \text{ concave}]$$

in which the $\geq$ sign appears instead of $\leq$. Geometrically, as shown in Fig. 11.10b for the one-variable case, the set $S^{\geq}$ contains all the x values corresponding to the segment of the $g(x)$ curve lying on or above the broken horizontal line. Thus it is again a line segment on the horizontal axis—a convex set.

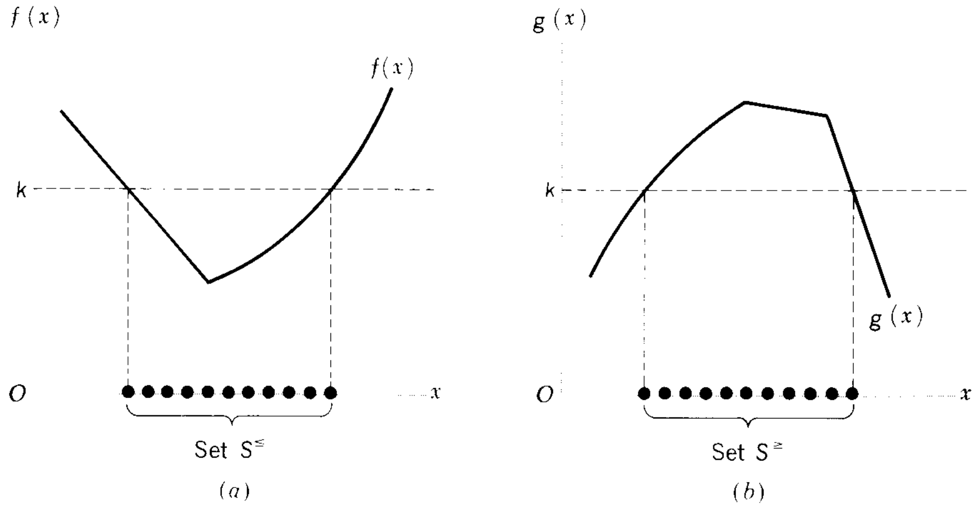


Figure 11.10

Although Fig. 11.10 specifically illustrates the one-variable case, the definitions of $S^{\leq}$ and $S^{\geq}$ in (11.27) and (11.28) are not limited to functions of a single variable. They are equally valid if we interpret x to be a vector, i.e., let $x = (x_1, \ldots, x_n)$. In that case, however, (11.27) and (11.28) will define convex sets in the n-space instead. It is important to remember that while a convex function implies (11.27), and a concave function implies (11.28), the converse is not true—for (11.27) can also be satisfied by a nonconvex function and (11.28) by a nonconcave function. This is discussed further in Sec. 12.4.

EXERCISE 11.5

1 Use (11.20) to check whether the following functions are concave, convex, strictly concave, strictly convex, or neither:

(*a*) $z = x^2$ (*b*) $z = x_1^2 + 2x_2^2$ (*c*) $z = 2x^2 - xy + y^2$

2 Use (11.24) or (11.24′) to check whether the following functions are concave, convex, strictly concave, strictly convex, or neither:

(*a*) $z = -x^2$ (*b*) $z = (x_1 + x_2)^2$ (*c*) $z = -xy$

3 In view of your answer to problem 2*c* above, could you have made use of Theorem III of this section to compartmentalize the task of checking the function $z = 2x^2 - xy + y^2$ in problem 1*c*? Explain your answer.

4 Do the following constitute convex sets in the 3-space?

(*a*) A doughnut (*b*) A bowling pin (*c*) A perfect marble

5 The equation $x^2 + y^2 = 4$ represents a circle with center at (0, 0) and with a radius of 2.

(*a*) Interpret geometrically the set $\{(x, y) \mid x^2 + y^2 \leq 4\}$.

(*b*) Is this set convex?

6 Graph each of the following sets, and indicate whether it is convex:

(*a*) $\{(x, y) \mid y = e^x\}$ (*c*) $\{(x, y) \mid y \leq 13 - x^2\}$

(*b*) $\{(x, y) \mid y \geq e^x\}$ (*d*) $\{(x, y) \mid xy \geq 1; x > 0, y > 0\}$

7 Given $u = \begin{bmatrix} 10 \\ 6 \end{bmatrix}$ and $v = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$, which of the following are *convex* combinations of u and v?

(*a*) $\begin{bmatrix} 7 \\ 7 \end{bmatrix}$ (*b*) $\begin{bmatrix} 5.2 \\ 7.6 \end{bmatrix}$ (*c*) $\begin{bmatrix} 6.2 \\ 8.2 \end{bmatrix}$

8 Given two vectors u and v in the 2-space, find and sketch:

(*a*) The set of all linear combinations of u and v

(*b*) The set of all nonnegative linear combinations of u and v

(*c*) The set of all convex combinations of u and v

9 (*a*) Rewrite (11.27) and (11.28) specifically for the cases where the f and g functions have n independent variables.

(*b*) Let $n = 2$, and let the function f be shaped like a (vertically held) ice-cream cone whereas the function g is shaped like a pyramid. Describe the sets $S^{\leq}$ and $S^{\geq}$.

11.6 ECONOMIC APPLICATIONS

At the beginning of this chapter, the case of a multiproduct firm was cited as an illustration of the general problem of optimization with more than one choice variable. We are now equipped to handle that problem and others of a similar nature.

Problem of a Multiproduct Firm

Example 1 Let us first postulate a two-product firm under circumstances of pure competition. Since with pure competition the prices of both commodities must be taken as exogenous, these will be denoted by P_{10} and P_{20}, respectively. Accordingly, the firm's revenue function will be

$$R = P_{10}Q_1 + P_{20}Q_2$$

where Q_i represents the output level of the ith product per unit of time. The firm's cost function is assumed to be

$$C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$$

Note that $\partial C/\partial Q_1 = 4Q_1 + Q_2$ (the marginal cost of the first product) is a function not only of Q_1 but also of Q_2. Similarly, the marginal cost of the second product also depends, in part, on the output level of the first product. Thus, according to the assumed cost function, the two commodities are seen to be technically related in production.

