CHAPTER

SEVEN

RULES OF DIFFERENTIATION AND THEIR USE IN COMPARATIVE STATICS

The central problem of comparative-static analysis, that of finding a rate of change, can be identified with the problem of finding the derivative of some function $y = f(x)$, provided only a small change in x is being considered. Even though the derivative dy/dx is defined as the limit of the difference quotient $q = g(v)$ as $v \to 0$, it is by no means necessary to undertake the process of limit-taking each time the derivative of a function is sought, for there exist various rules of differentiation (derivation) that will enable us to obtain the desired derivatives directly. Instead of going into comparative-static models immediately, therefore, let us begin by learning some rules of differentiation.

7.1 RULES OF DIFFERENTIATION FOR A FUNCTION OF ONE VARIABLE

First, let us discuss three rules that apply, respectively, to the following types of function of a single independent variable: $y = k$ (constant function), $y = x^n$, and $y = cx^n$ (power functions). All these have smooth, continuous graphs and are therefore differentiable everywhere.

Constant-Function Rule

The derivative of a constant function $y = f(x) = k$ is identically zero, i.e., is zero for all values of x. Symbolically, this may be expressed variously as

$$\frac{dy}{dx} = 0 \qquad \text{or} \qquad \frac{dk}{dx} = 0 \qquad \text{or} \qquad f'(x) = 0$$

In fact, we may also write these in the form

$$\frac{d}{dx}y = \frac{d}{dx}f(x) = \frac{d}{dx}k = 0$$

where the derivative symbol has been separated into two parts, d/dx on the one hand, and y [or $f(x)$ or k] on the other. The first part, d/dx, may be taken as an *operator symbol*, which instructs us to perform a particular mathematical operation. Just as the operator symbol $\sqrt{}$ instructs us to take a square root, the symbol d/dx represents an instruction to take the derivative of, or to differentiate, (some function) with respect to the variable x. The function to be operated on (to be differentiated) is indicated in the second part; here it is $y = f(x) = k$.

The proof of the rule is as follows. Given $f(x) = k$, we have $f(N) = k$ for any value of N. Thus the value of $f'(N)$—the value of the derivative at $x = N$—as defined in (6.13) will be

$$f'(N) = \lim_{x \to N} \frac{f(x) - f(N)}{x - N} = \lim_{x \to N} \frac{k - k}{x - N} = \lim_{x \to N} 0 = 0$$

Moreover, since N represents any value of x at all, the result $f'(N) = 0$ can be immediately generalized to $f'(x) = 0$. This proves the rule.

It is important to distinguish clearly between the statement $f'(x) = 0$ and the similar-looking but different statement $f'(x_0) = 0$. By $f'(x) = 0$, we mean that the derivative function f' has a zero value for *all* values of x; in writing $f'(x_0) = 0$, on the other hand, we are merely associating the zero value of the derivative with a particular value of x, namely, $x = x_0$.

As discussed before, the derivative of a function has its geometric counterpart in the slope of the curve. The graph of a constant function, say, a fixed-cost function $C_F = f(Q) = \$1200$, is a horizontal straight line with a zero slope throughout. Correspondingly, the derivative must also be zero for all values of Q:

$$\frac{d}{dQ}C_F = \frac{d}{dQ}1200 = 0 \qquad \text{or} \qquad f'(Q) = 0$$

Power-Function Rule

The derivative of a power function $y = f(x) = x^n$ is nx^{n-1}. Symbolically, this is expressed as

$$\text{(7.1)} \qquad \frac{d}{dx}x^n = nx^{n-1} \qquad \text{or} \qquad f'(x) = nx^{n-1}$$

Example 1 The derivative of $y = x^3$ is $\dfrac{dy}{dx} = \dfrac{d}{dx}x^3 = 3x^2$.

Example 2 The derivative of $y = x^9$ is $\dfrac{d}{dx}x^9 = 9x^8$.

This rule is valid for any real-valued power of x; that is, the exponent can be any real number. But we shall prove it only for the case where n is some positive integer. In the simplest case, that of $n = 1$, the function is $f(x) = x$, and according to the rule, the derivative is

$$f'(x) = \frac{d}{dx}x = 1(x^0) = 1$$

The proof of this result follows easily from the definition of $f'(N)$ in (6.14'). Given $f(x) = x$, the derivative value at any value of x, say, $x = N$, is

$$f'(N) = \lim_{x \to N} \frac{f(x) - f(N)}{x - N} = \lim_{x \to N} \frac{x - N}{x - N} = \lim_{x \to N} 1 = 1$$

Since N represents any value of x, it is permissible to write $f'(x) = 1$. This proves the rule for the case of $n = 1$. As the graphical counterpart of this result, we see that the function $y = f(x) = x$ plots as a 45° line, and it has a slope of $+1$ throughout.

For the cases of larger integers, $n = 2, 3, \ldots$, let us first note the following identities:

$$\frac{x^2 - N^2}{x - N} = x + N \qquad \text{[2 terms on the right]}$$

$$\frac{x^3 - N^3}{x - N} = x^2 + Nx + N^2 \qquad \text{[3 terms on the right]}$$

$$\vdots$$

(7.2) $$\frac{x^n - N^n}{x - N} = x^{n-1} + Nx^{n-2} + N^2x^{n-3} + \cdots + N^{n-1}$$

[n terms on the right]

On the basis of (7.2), we can express the derivative of a power function $f(x) = x^n$ at $x = N$ as follows:

(7.3) $$\begin{aligned} f'(N) &= \lim_{x \to N} \frac{f(x) - f(N)}{x - N} = \lim_{x \to N} \frac{x^n - N^n}{x - N} \\ &= \lim_{x \to N} \left(x^{n-1} + Nx^{n-2} + \cdots + N^{n-1}\right) \qquad \text{[by (7.2)]} \\ &= \lim_{x \to N} x^{n-1} + \lim_{x \to N} Nx^{n-2} + \cdots + \lim_{x \to N} N^{n-1} \\ & \qquad \text{[sum limit theorem]} \\ &= N^{n-1} + N^{n-1} + \cdots + N^{n-1} \qquad \text{[a total of } n \text{ terms]} \\ &= nN^{n-1} \end{aligned}$$

Again, N is any value of x; thus this last result can be generalized to

$$f'(x) = nx^{n-1}$$

which proves the rule for n, any positive integer.

As mentioned above, this rule applies even when the exponent n in the power expression x^n is not a positive integer. The following examples serve to illustrate its application to the latter cases.

