Lecture Notes for Chapter 8

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1 Chapter 8: Comparative Static Analysis of General Function Models

1. General Form

National Income model

1.0.1 Specific

$$\begin{aligned}(1) &: \quad Y = C + I_0 + G_0 \\ (2) &: \quad C = a + b(Y = T_0)\end{aligned}$$

By substitution

$$Y = a - b(Y - T_0) + I_0 + G_0$$

Solution

$$Y^e = \frac{a + I_0 + G_0 - bT_0}{1 - b}$$

1.0.2 General

$$\begin{aligned} Y &= Y(C, I_0, G_0) \\ C &= C(Y, T_0) \\ Y &= Y(C(Y, T_0), I_0, G_0) \\ Y^e &= Y^e(I_0, G_0, T_0) \end{aligned}$$

The general form can be expressed as:

$$Y^e = C(Y^e, T_0) + I_0 + G_0$$

$\frac{\partial Y^e}{\partial T_0}$ has a direct and indirect effect:

$$\frac{\partial C}{\partial T_0} \text{ and } \frac{\partial C}{\partial Y^e}\frac{\partial Y^e}{\partial T_0}$$

1.1 Differentials

$$\begin{aligned} \text{Given } y &= f(x) \\ \text{Then } \frac{dy}{dx} &= f'(x) \\ \text{But also } \underset{\text{Change in Y}}{dy} &= \underset{\text{A converter}}{f'(x)} \underset{\text{Change in X}}{dx} \end{aligned}$$

$f\prime(x)$ "converts" a Δ in x into a Δ in Y
Example:

$$\begin{aligned} y &= x^2 \Rightarrow dy = 2xdx \\ \text{at } x &= 2;\ y = 4,\ \text{if } dx = .01 \text{ then } dy = 2(2)(0.01) = 0.4 \end{aligned}$$

Therefore: as x $\Delta's$ from 2 to 2.01 then y $\Delta's$ from 4 to 4.04

1.1.1 Differentials and Point Elasticity

From ECON 200

Are Elasticity = $\frac{\Delta\frac{Q}{Q}}{\Delta\frac{P}{P}}$ or $\frac{\Delta Q}{\Delta P}\frac{P}{Q}$

Point Elasticity

$$\epsilon^d = \frac{dQ}{dP}\frac{P}{Q} = \frac{\frac{dQ}{dP}}{\frac{Q}{P}} = \frac{\text{Marginal}}{\text{Average}}$$

Example

$$\begin{aligned} \text{Let } Q &= a - bP \\ \text{Then } \frac{dQ}{dP} &= -b \ \text{ and } \ \frac{P}{Q} = \frac{P}{a - bP} \end{aligned}$$

Therefore

$$\epsilon^d = \frac{-bP}{a - bP}$$

$$\begin{aligned} \text{Let } Q &= 10 - 2P \\ \text{Then } \epsilon^d &= \frac{-2P}{10 - 2P} \\ \epsilon^d &= \frac{-P}{5 - P} \end{aligned}$$

1.2 Total Differentials

Consider the Utility Function:

$$U = U(x, y)$$

Totally differentiate

$$\begin{aligned} dU &= \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy \\ \text{or } dU &= MU_x dx + MU_y dy \end{aligned}$$

$$
\begin{aligned}
\text{Indifference Curve} &: \quad dU = 0 \\
MU_x dx + MU_y dy &= 0 \\
MU_y dy &= -MU_x dx \\
dy &= \frac{-MU_x}{MU_y} dx \quad (\text{iff } MU_y \neq 0!!) \\
\frac{dy}{dx} &= \frac{-MU_x}{MU_y} = MRS
\end{aligned}
$$

Graphically
[1]

Note: if $dx.0$ then $dy'0$ but both $MU_x, MU_y.0$ (from Economic Theory). Therefore minus sign (-) in front of $-\frac{MU_x}{MU_y}$

Example

1.

$$
\begin{aligned}
\text{if } U(x,y) &= xy \\
\text{then } dU &= ydx + xdy \\
\text{and } MRS &= \frac{dy}{dx} = -\frac{y}{x}
\end{aligned}
$$

2.