The profit function of this hypothetical firm can now be written readily as

$$\pi = R - C = P_{10}Q_1 + P_{20}Q_2 - 2Q_1^2 - Q_1Q_2 - 2Q_2^2$$

a function of two choice variables (Q_1 and Q_2) and two price parameters. It is our task to find the levels of Q_1 and Q_2 which, in combination, will maximize π. For this purpose, we first find the first-order partial derivatives of the profit function:

$$\begin{aligned} \pi_1 \left(\equiv \frac{\partial \pi}{\partial Q_1} \right) &= P_{10} - 4Q_1 - Q_2 \\ \pi_2 \left(\equiv \frac{\partial \pi}{\partial Q_2} \right) &= P_{20} - Q_1 - 4Q_2 \end{aligned} \tag{11.29}$$

Setting these both equal to zero, to satisfy the necessary condition for maximum, we get the two simultaneous equations

$$\begin{aligned} 4Q_1 + Q_2 &= P_{10} \\ Q_1 + 4Q_2 &= P_{20} \end{aligned}$$

which yield the unique solution

$$\bar{Q}_1 = \frac{4P_{10} - P_{20}}{15} \qquad \text{and} \qquad \bar{Q}_2 = \frac{4P_{20} - P_{10}}{15}$$

Thus, if $P_{10} = 12$ and $P_{20} = 18$, for example, we have $\bar{Q}_1 = 2$ and $\bar{Q}_2 = 4$, implying an optimal profit $\bar{\pi} = 48$ per unit of time.

To be sure that this does represent a maximum profit, let us check the second-order condition. The second partial derivatives, obtainable by partial differentiation of (11.29), give us the following Hessian:

$$|H| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} -4 & -1 \\ -1 & -4 \end{vmatrix}$$

Since $|H_1| = -4 < 0$ and $|H_2| = 15 > 0$, the Hessian matrix (or d^2z) is negative definite, and the solution does maximize the profit. In fact, since the signs of the principal minors do not depend on where they are evaluated, d^2z is in this case *everywhere* negative definite. Thus, according to (11.25), the objective function must be strictly concave, and the maximum profit found above is actually a unique absolute maximum.

Example 2 Let us now transplant the problem of Example 1 into the setting of a monopolistic market. By virtue of this new market-structure assumption, the revenue function must be modified to reflect the fact that the prices of the two products will now vary with their output levels (which are assumed to be identical with their sales levels, no inventory accumulation being contemplated in the model). The exact manner in which prices will vary with output levels is, of course, to be found in the demand functions for the firm's two products.

Suppose that the demands facing the monopolist firm are as follows:

$$\begin{aligned} Q_1 &= 40 - 2P_1 + P_2 \\ Q_2 &= 15 + P_1 - P_2 \end{aligned} \tag{11.30}$$

These equations reveal that the two commodities are related in *consumption*;

specifically, they are substitute goods, because an increase in the price of one will raise the demand for the other. As given, (11.30) expresses the quantities demanded Q_1 and Q_2 as functions of prices, but for our present purposes it will be more convenient to have prices P_1 and P_2 expressed in terms of the sales volumes Q_1 and Q_2, that is, to have average-revenue functions for the two products. Since (11.30) can be rewritten as

$$\begin{aligned} -2P_1 + P_2 &= Q_1 - 40 \\ P_1 - P_2 &= Q_2 - 15 \end{aligned}$$

we may (considering Q_1 and Q_2 as parameters) apply Cramer's rule to solve for P_1 and P_2 as follows:

$$\begin{aligned} P_1 &= 55 - Q_1 - Q_2 \\ P_2 &= 70 - Q_1 - 2Q_2 \end{aligned} \tag{11.30'}$$

These constitute the desired average-revenue functions, since $P_1 \equiv \mathrm{AR}_1$ and $P_2 \equiv \mathrm{AR}_2$.

Consequently, the firm's total-revenue function can be written as

$$\begin{aligned} R &= P_1Q_1 + P_2Q_2 \\ &= (55 - Q_1 - Q_2)Q_1 + (70 - Q_1 - 2Q_2)Q_2 \qquad [\text{by } (11.30')] \\ &= 55Q_1 + 70Q_2 - 2Q_1Q_2 - Q_1^2 - 2Q_2^2 \end{aligned}$$

If we again assume the total-cost function to be

$$C = Q_1^2 + Q_1Q_2 + Q_2^2$$

then the profit function will be

$$\pi = R - C = 55Q_1 + 70Q_2 - 3Q_1Q_2 - 2Q_1^2 - 3Q_2^2 \tag{11.31}$$

which is an objective function with two choice variables. Once the profit-maximizing output levels $\bar{Q}_1$ and $\bar{Q}_2$ are found, however, the optimal prices $\bar{P}_1$ and $\bar{P}_2$ are easy enough to find from (11.30′).

The objective function yields the following first and second partial derivatives:

$$\pi_1 = 55 - 3Q_2 - 4Q_1 \qquad \pi_2 = 70 - 3Q_1 - 6Q_2$$

$$\pi_{11} = -4 \qquad \pi_{12} = \pi_{21} = -3 \qquad \pi_{22} = -6$$

To satisfy the first-order condition for a maximum of π, we must have $\pi_1 = \pi_2 = 0$; that is,

$$\begin{aligned} 4Q_1 + 3Q_2 &= 55 \\ 3Q_1 + 6Q_2 &= 70 \end{aligned}$$

Thus the solution output levels (per unit of time) are

$$(\bar{Q}_1, \bar{Q}_2) = (8, 7\tfrac{2}{3})$$

Upon substitution of this result into (11.30′) and (11.31), respectively, we find that

$$\bar{P}_1 = 39\tfrac{1}{3} \qquad \bar{P}_2 = 46\tfrac{2}{3} \qquad \text{and} \qquad \bar{\pi} = 488\tfrac{1}{3} \qquad \text{(per unit of time)}$$

Inasmuch as the Hessian is $\begin{vmatrix} -4 & -3 \\ -3 & -6 \end{vmatrix}$, we have

$$|H_1| = -4 < 0 \qquad \text{and} \qquad |H_2| = 15 > 0$$

so that the value of $\bar{\pi}$ does represent the maximum profit. Here, the signs of the principal minors are again independent of where they are evaluated. Thus the Hessian matrix is everywhere negative definite, implying that the objective function is strictly concave and that it has a unique absolute maximum.