Example 3 Find the derivative of $y = x^0$. Applying (7.1), we find

$$\frac{d}{dx}x^0 = 0(x^{-1}) = 0$$

Example 4 Find the derivative of $y = 1/x^3$. This involves the reciprocal of a power, but by rewriting the function as $y = x^{-3}$, we can again apply (7.1) to get the derivative:

$$\frac{d}{dx}x^{-3} = -3x^{-4} \qquad \left[= \frac{-3}{x^4}\right]$$

Example 5 Find the derivative of $y = \sqrt{x}$. A square root is involved in this case, but since $\sqrt{x} = x^{1/2}$, the derivative can be found as follows:

$$\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} \qquad \left[= \frac{1}{2\sqrt{x}}\right]$$

Derivatives are themselves functions of the independent variable x. In Example 1, for instance, the derivative is $dy/dx = 3x^2$, or $f'(x) = 3x^2$, so that a different value of x will result in a different value of the derivative, such as

$$f'(1) = 3(1)^2 = 3 \qquad f'(2) = 3(2)^2 = 12$$

These specific values of the derivative can be expressed alternatively as

$$\left.\frac{dy}{dx}\right|_{x=1} = 3 \qquad \left.\frac{dy}{dx}\right|_{x=2} = 12$$

but the notations $f'(1)$ and $f'(2)$ are obviously preferable because of their simplicity.

It is of the utmost importance to realize that, to find the derivative values $f'(1)$, $f'(2)$, etc., we must *first* differentiate the function $f(x)$, in order to get the derivative function $f'(x)$, and *then* let x assume specific values in $f'(x)$. To substitute specific values of x into the primitive function $f(x)$ prior to differentiation is definitely not permissible. As an illustration, if we let $x = 1$ in the function of Example 1 before differentiation, the function will degenerate into $y = x = 1$ —a constant function—which will yield a zero derivative rather than the correct answer of $f'(x) = 3x^2$.

Power-Function Rule Generalized

When a multiplicative constant c appears in the power function, so that $f(x) = cx^n$, its derivative is

$$\frac{d}{dx}cx^n = cnx^{n-1} \qquad \text{or} \qquad f'(x) = cnx^{n-1}$$

This result shows that, in differentiating cx^n, we can simply retain the multiplicative constant c intact and then differentiate the term x^n according to (7.1).

Example 6 Given $y = 2x$, we have $dy/dx = 2x^0 = 2$.

Example 7 Given $f(x) = 4x^3$, the derivative is $f'(x) = 12x^2$.

Example 8 The derivative of $f(x) = 3x^{-2}$ is $f'(x) = -6x^{-3}$.

For a proof of this new rule, consider the fact that for any value of x, say, $x = N$, the value of the derivative of $f(x) = cx^n$ is

$$f'(N) = \lim_{x \to N} \frac{f(x) - f(N)}{x - N} = \lim_{x \to N} \frac{cx^n - cN^n}{x - N} = \lim_{x \to N} c\left(\frac{x^n - N^n}{x - N}\right)$$

$$= \lim_{x \to N} c \lim_{x \to N} \frac{x^n - N^n}{x - N} \qquad \text{[product limit theorem]}$$

$$= c \lim_{x \to N} \frac{x^n - N^n}{x - N} \qquad \text{[limit of a constant]}$$

$$= cnN^{n-1} \qquad \text{[from (7.3)]}$$

In view that N is any value of x, this last result can be generalized immediately to $f'(x) = cnx^{n-1}$, which proves the rule.

EXERCISE 7.1

1 Find the derivative of each of the following functions:
(a) $y = x^{13}$ (c) $y = 7x^6$ (e) $w = -4u^{1/2}$
(b) $y = 63$ (d) $w = 3u^{-1}$

2 Find the following:
(a) $\frac{d}{dx}(-x^{-4})$ (c) $\frac{d}{dw}9w^4$ (e) $\frac{d}{du}au^b$
(b) $\frac{d}{dx}7x^{1/3}$ (d) $\frac{d}{dx}cx^2$

3 Find $f'(1)$ and $f'(2)$ from the following functions:
(a) $y = f(x) = 18x$ (c) $f(x) = -5x^{-2}$ (e) $f(w) = 6w^{1/3}$
(b) $y = f(x) = cx^3$ (d) $f(x) = \frac{3}{4}x^{4/3}$

4 Graph a function $f(x)$ that gives rise to the derivative function $f'(x) = 0$. Then graph a function $g(x)$ characterized by $f'(x_0) = 0$.

7.2 RULES OF DIFFERENTIATION INVOLVING TWO OR MORE FUNCTIONS OF THE SAME VARIABLE

The three rules presented in the preceding section are each concerned with a single given function $f(x)$. Now suppose that we have two *differentiable* functions of the same variable x, say, $f(x)$ and $g(x)$, and we want to differentiate the sum, difference, product, or quotient formed with these two functions. In such circumstances, are there appropriate rules that apply? More concretely, given two functions—say, $f(x) = 3x^2$ and $g(x) = 9x^{12}$—how do we get the derivative of, say, $3x^2 + 9x^{12}$, or the derivative of $(3x^2)(9x^{12})$?

Sum-Difference Rule

The derivative of a sum (difference) of two functions is the sum (difference) of the derivatives of the two functions:

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) = f'(x) \pm g'(x)$$

The proof of this again involves the application of the definition of a derivative and of the various limit theorems. We shall omit the proof and, instead, merely verify its validity and illustrate its application.

Example 1 From the function $y = 14x^3$, we can obtain the derivative $dy/dx = 42x^2$. But $14x^3 = 5x^3 + 9x^3$, so that y may be regarded as the sum of two functions $f(x) = 5x^3$ and $g(x) = 9x^3$. According to the sum rule, we then have

$$\frac{dy}{dx} = \frac{d}{dx}(5x^3 + 9x^3) = \frac{d}{dx}5x^3 + \frac{d}{dx}9x^3 = 15x^2 + 27x^2 = 42x^2$$

which is identical with our earlier result.

This rule, stated above in terms of two functions, can easily be extended to more functions. Thus, it is also valid to write

$$\frac{d}{dx}[f(x) \pm g(x) \pm h(x)] = f'(x) \pm g'(x) \pm h'(x)$$

Example 2 The function cited in Example 1, $y = 14x^3$, can be written as $y = 2x^3 + 13x^3 - x^3$. The derivative of the latter, according to the sum-difference rule, is

$$\frac{dy}{dx} = \frac{d}{dx}(2x^3 + 13x^3 - x^3) = 6x^2 + 39x^2 - 3x^2 = 42x^2$$

which again checks with the previous answer.

This rule is of great practical importance. With it at our disposal, it is now possible to find the derivative of any polynomial function, since the latter is nothing but a sum of power functions.