$$
\begin{aligned}
\text{if } U(x,y) &= x^2y^2 \\
\text{then } dU &= 2xy^2dx + 2x^2ydy \\
\text{and } MRS &= \frac{dy}{dx} = -\frac{2xy^2}{2x^2y} = -\frac{y}{x}
\end{aligned}
$$

[1] Graph page 5 chapter 8

3.

$$\begin{aligned} \text{if } U(x,y) &= x+y \\ \text{then } dU &= dx+dy \\ \text{and } MRS &= \frac{dy}{dx} = -1 \end{aligned}$$

1.2.1 Total Differentials: Generally

$$\begin{aligned} \text{Let } U &= U(x_1, x_2, ...x_n) \\ \text{Then } dU &= \frac{\partial U}{\partial x_1}dx_1 + \frac{\partial U}{\partial x_2}dx_2 + ...\frac{\partial U}{\partial x_n}dx_n \\ \text{Or } dU &= U_1 dx_1 + U_2 dx_2 + ...U_n dx_n \\ \text{where } U_i &= \frac{\partial U}{\partial x_i} \text{ (the partial derivative)} \end{aligned}$$

$$\begin{aligned} \text{If } dx_2 &= dx_3 = ...dx_n = 0 \\ \text{Then } dU &= \frac{\partial U}{\partial x_1}dx_1 + (0) \\ \text{Then } \frac{dU}{dx_1} &= \frac{\partial U}{\partial x_1} = U_1 \end{aligned}$$

*The partial derivative of a function is simply the total differential with all but one of the dx_i's set equal to zero.

1.2.2 Rules of Differentials

1. $dk = 0$
2. $y = ax^n \Rightarrow dy = anx^{n-1}dx$
3. $y = x_1 + x_2 \Rightarrow dy = dx_1 + dx_2$

4. $y = x_1 x_2 \Rightarrow dy = x_2 dx_1 + x_1 dx_2$

5. $y = \frac{x_1}{x_2} \Rightarrow dy = \frac{x_2 dx_1 - x_1 dx_2}{x_2^2}$

Example

$$\begin{aligned}
y &= x_1^3 + 3x_2^2 + 4x_1 x_2 \\
dy &= \frac{dy}{dx_1} dx_1 + \frac{dy}{dx_2} dx_2 \\
\frac{\partial y}{\partial x_1} &= 3x_1^2 + 4x^2 \\
\frac{\partial y}{\partial x_2} &= 6x_2 + 4x_1 \\
dy &= \left(3x_1^2 + 4x^2\right) dx_1 + \left(6x_2 + 4x_1\right) dx_2
\end{aligned}$$

Further Examples

$$
\begin{aligned}
y &= \frac{(x_1 + x_2)^2}{x_2^3} \\
dy &= \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 \\
\frac{\partial y}{\partial x_1} &= \left(\frac{1}{x_2^3}\right) 2\left(x_1 + x_2\right)(1) = \frac{2\left(x_1 + x_2\right)}{x_2^3} \\
\frac{\partial y}{\partial x_2} &= \frac{\left[x_2^3\left(x_1 + x_2\right)(2)\right] - \left[\left(x_1 + x_2\right)^2 (3)(x_2^2)\right]}{\left(x_2^3\right)^2} \\
\frac{\partial y}{\partial x_2} &= \frac{2x_2^3\left(x_1 + x_2\right) - 3\left(x_1 + x_2\right)^2 x_2^2}{x_2^6} \\
\frac{\partial y}{\partial x_2} &= \frac{2x_2\left(x_1 + x_2\right) - 3\left(x_1 + x_2\right)^2}{x_2^4} \\
dy &= \left[\frac{2\left(x_1 + x_2\right)}{x_2^3}\right] dx_1 + \left[\frac{2x_2\left(x_1 + x_2\right) - 3\left(x_1 + x_2\right)^2}{x_2^4}\right] dx_2
\end{aligned}
$$