Price Discrimination

Even in a single-product firm, there can arise an optimization problem involving two or more choice variables. Such would be the case, for instance, when a monopolistic firm sells a single product in two or more separate markets (e.g., domestic and foreign) and therefore must decide upon the quantities (Q_1, Q_2, etc.) to be supplied to the respective markets in order to maximize profit. The several markets will, in general, have different demand conditions, and if demand elasticities differ in the various markets, profit maximization will entail the practice of price discrimination. Let us derive this familiar conclusion mathematically.

Example 3 For a change of pace, this time let us use three choice variables, i.e., assume three separate markets. Also, let us work with general rather than numerical functions. Accordingly, our monopolistic firm will simply be assumed to have total-revenue and total-cost functions as follows:

$$R = R_1(Q_1) + R_2(Q_2) + R_3(Q_3)$$

$$C = C(Q) \qquad \text{where } Q = Q_1 + Q_2 + Q_3$$

Note that the symbol R_i represents here the revenue function of the ith market, rather than a derivative in the sense of f_i. Each such revenue function naturally implies a particular demand structure, which will generally be different from those prevailing in the other two markets. On the cost side, on the other hand, only one cost function is postulated, since a single firm is producing for all three markets. In view of the fact that $Q = Q_1 + Q_2 + Q_3$, total cost C is also basically a function of Q_1, Q_2, and Q_3, which constitute the choice variables of the model. We can, of course, rewrite $C(Q)$ as $C(Q_1 + Q_2 + Q_3)$. It should be noted, however, that even though the latter version contains three independent variables, the function should nevertheless be considered as having a single argument only, because the sum of Q_i is really a single entity. In contrast, if the function appears in the form $C(Q_1, Q_2, Q_3)$, then there can be counted as many arguments as independent variables.

Now the profit function is

$$\pi = R_1(Q_1) + R_2(Q_2) + R_3(Q_3) - C(Q)$$

with first partial derivatives $\pi_i \equiv \partial\pi/\partial Q_i$ (for $i = 1, 2, 3$) as follows:*

$$\begin{aligned} \pi_1 &= R_1'(Q_1) - C'(Q)\frac{\partial Q}{\partial Q_1} = R_1'(Q_1) - C'(Q) && \left[\text{since } \frac{\partial Q}{\partial Q_1} = 1\right] \\ \pi_2 &= R_2'(Q_2) - C'(Q)\frac{\partial Q}{\partial Q_2} = R_2'(Q_2) - C'(Q) && \left[\text{since } \frac{\partial Q}{\partial Q_2} = 1\right] \\ \pi_3 &= R_3'(Q_3) - C'(Q)\frac{\partial Q}{\partial Q_3} = R_3'(Q_3) - C'(Q) && \left[\text{since } \frac{\partial Q}{\partial Q_3} = 1\right] \end{aligned} \tag{11.32}$$

Setting these equal to zero simultaneously will give us

$$C'(Q) = R_1'(Q_1) = R_2'(Q_2) = R_3'(Q_3)$$

That is,

$$\text{MC} = \text{MR}_1 = \text{MR}_2 = \text{MR}_3$$

Thus the levels of Q_1, Q_2, and Q_3 should be chosen such that the marginal revenue in each market is equated to the marginal cost of the total output Q.

To see the implications of this condition with regard to price discrimination, let us first find out how the MR in any market is specifically related to the price in that market. Since the revenue in each market is $R_i = P_iQ_i$, it follows that the marginal revenue must be

$$\begin{aligned} \text{MR}_i &\equiv \frac{dR_i}{dQ_i} = P_i\frac{dQ_i}{dQ_i} + Q_i\frac{dP_i}{dQ_i} \\ &= P_i\left(1 + \frac{dP_i}{dQ_i}\frac{Q_i}{P_i}\right) = P_i\left(1 + \frac{1}{\varepsilon_{di}}\right) \qquad \text{[by (8.4)]} \end{aligned}$$

where ε_{di}, the point elasticity of demand in the ith market, is normally negative. Consequently, the relationship between MR_i and P_i can be expressed alternatively by the equation

$$\text{MR}_i = P_i\left(1 - \frac{1}{|\varepsilon_{di}|}\right) \tag{11.33}$$

Recall that $|\varepsilon_{di}|$ is, in general, a function of P_i, so that when $\bar{Q}_i$ is chosen, and $\bar{P}_i$ thus specified, $|\varepsilon_{di}|$ will also assume a specific value, which can be either greater than, or less than, or equal to one. But if $|\varepsilon_{di}| < 1$ (demand being inelastic at a point), then its reciprocal will exceed one, and the parenthesized expression in (11.33) will be negative, thereby implying a negative value for MR_i. Similarly, if

* Note that, to find $\partial C/\partial Q_i$, the chain rule is used:

$$\frac{\partial C}{\partial Q_i} = \frac{dC}{dQ}\frac{\partial Q}{\partial Q_i}$$

$|\varepsilon_{di}| = 1$ (unitary elasticity), then MR_i will take a zero value. Inasmuch as a firm's MC is positive, the first-order condition $\mathrm{MC} = \mathrm{MR}_i$ requires the firm to operate at a positive level of MR_i. Hence the firm's chosen sales levels Q_i must be such that the corresponding point elasticity of demand in each market is greater than one.

The first-order condition $\mathrm{MR}_1 = \mathrm{MR}_2 = \mathrm{MR}_3$ can now be translated, via (11.33), into the following:

$$P_1\left(1 - \frac{1}{|\varepsilon_{d1}|}\right) = P_2\left(1 - \frac{1}{|\varepsilon_{d2}|}\right) = P_3\left(1 - \frac{1}{|\varepsilon_{d3}|}\right)$$

From this it can readily be inferred that the *smaller* the value of $|\varepsilon_d|$ (at the chosen level of output) in a particular market, the *higher* the price charged in that market must be—hence, price discrimination—if profit is to be maximized.