Example 3 $\frac{d}{dx}(ax^2 + bx + c) = 2ax + b$

Example 4

$$\frac{d}{dx}(7x^4 + 2x^3 - 3x + 37) = 28x^3 + 6x^2 - 3 + 0 = 28x^3 + 6x^2 - 3$$

Note that in the last two examples the constants c and 37 do not really produce any effect on the derivative, because the derivative of a constant term is zero. In contrast to the *multiplicative* constant, which is retained during differentiation, the *additive* constant drops out. This fact provides the mathematical explanation of the well-known economic principle that the fixed cost of a firm does not affect its marginal cost. Given a short-run total-cost function

$$C = Q^3 - 4Q^2 + 10Q + 75$$

the marginal-cost function (for infinitesimal output change) is the limit of the quotient $\Delta C/\Delta Q$, or the derivative of the C function:

$$\frac{dC}{dQ} = 3Q^2 - 8Q + 10$$

whereas the fixed cost is represented by the additive constant 75. Since the latter drops out during the process of deriving dC/dQ, the magnitude of the fixed cost obviously cannot affect the marginal cost.

In general, if a primitive function $y = f(x)$ represents a *total* function, then the derivative function dy/dx is its *marginal* function. Both functions can, of course, be plotted against the variable x graphically; and because of the correspondence between the derivative of a function and the slope of its curve, for each value of x the marginal function should show the slope of the total function at that value of x. In Fig. 7.1*a*, a linear (constant-slope) total function is seen to have a constant marginal function. On the other hand, the nonlinear (varying-slope) total function in Fig. 7.1*b* gives rise to a curved marginal function, which lies below (above) the horizontal axis when the total function is negatively (positively) sloped. And, finally, the reader may note from Fig. 7.1*c* (cf. Fig. 6.5) that "nonsmoothness" of a total function will result in a gap (discontinuity) in the marginal or derivative function. This is in sharp contrast to the everywhere-smooth total function in Fig. 7.1*b* which gives rise to a continuous marginal function. For this reason, the *smoothness* of a *primitive* function can be linked to the *continuity* of its *derivative* function. In particular, instead of saying that a certain function is smooth (and differentiable) everywhere, we may alternatively characterize it as a function with a continuous derivative function, and refer to it as a *continuously differentiable* function.

Product Rule

The derivative of the product of two (differentiable) functions is equal to the first function times the derivative of the second function plus the second function

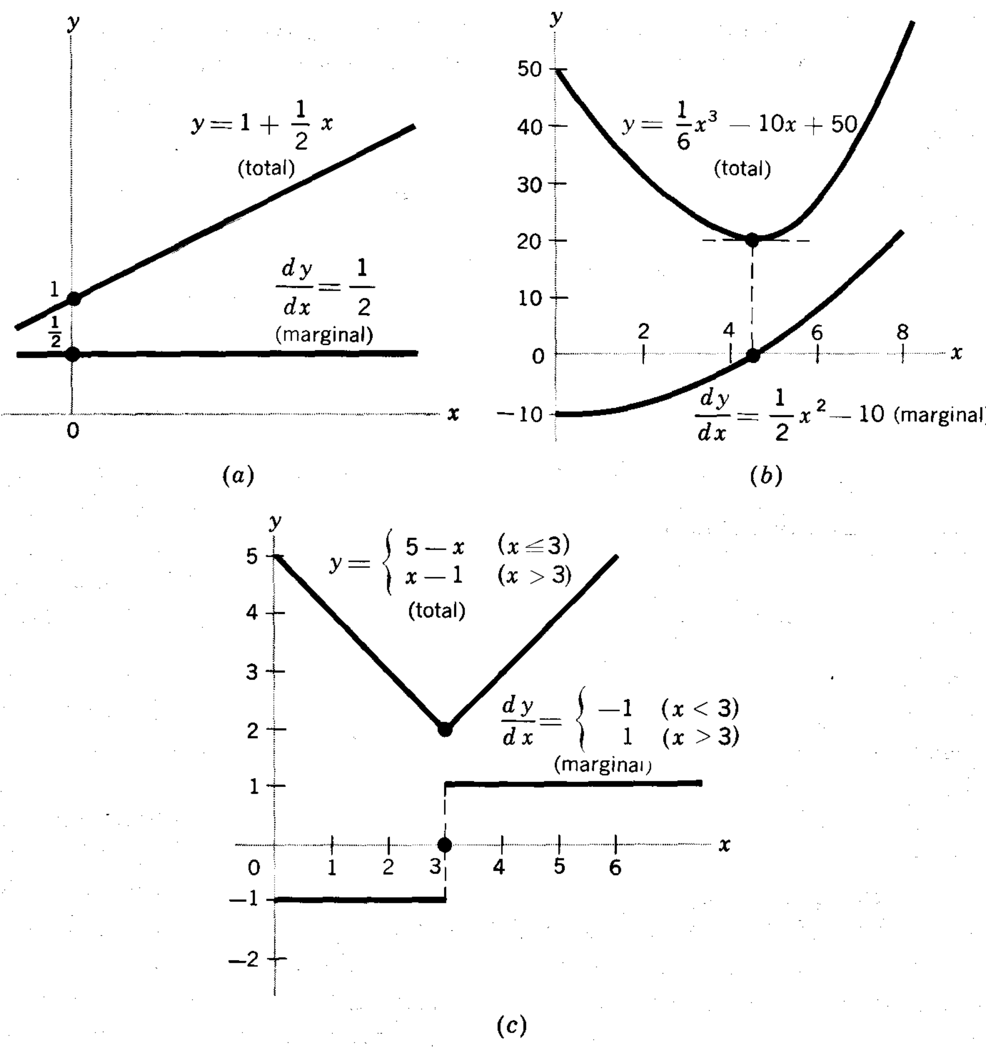

Figure 7.1

times the derivative of the first function:

$$\text{(7.4)} \qquad \frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$
$$= f(x)g'(x) + g(x)f'(x)$$

Example 5 Find the derivative of $y = (2x + 3)(3x^2)$. Let $f(x) = 2x + 3$ and $g(x) = 3x^2$. Then it follows that $f'(x) = 2$ and $g'(x) = 6x$, and according to (7.4) the desired derivative is

$$\frac{d}{dx}[(2x + 3)(3x^2)] = (2x + 3)(6x) + (3x^2)(2) = 18x^2 + 18x$$

This result can be checked by first multiplying out $f(x)g(x)$ and then taking the derivative of the product polynomial. The product polynomial is in this case $f(x)g(x) = (2x + 3)(3x^2) = 6x^3 + 9x^2$, and direct differentiation does yield the same derivative, $18x^2 + 18x$.