1.2.3 Cobb-Douglas Production Function

$$
\begin{aligned}
Q &= Q(K,L) = K^a L^b \\
dQ &= \frac{\partial Q}{\partial K} dK + \frac{\partial Q}{\partial L} dL = MP_K dK + MP_L dL \\
\frac{\partial Q}{\partial K} &= \left[aK^{a-1}L^b\right] = \left[a\frac{K^a L^b}{K}\right] = \left[a\frac{Q}{K}\right] \\
\frac{\partial Q}{\partial L} &= \left[bK^a L^{b-1}\right] = \left[b\frac{K^a L^b}{L}\right] = \left[b\frac{Q}{L}\right] \\
dQ &= \left[a\frac{K^a L^b}{K}\right] dK + \left[b\frac{K^a L^b}{L}\right] dL \\
dQ &= \left[a\frac{dK}{K} + b\frac{dL}{L}\right] \cdot K^a L^b \\
dQ &= \left[a\frac{dK}{K} + b\frac{dL}{L}\right] \cdot Q \\
\frac{dQ}{Q} &= (a+b)\,\frac{dS}{S} = \frac{\frac{dQ}{Q}}{\frac{dS}{S}} = (a+b) \quad \text{Elasticity of Scale}
\end{aligned}
$$

1.3 Total Derivatives and the Chain Rule

$$
\begin{aligned}
\text{Let } y &= y(x,z) \text{ and } x = x(z) \\
dy &= \left(\frac{\partial y}{\partial x}\right) dx + \left(\frac{\partial y}{\partial z}\right) dz \text{ and } dx = \frac{dx}{dz} dz \\
\text{Substitute } dy &= \left(\frac{\partial y}{\partial x}\right)\left(\frac{dx}{dz}\right) dz + \left(\frac{\partial y}{\partial z}\right) dz \\
\text{Divide by } dz \quad \frac{dy}{dz} &= \left(\frac{\partial y}{\partial x}\frac{dx}{dz} + \frac{\partial y}{\partial z}\right)\frac{dz}{dz} \quad \left\{\frac{dz}{dz} = 1\right\}
\end{aligned}
$$

The Total Derivative $\frac{dy}{dz}$ is:

$$\underbrace{\frac{dy}{dz}}_{\text{Total } \Delta \text{ in Y from } \Delta \text{ in z}} = \underbrace{\left(\frac{\partial y}{\partial x}\right)\left(\frac{dx}{dz}\right)}_{\text{The indirect effect of z on y through x}} + \underbrace{\left(\frac{\partial y}{\partial z}\right)}_{\text{The direct effect of z on y}}$$

1.3.1 Chain Rule

$$\begin{aligned} y &= y(x,z) \text{ but } x = x(z) \\ \text{Therefore} y &= y(x(z),z) \quad \{y = f(z)\} \end{aligned}$$

y is a function of one exogenous variable

$$\frac{dy}{dz} = \frac{\partial y}{\partial x}\frac{dx}{dz} + \frac{\partial y}{\partial z}$$

$$\overset{\text{Indirect}}{\underset{\text{Direct (z to y)}}{y \leftarrow x \leftarrow z}}$$

Example

$$
\begin{aligned}
y &= (x+2)^2 + zx + z^2 \\
x &= 2z+3 \\
\frac{dy}{dz} &= \frac{\partial y}{\partial x}\frac{dx}{dz} + \frac{\partial y}{\partial z}\frac{dz}{dz} \\
& \quad (1)\ \tfrac{\partial y}{\partial x} = [2(x+2)+z] \qquad (2)\ \tfrac{\partial y}{\partial z} = [x+2z] \qquad (3)\ \tfrac{dx}{dz} = 2 \\
\frac{dy}{dz} &= \underbrace{(2x+4+2)}_{\frac{\partial y}{\partial x}}\underbrace{(2)}_{\frac{dx}{dz}} + \underbrace{(x+2z)}_{\frac{\partial y}{\partial x}} \\
\text{sub in } x &= (2z+3) \\
\frac{dy}{dz} &= (2\,(2z+3)+4+2)\,(2) + ((2z+3)+2z) \\
\frac{dy}{dz} &= (10z+20) + (4z+3) = 14z+23
\end{aligned}
$$