To ensure maximization, let us examine the second-order condition. From (11.32), the second partial derivatives are found to be

$$\pi_{11} = R_1''(Q_1) - C''(Q)\frac{\partial Q}{\partial Q_1} = R_1''(Q_1) - C''(Q)$$

$$\pi_{22} = R_2''(Q_2) - C''(Q)\frac{\partial Q}{\partial Q_2} = R_2''(Q_2) - C''(Q)$$

$$\pi_{33} = R_3''(Q_3) - C''(Q)\frac{\partial Q}{\partial Q_3} = R_3''(Q_3) - C''(Q)$$

and $$\pi_{12} = \pi_{21} = \pi_{13} = \pi_{31} = \pi_{23} = \pi_{32} = -C''(Q) \qquad \left[\text{since } \frac{\partial Q}{\partial Q_i} = 1\right]$$

so that we have (after shortening the second-derivative notation)

$$|H| = \begin{vmatrix} R_1'' - C'' & -C'' & -C'' \\ -C'' & R_2'' - C'' & -C'' \\ -C'' & -C'' & R_3'' - C'' \end{vmatrix}$$

The second-order sufficient condition will thus be duly satisfied, provided we have:

1. $|H_1| = R_1'' - C'' < 0$; that is, the slope of MR_1 is less than the slope of MC of the entire output [cf. the situation of point L in Fig. 9.6c]. (Since any of the three markets can be taken as the "first" market, this in effect also implies $R_2'' - C'' < 0$ and $R_3'' - C'' < 0$.)
2. $|H_2| = (R_1'' - C'')(R_2'' - C'') - (C'')^2 > 0$; or, $R_1''R_2'' - (R_1'' + R_2'')C'' > 0$
3. $|H_3| = R_1''R_2''R_3'' - (R_1''R_2'' + R_1''R_3'' + R_2''R_3'')C'' < 0$

The last two parts of this condition are not as easy to interpret economically as the first. Note that had we assumed that the general $R_i(Q_i)$ functions are all concave and the general $C(Q)$ function is convex, so that $-C(Q)$ is concave, then the profit function—the sum of concave functions—could have been taken to be concave, thereby obviating the need to check the second-order condition.

Example 4 To make the above example more concrete, let us now give a numerical version. Suppose that our monopolistic firm has the specific average-revenue functions

$$P_1 = 63 - 4Q_1 \qquad \text{so that} \qquad R_1 = P_1Q_1 = 63Q_1 - 4Q_1^2$$
$$P_2 = 105 - 5Q_2 \qquad\qquad R_2 = P_2Q_2 = 105Q_2 - 5Q_2^2$$
$$P_3 = 75 - 6Q_3 \qquad\qquad R_3 = P_3Q_3 = 75Q_3 - 6Q_3^2$$

and that the total-cost function is

$$C = 20 + 15Q$$

Then the marginal functions will be

$$R_1' = 63 - 8Q_1 \qquad R_2' = 105 - 10Q_2 \qquad R_3' = 75 - 12Q_3 \qquad C' = 15$$

When each marginal revenue R_i' is set equal to the marginal cost C' of the total output, the equilibrium quantities are found to be

$$\bar{Q}_1 = 6 \qquad \bar{Q}_2 = 9 \qquad \text{and} \qquad \bar{Q}_3 = 5$$

$$\text{Thus} \qquad \bar{Q} = \sum_{i=1}^{3} \bar{Q}_i = 20$$

Substituting these solutions into the revenue and cost equations, we get $\bar{\pi} = 679$ as the total profit from the triple-market business operation.

Because this is a specific model, we do have to check the second-order condition (or the concavity of the objective function). Since the second derivatives are

$$R_1'' = -8 \qquad R_2'' = -10 \qquad R_3'' = -12 \qquad C'' = 0$$

all three parts of the second-order sufficient conditions given in Example 3 are duly satisfied.

It is easy to see from the average-revenue functions that the firm should charge the discriminatory prices $\bar{P}_1 = 39$, $\bar{P}_2 = 60$, and $\bar{P}_3 = 45$ in the three markets. As you can readily verify, the point elasticity of demand is lowest in the second market, in which the highest price is charged.

Input Decisions of a Firm

Instead of output levels Q_i, the choice variables of a firm may also appear in the guise of input levels.

Example 5 Let us assume the following circumstances: (1) Two inputs a and b are used in the production of a single product Q of a hypothetical firm. (2) The prices of both inputs, P_a and P_b, are beyond the control of the firm, as is the output price P; hence we shall denote them by P_{a0}, P_{b0}, and P_0, respectively. (3) The production process takes t_0 years (t_0 being some positive constant) to complete; thus the revenue from sales should be duly discounted before it can be properly compared with the cost of production incurred at the present time. The rate of discount, on a continuous basis, is assumed to be given at r_0.

Upon assumption 1, we can write a general production function $Q = Q(a, b)$, with marginal physical products Q_a and Q_b. Assumption 2 enables us to express the total cost as

$$C = aP_{a0} + bP_{b0}$$

and the total revenue as

$$R = P_0 Q(a, b)$$

To write the profit function, however, we must first discount the revenue by multiplying it by the constant $e^{-r_0 t_0}$—which, to avoid complicated superscripts with subscripts, we shall write as e^{-rt}. Thus, the profit function is

$$\pi = P_0 Q(a, b) e^{-rt} - aP_{a0} - bP_{b0}$$

in which a and b are the only choice variables.

To maximize profit, it is necessary that the first partial derivatives

$$\begin{aligned} \pi_a\left(\equiv \frac{\partial \pi}{\partial a}\right) &= P_0 Q_a e^{-rt} - P_{a0} \\ \pi_b\left(\equiv \frac{\partial \pi}{\partial b}\right) &= P_0 Q_b e^{-rt} - P_{b0} \end{aligned} \tag{11.34}$$

both be zero. This means that

$$P_0 Q_a e^{-rt} = P_{a0} \qquad \text{and} \qquad P_0 Q_b e^{-rt} = P_{b0} \tag{11.35}$$

Since $P_0 Q_a$ (the price of the product times the marginal product of input a) represents the *value of marginal product of input a* (VMP_a), the first equation merely says that the present value of VMP_a should be equated to the given price of input a. The second equation is the same prerequisite applied to input b.