The important point to remember is that the derivative of a product of two functions is *not* the simple product of the two separate derivatives. Since this differs from what intuitive generalization leads one to expect, let us produce a proof for (7.4). According to (6.13), the value of the derivative of $f(x)g(x)$ when $x = N$ should be

$$\frac{d}{dx}[f(x)g(x)]\Big|_{x=N} = \lim_{x \to N} \frac{f(x)g(x) - f(N)g(N)}{x - N} \tag{7.5}$$

But, by adding *and* subtracting $f(x)g(N)$ in the numerator (thereby leaving the original magnitude unchanged), we can transform the quotient on the right of (7.5) as follows:

$$\frac{f(x)g(x) - f(x)g(N) + f(x)g(N) - f(N)g(N)}{x - N}$$

$$= f(x)\frac{g(x) - g(N)}{x - N} + g(N)\frac{f(x) - f(N)}{x - N}$$

Substituting this for the quotient on the right of (7.5) and taking its limit, we then get

$$\frac{d}{dx}[f(x)g(x)]\Big|_{x=N} = \lim_{x \to N} f(x) \lim_{x \to N} \frac{g(x) - g(N)}{x - N} + \lim_{x \to N} g(N) \lim_{x \to N} \frac{f(x) - f(N)}{x - N} \tag{7.5'}$$

The four limit expressions in (7.5′) are easily evaluated. The first one is $f(N)$, and the third is $g(N)$ (limit of a constant). The remaining two are, according to (6.13), respectively, $g'(N)$ and $f'(N)$. Thus (7.5′) reduces to

$$\frac{d}{dx}[f(x)g(x)]\Big|_{x=N} = f(N)g'(N) + g(N)f'(N) \tag{7.5''}$$

And, since N represents any value of x, (7.5″) remains valid if we replace every N symbol by x. This proves the rule.

As an extension of the rule to the case of *three* functions, we have

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x) \tag{7.6}$$

In words, the derivative of the product of three functions is equal to the product of the second and third functions times the derivative of the first, plus the product of the first and third functions times the derivative of the second, plus the product of the first and second functions times the derivative of the third. This result can be derived by the repeated application of (7.4). First treat the product $g(x)h(x)$ as a single function, say, $\phi(x)$, so that the original product of three functions will become a product of *two* functions, $f(x)\phi(x)$. To this, (7.4) is applicable. After the derivative of $f(x)\phi(x)$ is obtained, we may reapply (7.4) to the product $g(x)h(x) \equiv \phi(x)$ to get $\phi'(x)$. Then (7.6) will follow. The details are left to you as an exercise.

The validity of a rule is one thing; its serviceability is something else. Why do we need the product rule when we can resort to the alternative procedure of multiplying out the two functions $f(x)$ and $g(x)$ and then taking the derivative of the product directly? One answer to that question is that the alternative procedure is applicable only to *specific* (numerical or parametric) functions, whereas the product rule is applicable even when the functions are given in the *general* form. Let us illustrate with an economic example.

Finding Marginal-Revenue Function from Average-Revenue Function

If we are given an average-revenue (AR) function in specific form,

$$\text{AR} = 15 - Q$$

the marginal-revenue (MR) function can be found by first multiplying AR by Q to get the total-revenue (R) function:

$$R \equiv \text{AR} \cdot Q = (15 - Q)Q = 15Q - Q^2$$

and then differentiating R:

$$\text{MR} \equiv \frac{dR}{dQ} = 15 - 2Q$$

But if the AR function is given in the general form $\text{AR} = f(Q)$, then the total-revenue function will also be in a general form:

$$R \equiv \text{AR} \cdot Q = f(Q) \cdot Q$$

and therefore the "multiply out" approach will be to no avail. However, because R is a product of two functions of Q, namely, $f(Q)$ and Q itself, the product rule may be put to work. Thus we can differentiate R to get the MR function as follows:

$$\text{(7.7)} \qquad \text{MR} \equiv \frac{dR}{dQ} = f(Q) \cdot 1 + Q \cdot f'(Q) = f(Q) + Qf'(Q)$$

However, can such a general result tell us anything significant about the MR? Indeed it can. Recalling that $f(Q)$ denotes the AR function, let us rearrange (7.7) and write

$$\text{(7.7')} \qquad \text{MR} - \text{AR} = \text{MR} - f(Q) = Qf'(Q)$$

This gives us an important relationship between MR and AR: namely, they will always differ by the amount $Qf'(Q)$.

It remains to examine the expression $Qf'(Q)$. Its first component Q denotes output and is always nonnegative. The other component, $f'(Q)$, represents the slope of the AR curve plotted against Q. Since "average revenue" and "price" are but different names for the same thing:

$$\text{AR} \equiv \frac{R}{Q} \equiv \frac{PQ}{Q} \equiv P$$

the AR curve can also be regarded as a curve relating price P to output Q: $P = f(Q)$. Viewed in this light, the AR curve is simply the *inverse* of the demand curve for the product of the firm, i.e., the demand curve plotted after the P and Q axes are reversed. Under pure competition, the AR curve is a horizontal straight line, so that $f'(Q) = 0$ and, from (7.7′), MR − AR = 0 for all possible values of Q. Thus the MR curve and the AR curve must coincide. Under imperfect competition, on the other hand, the AR curve is normally downward-sloping, as in Fig. 7.2, so that $f'(Q) < 0$ and, from (7.7′), MR − AR < 0 for all positive levels of output. In this case, the MR curve must lie below the AR curve.

The conclusion just stated is *qualitative* in nature; it concerns only the relative positions of the two curves. But (7.7′) also furnishes the *quantitative* information that the MR curve will fall short of the AR curve at any output level Q by precisely the amount $Qf'(Q)$. Let us look at Fig. 7.2 again and consider the particular output level N. For that output, the expression $Qf'(Q)$ specifically becomes $Nf'(N)$; if we can find the magnitude of $Nf'(N)$ in the diagram, we shall know how far below the average-revenue point G the corresponding marginal-revenue point must lie.

The magnitude of N is already specified. And $f'(N)$ is simply the slope of the AR curve at point G (where $Q = N$), that is, the slope of the tangent line JM measured by the ratio of two distances OJ/OM. However, we see that $OJ/OM =$

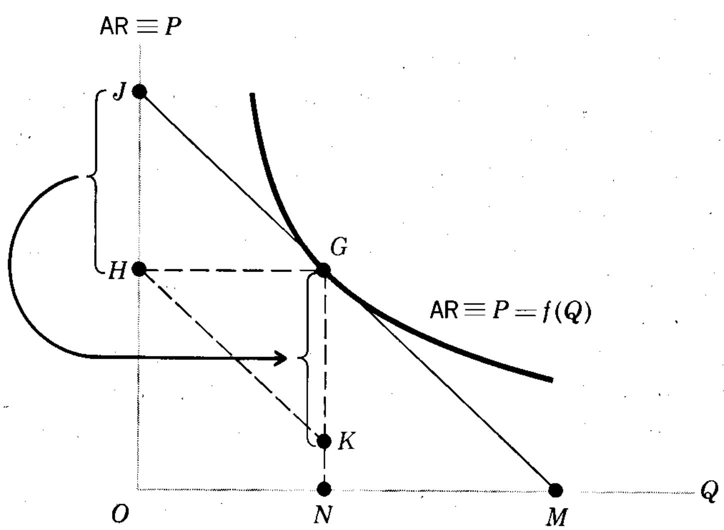

Figure 7.2

HJ/HG; besides, distance HG is precisely the amount of output under consideration, N. Thus the distance $Nf'(N)$, by which the MR curve must lie below the AR curve at output N, is

$$Nf'(N) = HG\frac{HJ}{HG} = HJ$$

Accordingly, if we mark a vertical distance $KG = HJ$ directly below point G, then point K must be a point on the MR curve. (A simple way of accurately plotting KG is to draw a straight line passing through point H and parallel to JG; point K is where that line intersects the vertical line NG.)