Alternative Method

$$
\begin{aligned}
y &= (x+2)^2 + zx + z^2 \\
x &= 2z+3 \\
y &= ((2z+3)+2)^2 + z\,(2z+3) + z^2 \\
y &= (2z+5)^2 + 3z^2 + 3z \\
\frac{dy}{dz} &= 2(2z+5)(2) + 6z + 3 \\
\frac{dy}{dz} &= 8z + 20 + 6z + +3 = 14z + 23
\end{aligned}
$$

2 approaches for $y = y(x,z)$ and $x = x(z)$

1.

$$\begin{aligned} dy &= \frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial z}dz \\ \text{sub in for } dx &= \frac{dx}{dz}dz \\ dy &= \left[\frac{\partial y}{\partial x}\frac{dx}{dz} + \frac{\partial y}{\partial z}\right] dz \end{aligned}$$

2.

$$\begin{aligned} & \text{sub } x\left(z\right) \text{ into } y(x,z) \\ y &= y\left(x(z), z\right) \\ y &= g(z) \text{ "g" is a new function} \\ \frac{dy}{dz} &= g'(z) \end{aligned}$$

Further Examples

$$\begin{aligned} y &= y\left(x_1, x_2, \alpha, \beta\right) \\ \text{and } x_1 &= x_1(\alpha,\beta) \qquad x_2 = x_2(\alpha,\beta) \\ dy &= \left[\left(\tfrac{\partial y}{\partial x_1}\right)\left(\tfrac{dx_1}{d\alpha}\right) + \left(\tfrac{\partial y}{\partial x_2}\right)\left(\tfrac{dx_2}{d\alpha}\right) + \tfrac{\partial y}{\partial \alpha}\right] d\alpha \\ & \quad + \left[\left(\tfrac{\partial y}{\partial x_1}\right)\left(\tfrac{dx_1}{d\beta}\right) + \left(\tfrac{\partial y}{\partial x_2}\right)\left(\tfrac{dx_2}{d\beta}\right) + \tfrac{\partial y}{\partial \beta}\right] d\beta \end{aligned}$$

y is a function of 4 variables but only 2 exogenous variables (α, β)
Find $\frac{dy}{d\alpha}$, (the total derivative w.r.t. α)

1. set $d\beta = 0$ (the second term drops out)
2. divide by $d\alpha$

$$\frac{dy}{d\alpha} = \left[\left(\frac{\partial y}{\partial x_1}\frac{dx_1}{d\alpha}\right) + \left(\frac{\partial y}{\partial x_2}\frac{dx_2}{d\alpha}\right) + \frac{\partial y}{\partial \alpha}\right]\frac{d\alpha}{d\alpha} \quad \left(\frac{d\alpha}{d\alpha} = 1\right)$$

1.3.2 Differentials and Derivatives

$$\begin{aligned} y &= y(x) \\ dy &= y'(x)dx \\ \text{or } dy &= \frac{dy}{dx}dx \end{aligned}$$

Divide both sides by dx

$\frac{dy}{dx}$	$=$	$\frac{dy}{dx}$
LHS: is a ratio of two differentials		RHS: is NOT a ratio of two differentials. RHS is the derivative $\frac{dy}{dx} = y'(x)$

1.4 Implicit Functions

Explicit Function

$$y = f(x)$$

Rewritten as an Implicit Function

$$y - f(x) = 0$$

In General:

$$\begin{aligned} F(y,x) &= 0 \\ F(y,x) &= k \text{ (where k is some constant or parameter} \end{aligned}$$

Any explicit function, y=f(x), can be expressed as an implicit function, F(y,x)=0, however, not all implicit functions can be expressed as explicit functions directly.

An implicit function: $F(y, x_1, ...x_n) = 0$ may define y as a function of $x_1, ...x_n$, yet cannot be solved directly for $y = f(x_1, ..., x_n)$ (this may hold only over a limited range of F, but not everywhere).

We can tell if $F(y, x_1, ...x_n)$ does indeed implicitly define y as a function of $x_1, ...x_n$ by us of the IMPLICIT FUNCTION THEOREM.