Note that, to satisfy (11.35), the marginal physical products Q_a and Q_b must both be positive, because P_0, P_{a0}, P_{b0}, and e^{-rt} all have positive values. This has an important interpretation in terms of an *isoquant*, defined as the locus of input combinations that yield the same output level. When plotted in the ab plane, isoquants will generally appear like those drawn in Fig. 11.11. Inasmuch as each of them pertains to a fixed output level, along any isoquant we must have

$$dQ = Q_a\,da + Q_b\,db = 0$$

which implies that the slope of an isoquant is expressible as

$$\frac{db}{da} = -\frac{Q_a}{Q_b} \qquad \left(= -\frac{\text{MPP}_a}{\text{MPP}_b}\right) \tag{11.36}$$

Thus, to have Q_a and Q_b both positive is to confine the firm's input choice to the negatively sloped segments of the isoquants only. In Fig. 11.11, the relevant region of operation is accordingly restricted to the shaded area defined by the two so-called "ridge lines." Outside the shaded area, where the isoquants are characterized by positive slopes, the marginal product of one input must be negative.

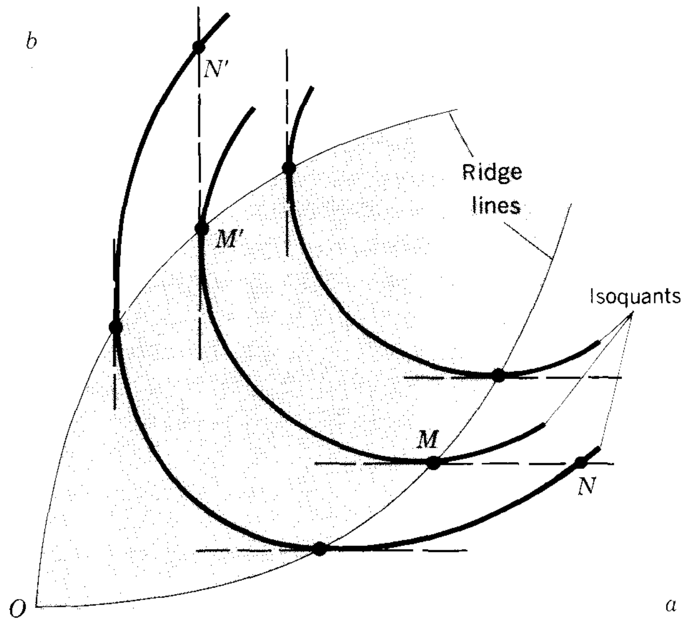


Figure 11.11

The movement from the input combination at M to the one at N, for instance, indicates that with input b held constant the *increase* in input a leads us to a *lower* isoquant (a smaller output); thus, Q_a must be negative. Similarly, a movement from M' to N' illustrates the negativity of Q_b. Note that when we confine our attention to the shaded area, each isoquant can be taken as a function of the form $b = \phi(a)$, because for every admissible value of a, the isoquant determines a unique value of b.

The second-order condition revolves around the second partial derivatives of π, obtainable from (11.34). Bearing in mind that Q_a and Q_b, being derivatives, are themselves functions of the variables a and b, we can find π_{aa}, $\pi_{ab} = \pi_{ba}$, and π_{bb}, and arrange them into a Hessian:

$$(11.37) \qquad |H| = \begin{vmatrix} \pi_{aa} & \pi_{ab} \\ \pi_{ab} & \pi_{bb} \end{vmatrix} = \begin{vmatrix} P_0 Q_{aa} e^{-rt} & P_0 Q_{ab} e^{-rt} \\ P_0 Q_{ab} e^{-rt} & P_0 Q_{bb} e^{-rt} \end{vmatrix}$$

For a stationary value of π to be a maximum, it is sufficient that

$$|H_1| < 0 \qquad [\text{that is, } \pi_{aa} < 0, \text{ which can obtain iff } Q_{aa} < 0]$$

$$|H_2| = |H| > 0 \quad [\text{that is, } \pi_{aa}\pi_{bb} > \pi_{ab}^2, \text{ which can obtain iff } Q_{aa}Q_{bb} > Q_{ab}^2]$$

Thus, we note, the second-order condition can be tested either with the π_{ij} derivatives or the Q_{ij} derivatives, whichever are more convenient.

The symbol Q_{aa} denotes the rate of change of Q_a ($\equiv \text{MPP}_a$) as input a changes while input b is fixed; similarly, Q_{bb} denotes the rate of change of Q_b ($\equiv \text{MPP}_b$) as input b changes alone. So the second-order sufficient condition stipulates, in part, that the MPP of both inputs be *diminishing* at the chosen input levels $\bar{a}$ and $\bar{b}$. Observe, however, that diminishing MPP_a and MPP_b do *not* guarantee the satisfaction of the second-order condition, because the latter condition also involves the magnitude of $Q_{ab} = Q_{ba}$, which measures the rate of change of MPP of one input as the amount of the other input varies.

Upon further examination it emerges that, just as the first-order condition specifies the isoquant to be negatively sloped at the chosen input combination (as shown in the shaded area of Fig. 11.11), the second-order sufficient condition serves to specify that same isoquant to be strictly convex at the chosen input combination. The curvature of the isoquant is associated with the sign of the second derivative d^2b/da^2. To obtain the latter, (11.36) must be differentiated totally with respect to a, bearing in mind that Q_a and Q_b are both derivative functions of a and b and yet, on an isoquant, b is itself a function of a; that is,

$$Q_a = Q_a(a, b) \qquad Q_b = Q_b(a, b) \qquad \text{and} \qquad b = \phi(a)$$

The total differentiation thus proceeds as follows:

$$\text{(11.38)} \qquad \frac{d^2b}{da^2} = \frac{d}{da}\left(-\frac{Q_a}{Q_b}\right) = -\frac{1}{Q_b^2}\left[Q_b\frac{dQ_a}{da} - Q_a\frac{dQ_b}{da}\right]$$