The same procedure can be used to locate other points on the MR curve. All we must do, for any chosen point G' on the curve, is first to draw a tangent to the AR curve at G' that will meet the vertical axis at some point J'. Then draw a horizontal line from G' to the vertical axis, and label the intersection with the axis as H'. If we mark a vertical distance $K'G' = H'J'$ directly below point G', then the point K' will be a point on the MR curve. This is the graphical way of deriving an MR curve from a given AR curve. Strictly speaking, the accurate drawing of a tangent line requires a knowledge of the value of the derivative at the relevant output, that is, $f'(N)$; hence the graphical method just outlined cannot quite exist by itself. An important exception is the case of a linear AR curve, where the tangent to any point on the curve is simply the given line itself, so that there is in effect no need to draw any tangent at all. Then the above graphical method will apply in a straightforward way.

Quotient Rule

The derivative of the quotient of two functions, $f(x)/g(x)$, is

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

In the numerator of the right-hand expression, we find two product terms, each involving the derivative of only one of the two original functions. Note that $f'(x)$ appears in the positive term, and $g'(x)$ in the negative term. The denominator consists of the square of the function $g(x)$; that is, $g^2(x) \equiv [g(x)]^2$.

Example 6 $$\frac{d}{dx}\left(\frac{2x-3}{x+1}\right) = \frac{2(x+1)-(2x-3)(1)}{(x+1)^2} = \frac{5}{(x+1)^2}$$

Example 7 $$\frac{d}{dx}\left(\frac{5x}{x^2+1}\right) = \frac{5(x^2+1)-5x(2x)}{(x^2+1)^2} = \frac{5(1-x^2)}{(x^2+1)^2}$$

Example 8 $$\frac{d}{dx}\left(\frac{ax^2+b}{cx}\right) = \frac{2ax(cx)-(ax^2+b)(c)}{(cx)^2} = \frac{c(ax^2-b)}{(cx)^2} = \frac{ax^2-b}{cx^2}$$

This rule can be proved as follows. For any value of $x = N$, we have

$$(7.8) \qquad \frac{d}{dx}\frac{f(x)}{g(x)}\bigg|_{x=N} = \lim_{x \to N} \frac{f(x)/g(x) - f(N)/g(N)}{x - N}$$

The quotient expression following the limit sign can be rewritten in the form

$$\frac{f(x)g(N) - f(N)g(x)}{g(x)g(N)} \frac{1}{x - N}$$

By adding *and* subtracting $f(N)g(N)$ in the numerator and rearranging, we can further transform the expression to

$$\frac{1}{g(x)g(N)}\left[\frac{f(x)g(N) - f(N)g(N) + f(N)g(N) - f(N)g(x)}{x - N}\right]$$

$$= \frac{1}{g(x)g(N)}\left[g(N)\frac{f(x) - f(N)}{x - N} - f(N)\frac{g(x) - g(N)}{x - N}\right]$$

Substituting this result into (7.8) and taking the limit, we then have

$$\frac{d}{dx}\frac{f(x)}{g(x)}\bigg|_{x=N} = \lim_{x \to N}\frac{1}{g(x)g(N)}\left[\lim_{x \to N} g(N) \lim_{x \to N}\frac{f(x) - f(N)}{x - N} - \lim_{x \to N} f(N) \lim_{x \to N}\frac{g(x) - g(N)}{x - N}\right]$$

$$= \frac{1}{g^2(N)}[g(N)f'(N) - f(N)g'(N)] \qquad [\text{by } (6.13)]$$

which can be generalized by replacing the symbol N with x, because N represents any value of x. This proves the quotient rule.

Relationship Between Marginal-Cost and Average-Cost Functions

As an economic application of the quotient rule, let us consider the rate of change of average cost when output varies.

Given a total-cost function $C = C(Q)$, the average-cost (AC) function will be a quotient of two functions of Q, since $AC \equiv C(Q)/Q$, defined as long as $Q > 0$. Therefore, the rate of change of AC with respect to Q can be found by differentiating AC:

$$(7.9) \qquad \frac{d}{dQ}\frac{C(Q)}{Q} = \frac{[C'(Q) \cdot Q - C(Q) \cdot 1]}{Q^2} = \frac{1}{Q}\left[C'(Q) - \frac{C(Q)}{Q}\right]$$

From this it follows that, for $Q > 0$,

$$(7.10) \qquad \frac{d}{dQ}\frac{C(Q)}{Q} \gtreqless 0 \quad \text{iff} \quad C'(Q) \gtreqless \frac{C(Q)}{Q}$$

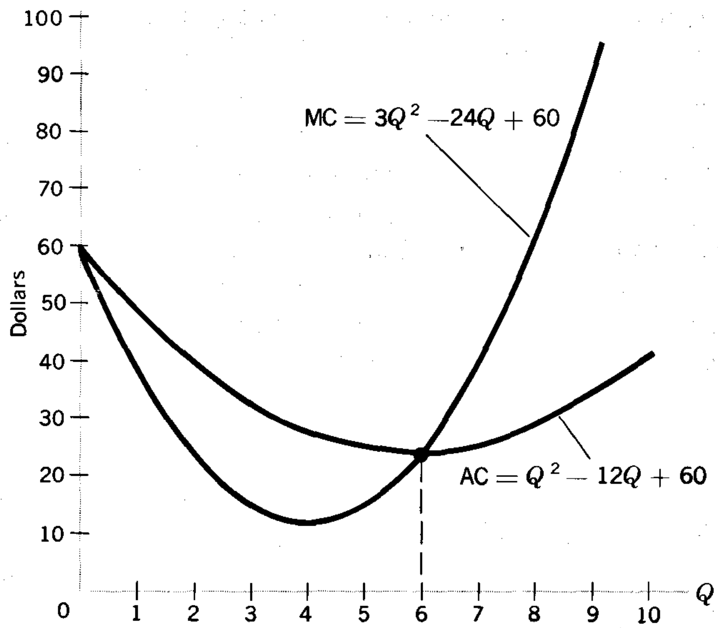

Figure 7.3

Since the derivative $C'(Q)$ represents the marginal-cost (MC) function, and $C(Q)/Q$ represents the AC function, the economic meaning of (7.10) is: The slope of the AC curve will be positive, zero, or negative if and only if the marginal-cost curve lies above, intersects, or lies below the AC curve. This is illustrated in Fig. 7.3, where the MC and AC functions plotted are based on the specific total-cost function

$$C = Q^3 - 12Q^2 + 60Q$$

To the left of $Q = 6$, AC is declining, and thus MC lies below it; to the right, the opposite is true. At $Q = 6$, AC has a slope of zero, and MC and AC have the same value.*

The qualitative conclusion in (7.10) is stated explicitly in terms of cost functions. However, its validity remains unaffected if we interpret $C(Q)$ as *any other* differentiable total function, with $C(Q)/Q$ and $C'(Q)$ as its corresponding average and marginal functions. Thus this result gives us a *general* marginal-average relationship. In particular, we may point out, the fact that MR lies below AR when AR is downward-sloping, as discussed in connection with Fig. 7.2, is nothing but a special case of the general result in (7.10).