THEOREM:

1. (a) if F has continuous partial derivatives $F_y, F_1, F_2, ...F_n$ and
   (b) at the point we are interested in $F_y \neq 0$ at $y = y_0$

Then at $y = y_0$ F implicitly defines y as a function of $x_1, ...x_n$.(at some value $y = y_0$ F=0 is an identity)

Suppose:

$$F(y, x_1, x_2) = 0$$

(if the values of y, x_1, x_2 are the onesthat satisfy this equiation, then this equation is an identity)

However, this function cannot be solved explicity for

$$y = f(x_1, x_2)$$

We can still find

$$\frac{\partial y}{\partial x_1} \text{ and } \frac{\partial y}{\partial x_2}$$

Through the use of Total Differentials

$$dF = F_y dy + F_1 dx_1 + F_2 dx_2 = 0$$

Let $dx_2 = 0$

Then

$$\begin{aligned} F_y dy + F_1 dx_1 &= 0 \\ F_y dy &= -F_1 dx_1 \\ \frac{\partial y}{\partial x} &= \left.\frac{dy}{dx_1}\right|_{dx_2=0} = \frac{-F_1}{F_y} \quad \{F_y \neq 0\} \end{aligned}$$

1.4.1 Implicit Function Rule

Given:

$$F(y, x_1, ...x_n) = 0$$

Then:

$$\underset{\text{The partial derivative is interpreted as a ratio of two differentials}}{\frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y}} \left\{ = \frac{\overbrace{\frac{\partial F}{\partial x_i}}^{F_y \neq 0}}{\frac{\partial F}{\partial y}} \right\}$$

Example:

$$\bar{U} = U(y, x) = x^{1/2}y^{1/2}$$

For $dU = 0$

$$\frac{dy}{dx} = -\frac{U_x}{U_y} = -\frac{(\frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}})}{(\frac{1}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}})} = -\frac{y}{x} = MRS$$

Explicitly:

$$\begin{aligned} y &= \frac{\bar{U}^2}{x} \quad \left\{\bar{U}^2 = \text{constant}\right\} \\ \frac{dy}{dx} &= -\frac{\bar{U}^2}{x^2} = -\left(\frac{\bar{U}^2}{x}\right)\frac{1}{x} = -\frac{y}{x} \end{aligned}$$

Or:

$$\begin{aligned} \frac{\partial F}{\partial y_1}dy_1 + \frac{\partial F}{\partial y_2}dy_2 &= \left(-\frac{\partial F}{\partial x_1}dx_1\right) + \left(-\frac{\partial F}{\partial x_2}dx_2\right) \\ \frac{\partial G}{\partial y_1}dy_1 + \frac{\partial G}{\partial y_2}dy_2 &= \left(-\frac{\partial G}{\partial x_1}dx_1\right) + \left(-\frac{\partial G}{\partial x_2}dx_2\right) \end{aligned}$$

In Matrix Form:

$$\underbrace{\overset{\displaystyle A}{\overset{(2x2)}{\begin{bmatrix} \frac{\partial F}{\partial y_1} & \frac{\partial F}{\partial y_2} \\ \frac{\partial G}{\partial y_1} & \frac{\partial G}{\partial y_2} \end{bmatrix}}}}_{"Jacobian"} \overset{\displaystyle X}{\overset{(2x1)}{\begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix}}} = \overset{\displaystyle d}{\overset{(2x1)}{\begin{bmatrix} -\frac{\partial F}{\partial x_1}dx_1 & -\frac{\partial F}{\partial x_2}dx_2 \\ -\frac{\partial G}{\partial x_1}dx_1 & -\frac{\partial G}{\partial x_2}dx_2 \end{bmatrix}}}$$

Test for existance by the Determinant

$$|J| = \left(\frac{\partial F}{\partial y_1}\right)\left(\frac{\partial G}{\partial y_2}\right) - \left(\frac{\partial F}{\partial y_2}\right)\left(\frac{\partial G}{\partial y_1}\right) \neq 0$$

If $|J| = 0$ then y_1 and y_2 are not functions of x_1 and x_2

$|J| = 0$ is the same as $f_y \neq$ in single equation case.

Jacobian: Matrix of "Partial Derivatives" with respect ot the "Endogenous variables" where the partial derivative and are treated as constants.