Since b is a function of a on the isoquant, the total-derivative formula (8.9) gives us

$$\text{(11.39)} \qquad \begin{aligned} \frac{dQ_a}{da} &= \frac{\partial Q_a}{\partial b}\frac{db}{da} + \frac{\partial Q_a}{\partial a} = Q_{ba}\frac{db}{da} + Q_{aa} \\ \frac{dQ_b}{da} &= \frac{\partial Q_b}{\partial b}\frac{db}{da} + \frac{\partial Q_b}{\partial a} = Q_{bb}\frac{db}{da} + Q_{ab} \end{aligned}$$

After substituting (11.36) into (11.39) and then substituting the latter into (11.38), we can rewrite the second derivative as

$$\text{(11.40)} \qquad \begin{aligned} \frac{d^2b}{da^2} &= -\frac{1}{Q_b^2}\left[Q_{aa}Q_b - Q_{ba}Q_a - Q_{ab}Q_a + Q_{bb}Q_a^2\left(\frac{1}{Q_b}\right)\right] \\ &= -\frac{1}{Q_b^3}\left[Q_{aa}(Q_b)^2 - 2Q_{ab}(Q_a)(Q_b) + Q_{bb}(Q_a)^2\right] \end{aligned}$$

It is to be noted that the expression in brackets (last line) is a quadratic form in the two variables Q_a and Q_b. If the second-order sufficient condition is satisfied, so that

$$Q_{aa} < 0 \qquad \text{and} \qquad \begin{vmatrix} Q_{aa} & -Q_{ab} \\ -Q_{ab} & Q_{bb} \end{vmatrix} > 0$$

then, by virtue of (11.11′), the said quadratic form must be negative definite. This will in turn make d^2b/da^2 positive, because Q_b has been constrained to be positive by the first-order condition. Thus the satisfaction of the second-order sufficient condition means that the relevant (negatively sloped) isoquant is strictly convex at the chosen input combination, as was asserted.

The concept of strict convexity, as applied to an isoquant $b = \phi(a)$, which is drawn in the two-dimensional ab plane, should be carefully distinguished from the same concept as applied to the production function $Q(a, b)$ itself, which is drawn in the three-dimensional abQ space. Note, in particular, that if we are to apply the concept of strict concavity or convexity to the production function in the present context, then, to produce the desired isoquant shape, the appropriate stipulation is that $Q(a, b)$ be strictly *concave* in the 3-space (be dome-shaped), which is in sharp contradistinction to the stipulation that the relevant isoquant be strictly *convex* in the 2-space (be U-shaped, or shaped like a part of a U).

Example 6 Next, suppose that interest is compounded *quarterly* instead, at a given interest rate of i_0 per quarter. Also suppose that the production process takes exactly a quarter of a year. The profit function then becomes

$$\pi = P_0 Q(a, b)(1 + i_0)^{-1} - aP_{a0} - bP_{b0}$$

The first-order condition is now found to be

$$P_0 Q_a (1 + i_0)^{-1} - P_{a0} = 0$$

$$P_0 Q_b (1 + i_0)^{-1} - P_{b0} = 0$$

with an analytical interpretation entirely the same as in Example 5, except for the different manner of discounting.

You can readily see that the same sufficient condition derived in the preceding example must apply here as well.

EXERCISE 11.6

1 If the competitive firm of Example 1 has the cost function $C = 2Q_1^2 + 2Q_2^2$ instead, then:

(a) Will the production of the two goods still be technically related?

(b) What will be the new optimal levels of Q_1 and Q_2?

(c) What is the value of π_{12}? What does this imply economically?

2 A two-product firm faces the demand and cost functions below:

$$Q_1 = 40 - 2P_1 - P_2 \qquad Q_2 = 35 - P_1 - P_2 \qquad C = Q_1^2 + 2Q_2^2 + 10$$

(a) Find the output levels that satisfy the first-order condition for maximum profit. (Use fractions.)

(b) Check the second-order sufficient condition. Can you conclude that this problem possesses a unique absolute maximum?

(c) What is the maximal profit?

3 On the basis of the equilibrium price and quantity in Example 4, calculate the point elasticity of demand $|\varepsilon_{di}|$ (for $i = 1, 2, 3$). Which market has the highest and the lowest demand elasticities?

4 If the cost function of Example 4 is changed to $C = 20 + 15Q + Q^2$:
(*a*) Find the new marginal-cost function.
(*b*) Find the new equilibrium quantities. (Use fractions).
(*c*) Find the new equilibrium prices.
(*d*) Verify that the second-order sufficient condition is met.

5 In Example 6, how would you rewrite the profit function if the following conditions hold?
(*a*) Interest is compounded semiannually at an interest rate of i_0 per annum, and the production process takes 1 year.
(*b*) Interest is compounded quarterly at an interest rate of i_0 per annum, and the production process takes 9 months.

6 Given $Q = Q(a, b)$, how would you express algebraically the isoquant for the output level of, say, 260?

11.7 COMPARATIVE-STATIC ASPECTS OF OPTIMIZATION

Optimization, which is a special variety of static equilibrium analysis, is naturally also subject to investigations of the comparative-static sort. The idea is, again, to find out how a change in any parameter will affect the equilibrium position of the model, which in the present context refers to the optimal values of the choice variables (and the optimal value of the objective function). Since no new technique is involved beyond those discussed in Part 3, we may proceed directly with some illustrations, based on the examples introduced in the preceding section.

Reduced-Form Solutions

Example 1 of Sec. 11.6 contains two parameters (or exogenous variables), P_{10} and P_{20}; it is not surprising, therefore, that the optimal output levels of this two-product firm are expressed strictly in terms of these parameters:

$$\bar{Q}_1 = \frac{4P_{10} - P_{20}}{15} \qquad \text{and} \qquad \bar{Q}_2 = \frac{4P_{20} - P_{10}}{15}$$

These are reduced-form solutions, and simple partial differentiation alone is sufficient to tell us all the comparative-static properties of the model, namely,

$$\frac{\partial \bar{Q}_1}{\partial P_{10}} = \frac{4}{15} \qquad \frac{\partial \bar{Q}_1}{\partial P_{20}} = -\frac{1}{15} \qquad \frac{\partial \bar{Q}_2}{\partial P_{10}} = -\frac{1}{15} \qquad \frac{\partial \bar{Q}_2}{\partial P_{20}} = \frac{4}{15}$$

For maximum profit, each product of the firm should be produced in a larger quantity if its market price rises or if the market price of the other product falls.