* Note that (7.10) does *not* state that, when AC is negatively sloped, MC must also be negatively sloped; it merely says that AC must exceed MC in that circumstance. At $Q = 5$ in Fig. 7.3, for instance, AC is declining but MC is rising, so that their slopes will have opposite signs.

EXERCISE 7.2

1 Given the total-cost function $C = Q^3 - 5Q^2 + 14Q + 75$, write out a variable-cost (VC) function. Find the derivative of the VC function, and interpret the economic meaning of that derivative.

2 Given the average-cost function $\text{AC} = Q^2 - 4Q + 214$, find the MC function. Is the given function more appropriate as a long-run or a short-run function? Why?

3 Differentiate the following by using the product rule:

(*a*) $(9x^2 - 2)(3x + 1)$ (*d*) $(ax - b)(cx^2)$
(*b*) $(3x + 11)(6x^2 - 5x)$ (*e*) $(2 - 3x)(1 + x)(x + 2)$
(*c*) $x^2(4x + 6)$ (*f*) $(x^2 + 3)x^{-1}$

4 (*a*) Given $\text{AR} = 60 - 3Q$, plot the average-revenue curve, and then find the MR curve by the method used in Fig. 7.2.

(*b*) Find the total-revenue function and the marginal-revenue function mathematically from the given AR function.

(*c*) Does the graphically derived MR curve in (*a*) check with the mathematically derived MR function in (*b*)?

(*d*) Comparing the AR and MR functions, what can you conclude about their relative slopes?

5 Provide a mathematical proof for the general result that, given a *linear* average curve, the corresponding marginal curve must have the same vertical intercept but will be twice as steep as the average curve.

6 Prove the result in (7.6) by first treating $g(x)h(x)$ as a single function, $g(x)h(x) \equiv \phi(x)$, and then applying the product rule (7.4).

7 Find the derivatives of:

(*a*) $(x^2 + 3)/x$ (*c*) $4x/(x + 5)$
(*b*) $(x + 7)/x$ (*d*) $(ax^2 + b)/(cx + d)$

8 Given the function $f(x) = ax + b$, find the derivatives of:

(a) $f(x)$ (*b*) $xf(x)$ (*c*) $1/f(x)$ (*d*) $f(x)/x$

7.3 RULES OF DIFFERENTIATION INVOLVING FUNCTIONS OF DIFFERENT VARIABLES

In the preceding section, we discussed the rules of differentiation of a sum, difference, product, or quotient of two (or more) differentiable functions of the same variable. Now we shall consider cases where there are two or more differentiable functions, each of which has a *distinct* independent variable.

Chain Rule

If we have a function $z = f(y)$, where y is in turn a function of another variable x, say, $y = g(x)$, then the derivative of z with respect to x is equal to the derivative of z with respect to y, times the derivative of y with respect to x. Expressed symbolically,

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = f'(y)g'(x) \tag{7.11}$$

This rule, known as the *chain rule*, appeals easily to intuition. Given a Δx, there must result a corresponding Δy via the function $y = g(x)$, but this Δy will in turn bring about a Δz via the function $z = f(y)$. Thus there is a "chain reaction" as follows:

$$\Delta x \overset{\text{via } g}{\rightarrow} \Delta y \overset{\text{via } f}{\rightarrow} \Delta z$$

The two links in this chain entail two difference quotients, $\Delta y/\Delta x$ and $\Delta z/\Delta y$, but when they are multiplied, the Δy will cancel itself out, and we end up with

$$\frac{\Delta z}{\Delta y}\frac{\Delta y}{\Delta x} = \frac{\Delta z}{\Delta x}$$

a difference quotient that relates Δz to Δx. If we take the limit of these difference quotients as $\Delta x \to 0$ (which implies $\Delta y \to 0$), each difference quotient will turn into a derivative; i.e., we shall have $(dz/dy)(dy/dx) = dz/dx$. This is precisely the result in (7.11).

In view of the function $y = g(x)$, we can express the function $z = f(y)$ as $z = f[g(x)]$, where the contiguous appearance of the two function symbols f and g indicates that this is a *composite function* (function of a function). It is for this reason that the chain rule is also referred to as the *composite-function rule* or *function-of-a-function rule*.

The extension of the chain rule to three or more functions is straightforward. If we have $z = f(y)$, $y = g(x)$, and $x = h(w)$, then

$$\frac{dz}{dw} = \frac{dz}{dy}\frac{dy}{dx}\frac{dx}{dw} = f'(y)g'(x)h'(w)$$

and similarly for cases in which more functions are involved.

Example 1 If $z = 3y^2$, where $y = 2x + 5$, then

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = 6y(2) = 12y = 12(2x + 5)$$

Example 2 If $z = y - 3$, where $y = x^3$, then

$$\frac{dz}{dx} = 1(3x^2) = 3x^2$$

Example 3 The usefulness of this rule can best be appreciated when we must differentiate a function such as $z = (x^2 + 3x - 2)^{17}$. Without the chain rule at our disposal, dz/dx can be found only via the laborious route of first multiplying out the 17th-power expression. With the chain rule, however, we can take a shortcut by defining a new, *intermediate* variable $y = x^2 + 3x - 2$, so that we get in effect two functions linked in a chain:

$$z = y^{17} \qquad \text{and} \qquad y = x^2 + 3x - 2$$

The derivative dz/dx can then be found as follows:

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = 17y^{16}(2x + 3) = 17(x^2 + 3x - 2)^{16}(2x + 3)$$

Example 4 Given a total-revenue function of a firm $R = f(Q)$, where output Q is a function of labor input L, or $Q = g(L)$, find dR/dL. By the chain rule, we have

$$\frac{dR}{dL} = \frac{dR}{dQ}\frac{dQ}{dL} = f'(Q)g'(L)$$

Translated into economic terms, dR/dQ is the MR function and dQ/dL is the marginal-physical-product-of-labor (MPP_L) function. Similarly, dR/dL has the connotation of the marginal-revenue-product-of-labor (MRP_L) function. Thus the result shown above constitutes the mathematical statement of the well-known result in economics that $\text{MRP}_L = \text{MR} \cdot \text{MPP}_L$.