Of course, these conclusions follow only from the particular assumptions of the model in question. We may point out, in particular, that the effects of a change in P_{10} on $\bar{Q}_2$ and of P_{20} on $\bar{Q}_1$, are consequences of the assumed technical relation on the production side of these two commodities, and that in the absence of such a relation we shall have

$$\frac{\partial \bar{Q}_1}{\partial P_{20}} = \frac{\partial \bar{Q}_2}{\partial P_{10}} = 0$$

Moving on to Example 2, we note that the optimal output levels are there stated, numerically, as $\bar{Q}_1 = 8$ and $\bar{Q}_2 = 7\frac{2}{3}$—no parameters appear. In fact, all the constants in the equations of the model are numerical rather than parametric, so that by the time we reach the solution stage those constants have all lost their respective identities through the process of arithmetic manipulation. What this serves to underscore is the fundamental lack of generality in the use of numerical constants and the consequent lack of comparative-static content in the equilibrium solution.

On the other hand, the *non*use of numerical constants is no guarantee that a problem will automatically become amenable to comparative-static analysis. The price-discrimination problem (Example 3), for instance, was primarily set up for the study of the equilibrium (profit-maximization) condition, and no parameter was introduced at all. Accordingly, even though stated in terms of general functions, a reformulation will be necessary if a comparative-static study is contemplated.

General-Function Models

The input-decision problem of Example 5 illustrates the case where a general-function formulation does embrace several parameters—in fact, no less than five (P_0, P_{a0}, P_{b0}, r, and t), where we have, as before, omitted the 0 subscripts from the exogenous variables r_0 and t_0. How do we derive the comparative-static properties of this model?

The answer lies again in the application of the implicit-function theorem. But, unlike the cases of nongoal-equilibrium models of the market or of national-income determination, where we worked with the equilibrium conditions of the model, the present context of goal equilibrium dictates that we work with the first-order conditions of optimization. For Example 5, these conditions are stated in (11.35). Collecting all terms in (11.35) to the left of the equals signs, and making explicit that Q_a and Q_b are both functions of the endogenous (choice) variables a and b, we can rewrite the first-order conditions in the format of (8.20) as follows:

$$\begin{aligned} F^1(a, b; P_0, P_{a0}, P_{b0}, r, t) &= P_0 Q_a(a, b) e^{-rt} - P_{a0} = 0 \\ F^2(a, b; P_0, P_{a0}, P_{b0}, r, t) &= P_0 Q_b(a, b) e^{-rt} - P_{b0} = 0 \end{aligned} \tag{11.41}$$

The functions F^1 and F^2 are assumed to possess continuous derivatives. Thus it would be possible to apply the implicit-function theorem, provided the Jacobian of this system with respect to the endogenous variables a and b does not vanish at the initial equilibrium. The said Jacobian turns out to be nothing but the Hessian determinant of the π function of Example 5:

$$(11.42)\qquad |J| = \begin{vmatrix} \dfrac{\partial F^1}{\partial a} & \dfrac{\partial F^1}{\partial b} \\ \dfrac{\partial F^2}{\partial a} & \dfrac{\partial F^2}{\partial b} \end{vmatrix} = \begin{vmatrix} P_0Q_{aa}e^{-rt} & P_0Q_{ab}e^{-rt} \\ P_0Q_{ab}e^{-rt} & P_0Q_{bb}e^{-rt} \end{vmatrix} = |H|$$

[by (11.37)]

Hence, if we assume that the second-order sufficient condition for profit-maximization is satisfied, then $|H|$ must be positive, and so must be $|J|$, at the initial equilibrium or optimum. In that event, the implicit-function theorem will enable us to write the pair of implicit functions

$$(11.43)\qquad \begin{aligned} \bar{a} &= \bar{a}(P_0, P_{a0}, P_{b0}, r, t) \\ \bar{b} &= \bar{b}(P_0, P_{a0}, P_{b0}, r, t) \end{aligned}$$

as well as the pair of identities

$$(11.44)\qquad \begin{aligned} P_0Q_a(\bar{a}, \bar{b})e^{-rt} - P_{a0} &\equiv 0 \\ P_0Q_b(\bar{a}, \bar{b})e^{-rt} - P_{b0} &\equiv 0 \end{aligned}$$

To study the comparative statics of the model, first take the total differential of each identity in (11.44). For the time being, we shall permit all the exogenous variables to vary, so that the result of total differentiation will involve $d\bar{a}$, $d\bar{b}$, as well as dP_0, dP_{a0}, dP_{b0}, dr, and dt. If we place on the left side of the equals sign only those terms involving $d\bar{a}$ and $d\bar{b}$, the result will be

$$(11.45)\qquad \begin{aligned} & P_0Q_{aa}e^{-rt}\,d\bar{a} + P_0Q_{ab}e^{-rt}\,d\bar{b} \\ &\qquad = -Q_ae^{-rt}\,dP_0 + dP_{a0} + P_0Q_ate^{-rt}\,dr + P_0Q_are^{-rt}\,dt \\ & P_0Q_{ab}e^{-rt}\,d\bar{a} + P_0Q_{bb}e^{-rt}\,d\bar{b} \\ &\qquad = -Q_be^{-rt}\,dP_0 + dP_{b0} + P_0Q_bte^{-rt}\,dr + P_0Q_bre^{-rt}\,dt \end{aligned}$$

where, be it noted, the first and second derivatives of Q are all to be evaluated at the equilibrium, i.e., at $\bar{a}$ and $\bar{b}$. You will also note that the coefficients of $d\bar{a}$ and $d\bar{b}$ on the left are precisely the elements of the Jacobian in (11.42).