Inverse-Function Rule

If the function $y = f(x)$ represents a one-to-one mapping, i.e., if the function is such that a different value of x will always yield a different value of y, the function f will have an *inverse function* $x = f^{-1}(y)$ (read: "x is an inverse function of y"). Here, the symbol f^{-1} is a function symbol which, like the derivative-function symbol f', signifies a function related to the function f; it does *not* mean the reciprocal of the function $f(x)$.

What the existence of an inverse function essentially means is that, in this case, not only will a given value of x yield a unique value of y [that is, $y = f(x)$], but also a given value of y will yield a unique value of x. To take a nonnumerical instance, we may exemplify the one-to-one mapping by the mapping from the set of all husbands to the set of all wives in a monogamous society. Each husband has a unique wife, and each wife has a unique husband. In contrast, the mapping from the set of all fathers to the set of all sons is not one-to-one, because a father may have more than one son, albeit each son has a unique father.

When x and y refer specifically to numbers, the property of one-to-one mapping is seen to be unique to the class of functions known as *monotonic functions*. Given a function $f(x)$, if successively larger values of the independent variable x *always* lead to successively larger values of $f(x)$, that is, if

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) > f(x_2)$$

then the function f is said to be an *increasing* (or *monotonically increasing*) function.* If successive increases in x *always* lead to successive decreases in $f(x)$, that is, if

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) < f(x_2)$$

on the other hand, the function is said to be a *decreasing* (or *monotonically decreasing*) function. In either of these cases, an inverse function f^{-1} exists.

A practical way of ascertaining the monotonicity of a given function $y = f(x)$ is to check whether the derivative $f'(x)$ always adheres to the same algebraic sign (not zero) for all values of x. Geometrically, this means that its slope is either always upward or always downward. Thus a firm's demand curve $Q = f(P)$ that has a negative slope throughout is monotonic. As such, it has an inverse function $P = f^{-1}(Q)$, which, as mentioned previously, gives the average-revenue curve of the firm, since $P \equiv \text{AR}$.

Example 5 The function

$$y = 5x + 25$$

has the derivative $dy/dx = 5$, which is positive regardless of the value of x; thus the function is monotonic. (In this case it is increasing, because the derivative is positive.) It follows that an inverse function exists. In the present case, the inverse function is easily found by solving the given equation $y = 5x + 25$ for x. The result is the function

$$x = \tfrac{1}{5}y - 5$$

It is interesting to note that this inverse function is also monotonic, and increasing, because $dx/dy = \frac{1}{5} > 0$ for all values of y.

Generally speaking, if an inverse function exists, the original and the inverse functions must both be monotonic. Moreover, if f^{-1} is the inverse function of f, then f must be the inverse function of f^{-1}; that is, f and f^{-1} must be inverse functions of each other.

It is easy to verify that the graph of $y = f(x)$ and that of $x = f^{-1}(y)$ are one and the same, only with the axes reversed. If one lays the x axis of the f^{-1} graph over the x axis of the f graph (and similarly for the y axis), the two curves will coincide. On the other hand, if the x axis of the f^{-1} graph is laid over the y axis of

* Some writers prefer to define an *increasing function* as a function with the property that

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) \geq f(x_2) \qquad \text{[with a weak inequality]}$$

and then reserve the term *strictly increasing function* for the case where

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) > f(x_2) \qquad \text{[with a strict inequality]}$$

Under this usage, an ascending step function qualifies as an increasing (though not strictly increasing) function, despite the fact that its graph contains horizontal segments. We shall not follow this usage in the present book. Instead, we shall consider an ascending step function to be, not an increasing function, but a *nondecreasing* one. By the same token, we shall regard a descending step function not as a decreasing function, but as a *nonincreasing* one.

the f graph (and vice versa), the two curves will become *mirror images* of each other with reference to the 45° line drawn through the origin. This mirror-image relationship provides us with an easy way of graphing the inverse function f^{-1}, once the graph of the original function f is given. (You should try this with the two functions in Example 5.)

For inverse functions, the rule of differentiation is

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

This means that the derivative of the inverse function is the reciprocal of the derivative of the original function; as such, dx/dy must take the same sign as dy/dx, so that if f is increasing (decreasing), then so must be f^{-1}.

As a verification of this rule, we can refer back to Example 5, where dy/dx was found to be 5, and dx/dy equal to $\frac{1}{5}$. These two derivatives are indeed reciprocal to each other and have the same sign.

In that simple example, the inverse function is relatively easy to obtain, so that its derivative dx/dy can be found directly from the inverse function. As the next example shows, however, the inverse function is sometimes difficult to express explicitly, and thus direct differentiation may not be practicable. The usefulness of the inverse-function rule then becomes more fully apparent.

Example 6 Given $y = x^5 + x$, find dx/dy. First of all, since

$$\frac{dy}{dx} = 5x^4 + 1 > 0$$

for any value of x, the given function is monotonically increasing, and an inverse function exists. To solve the given equation for x may not be such an easy task, but the derivative of the inverse function can nevertheless be found quickly by use of the inverse-function rule:

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{5x^4 + 1}$$

The inverse-function rule is, strictly speaking, applicable only when the function involved is a one-to-one mapping. In fact, however, we do have some leeway. For instance, when dealing with a U-shaped curve (not monotonic), we may consider the downward- and the upward-sloping segments of the curve as representing two *separate* functions, each with a restricted domain, and each being monotonic in the restricted domain. To each of these, the inverse-function rule can then again be applied.

EXERCISE 7.3

1 Given $y = u^3 + 1$, where $u = 5 - x^2$, find dy/dx by the chain rule.

2 Given $w = ay^2$ and $y = bx^2 + cx$, find dw/dx by the chain rule.

3 Use the chain rule to find dy/dx for the following:
(a) $y = (3x^2 - 13)^3$ (b) $y = (8x^3 - 5)^9$ (c) $y = (ax + b)^4$

4 Given $y = (16x + 3)^{-2}$, use the chain rule to find dy/dx. Then rewrite the function as $y = 1/(16x + 3)^2$ and find dy/dx by the quotient rule. Are the answers identical?

5 Given $y = 7x + 21$, find its inverse function. Then find dy/dx and dx/dy, and verify the inverse-function rule. Also verify that the graphs of the two functions bear a mirror-image relationship to each other.

6 Are the following functions monotonic?
(a) $y = -x^6 + 5 \quad (x > 0)$ (b) $y = 4x^5 + x^3 + 3x$
For each monotonic function, find dx/dy by the inverse-function rule.