To derive the specific comparative-static derivatives—of which there are a total of ten (why?)—we now shall allow only a single exogenous variable to vary at a time. Suppose we let P_0 vary, alone. Then $dP_0 \neq 0$, but $dP_{a0} = dP_{b0} = dr = dt = 0$, so that only the first term will remain on the right side of each equation in (11.45). Dividing through by dP_0, and interpreting the ratio $d\bar{a}/dP_0$ to be the comparative-static derivative $(\partial\bar{a}/\partial P_0)$, and similarly for the ratio $d\bar{b}/dP_0$, we can write the matrix equation

$$\begin{bmatrix} P_0Q_{aa}e^{-rt} & P_0Q_{ab}e^{-rt} \\ P_0Q_{ab}e^{-rt} & P_0Q_{bb}e^{-rt} \end{bmatrix} \begin{bmatrix} (\partial\bar{a}/\partial P_0) \\ (\partial\bar{b}/\partial P_0) \end{bmatrix} = \begin{bmatrix} -Q_ae^{-rt} \\ -Q_be^{-rt} \end{bmatrix}$$

The solution, by Cramer's rule, is found to be

$$(11.46)\qquad \begin{aligned} \left(\frac{\partial \bar{a}}{\partial P_0}\right) &= \frac{(Q_b Q_{ab} - Q_a Q_{bb}) P_0 e^{-2rt}}{|J|} \\ \left(\frac{\partial \bar{b}}{\partial P_0}\right) &= \frac{(Q_a Q_{ab} - Q_b Q_{aa}) P_0 e^{-2rt}}{|J|} \end{aligned}$$

If you prefer, an alternative method is available for obtaining these results: You may simply differentiate the two identities in (11.44) *totally* with respect to P_0 (while holding the other four exogenous variables fixed), bearing in mind that P_0 can affect $\bar{a}$ and $\bar{b}$ via (11.43).

Let us now analyze the signs of the comparative-static derivatives in (11.46). On the assumption that the second-order sufficient condition is satisfied, the Jacobian in the denominator must be positive. The second-order condition also implies that Q_{aa} and Q_{bb} are negative, just as the first-order condition implies that Q_a and Q_b are positive. Moreover, the expression $P_0 e^{-2rt}$ is certainly positive. Thus, if $Q_{ab} > 0$ (if increasing one input will raise the MPP of the other input), we can conclude that both $(\partial \bar{a}/\partial P_0)$ and $(\partial \bar{b}/\partial P_0)$ will be positive, implying that an increase in the product price will result in increased employment of both inputs in equilibrium. If $Q_{ab} < 0$, on the other hand, the sign of each derivative in (11.46) will depend on the relative strength of the negative force and the positive force in the parenthetical expression on the right.

Next, let the exogenous variable r vary, alone. Then all the terms on the right of (11.45) will vanish except those involving dr. Dividing through by $dr \neq 0$, we now obtain the following matrix equation

$$\begin{bmatrix} P_0 Q_{aa} e^{-rt} & P_0 Q_{ab} e^{-rt} \\ P_0 Q_{ab} e^{-rt} & P_0 Q_{bb} e^{-rt} \end{bmatrix} \begin{bmatrix} (\partial \bar{a}/\partial r) \\ (\partial \bar{b}/\partial r) \end{bmatrix} = \begin{bmatrix} P_0 Q_a t e^{-rt} \\ P_0 Q_b t e^{-rt} \end{bmatrix}$$

with the solution

$$(11.47)\qquad \begin{aligned} \left(\frac{\partial \bar{a}}{\partial r}\right) &= \frac{t(Q_a Q_{bb} - Q_b Q_{ab})(P_0 e^{-rt})^2}{|J|} \\ \left(\frac{\partial \bar{b}}{\partial r}\right) &= \frac{t(Q_b Q_{aa} - Q_a Q_{ab})(P_0 e^{-rt})^2}{|J|} \end{aligned}$$

Both of these comparative-static derivatives will be negative if Q_{ab} is positive, but indeterminate in sign if Q_{ab} is negative.

By a similar procedure, we may find the effects of changes in the remaining parameters. Actually, in view of the symmetry between r and t in (11.44) it is immediately obvious that both $(\partial \bar{a}/\partial t)$ and $(\partial \bar{b}/\partial t)$ must be similar in appearance to (11.47).

The effects of changes in P_{a0} and P_{b0} are left to you to analyze. As you will find, the sign restriction of the second-order sufficient condition will again be useful in evaluating the comparative-static derivatives, because it can tell us the signs of Q_{aa} and Q_{bb} as well as the Jacobian $|J|$ at the initial equilibrium (optimum). Thus, aside from distinguishing between maximum and minimum, the second-order condition also has a vital role to play in the study of shifts in equilibrium positions as well.

EXERCISE 11.7

For the following three problems, assume that $Q_{ab} > 0$.

1 On the basis of the model described in (11.41) through (11.44), find the comparative-static derivatives $(\partial\bar{a}/\partial P_{a0})$ and $(\partial\bar{b}/\partial P_{a0})$. Interpret the economic meaning of the result. Then analyze the effects on $\bar{a}$ and $\bar{b}$ of a change in P_{b0}.

2 For the problem of Example 6 in Sec. 11.6:

(*a*) How many parameters are there? Enumerate them.

(*b*) Following the procedure described in (11.41) through (11.46), and assuming that the second-order sufficient condition is satisfied, find the comparative-static derivatives $(\partial\bar{a}/\partial P_0)$ and $(\partial\bar{b}/\partial P_0)$. Evaluate their signs and interpret their economic meanings.

(*c*) Find $(\partial\bar{a}/\partial i_0)$ and $(\partial\bar{b}/\partial i_0)$, evaluate their signs, and interpret their economic meanings.

3 Show that the results in (11.46) can be obtained alternatively by differentiating the two identities in (11.44) *totally* with respect to P_0, while holding the other exogenous variables fixed. Bear in mind that P_0 can affect $\bar{a}$ and $\bar{b}$ by virtue of (11.43).

4 A Jacobian determinant, as defined in (7.27), is made up of *first*-order partial derivatives. On the other hand, a Hessian determinant, as defined in Secs. 11.3 and 11.4, has as its elements *second*-order partial derivatives. How, then, can it turn out that $|J| = |H|$, as in (11.42)?