7.4 PARTIAL DIFFERENTIATION

Hitherto, we have considered only the derivatives of functions of a single independent variable. In comparative-static analysis, however, we are likely to encounter the situation in which several parameters appear in a model, so that the equilibrium value of each endogenous variable may be a function of more than one parameter. Therefore, as a final preparation for the application of the concept of derivative to comparative statics, we must learn how to find the derivative of a function of more than one variable.

Partial Derivatives

Let us consider a function

$$y = f(x_1, x_2, \ldots, x_n) \tag{7.12}$$

where the variables x_i $(i = 1, 2, \ldots, n)$ are all *independent* of one another, so that each can vary by itself without affecting the others. If the variable x_1 undergoes a change Δx_1 while $x_2, \ldots, x_n$ all remain fixed, there will be a corresponding change in y, namely, Δy. The difference quotient in this case can be expressed as

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_1} \tag{7.13}$$

If we take the limit of $\Delta y/\Delta x_1$ as $\Delta x_1 \to 0$, that limit will constitute a derivative. We call it the *partial derivative* of y with respect to x_1, to indicate that all the other independent variables in the function are held constant when taking this particular derivative. Similar partial derivatives can be defined for infinitesimal changes in the other independent variables. The process of taking partial derivatives is called *partial differentiation*.

Partial derivatives are assigned distinctive symbols. In lieu of the letter d (as in dy/dx), we employ the symbol ∂, which is a variant of the Greek δ (lower case delta). Thus we shall now write $\partial y/\partial x_i$, which is read: "the partial derivative of y

with respect to x_i." The partial-derivative symbol sometimes is also written as $\frac{\partial}{\partial x_i} y$; in that case, its $\partial/\partial x_i$ part can be regarded as an operator symbol instructing us to take the partial derivative of (some function) with respect to the variable x_i. Since the function involved here is denoted in (7.12) by f, it is also permissible to write $\partial f/\partial x_i$.

Is there also a partial-derivative counterpart for the symbol $f'(x)$ that we used before? The answer is yes. Instead of f', however, we now use f_1, f_2, etc., where the subscript indicates which independent variable (alone) is being allowed to vary. If the function in (7.12) happens to be written in terms of unsubscripted variables, such as $y = f(u, v, w)$, then the partial derivatives may be denoted by f_u, f_v, and f_w rather than f_1, f_2, and f_3.

In line with these notations, and on the basis of (7.12) and (7.13), we can now define

$$f_1 \equiv \frac{\partial y}{\partial x_1} \equiv \lim_{\Delta x_1 \to 0} \frac{\Delta y}{\Delta x_1}$$

as the first in the set of n partial derivatives of the function f.

Techniques of Partial Differentiation

Partial differentiation differs from the previously discussed differentiation primarily in that we must hold $(n - 1)$ independent variables *constant* while allowing *one* variable to vary. Inasmuch as we have learned how to handle *constants* in differentiation, the actual differentiation should pose little problem.

Example 1 Given $y = f(x_1, x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$, find the partial derivatives. When finding $\partial y/\partial x_1$ (or f_1), we must bear in mind that x_2 is to be treated as a constant during differentiation. As such, x_2 will drop out in the process if it is an *additive* constant (such as the term $4x_2^2$) but will be retained if it is a *multiplicative* constant (such as in the term x_1x_2). Thus we have

$$\frac{\partial y}{\partial x_1} \equiv f_1 = 6x_1 + x_2$$

Similarly, by treating x_1 as a constant, we find that

$$\frac{\partial y}{\partial x_2} \equiv f_2 = x_1 + 8x_2$$

Note that, like the primitive function f, both partial derivatives are themselves functions of the variables x_1 and x_2. That is, we may write them as two derived functions

$$f_1 = f_1(x_1, x_2) \qquad \text{and} \qquad f_2 = f_2(x_1, x_2)$$

For the point $(x_1, x_2) = (1, 3)$ in the domain of the function f, for example, the partial derivatives will take the following specific values:

$$f_1(1, 3) = 6(1) + 3 = 9 \qquad \text{and} \qquad f_2(1, 3) = 1 + 8(3) = 25$$

Example 2 Given $y = f(u, v) = (u + 4)(3u + 2v)$, the partial derivatives can be found by use of the product rule. By holding v constant, we have

$$f_u = (u + 4)(3) + 1(3u + 2v) = 2(3u + v + 6)$$

Similarly, by holding u constant, we find that

$$f_v = (u + 4)(2) + 0(3u + 2v) = 2(u + 4)$$

When $u = 2$ and $v = 1$, these derivatives will take the following values:

$$f_u(2, 1) = 2(13) = 26 \qquad \text{and} \qquad f_v(2, 1) = 2(6) = 12$$

Example 3 Given $y = (3u - 2v)/(u^2 + 3v)$, the partial derivatives can be found by use of the quotient rule:

$$\frac{\partial y}{\partial u} = \frac{3(u^2 + 3v) - 2u(3u - 2v)}{(u^2 + 3v)^2} = \frac{-3u^2 + 4uv + 9v}{(u^2 + 3v)^2}$$

$$\frac{\partial y}{\partial v} = \frac{-2(u^2 + 3v) - 3(3u - 2v)}{(u^2 + 3v)^2} = \frac{-u(2u + 9)}{(u^2 + 3v)^2}$$

Geometric Interpretation of Partial Derivatives

As a special type of derivative, a partial derivative is a measure of the instantaneous rates of change of some variable, and in that capacity it again has a geometric counterpart in the slope of a particular curve.

Let us consider a production function $Q = Q(K, L)$, where Q, K, and L denote output, capital input, and labor input, respectively. This function is a particular two-variable version of (7.12), with $n = 2$. We can therefore define two partial derivatives $\partial Q/\partial K$ (or Q_K) and $\partial Q/\partial L$ (or Q_L). The partial derivative Q_K relates to the rates of change in output with respect to infinitesimal changes in capital, while labor input is held constant. Thus Q_K symbolizes the marginal-physical-product-of-capital (MPP_K) function. Similarly, the partial derivative Q_L is the mathematical representation of the MPP_L function.

Geometrically, the production function $Q = Q(K, L)$ can be depicted by a *production surface* in a 3-space, such as is shown in Fig. 7.4. The variable Q is plotted vertically, so that for any point (K, L) in the base plane (KL plane), the height of the surface will indicate the output Q. The domain of the function should consist of the entire nonnegative quadrant of the base plane, but for our purposes it is sufficient to consider a subset of it, the rectangle OK_0BL_0. As a consequence, only a small portion of the production surface is shown in the figure.

Let us now hold capital fixed at the level K_0 and consider only variations in the input L. By setting $K = K_0$, all points in our (curtailed) domain become irrelevant except those on the line segment K_0B. By the same token, only the curve K_0CDA (a cross section of the production surface) will be germane to the present discussion. This curve represents a total-physical-product-of-labor (TPP_L)

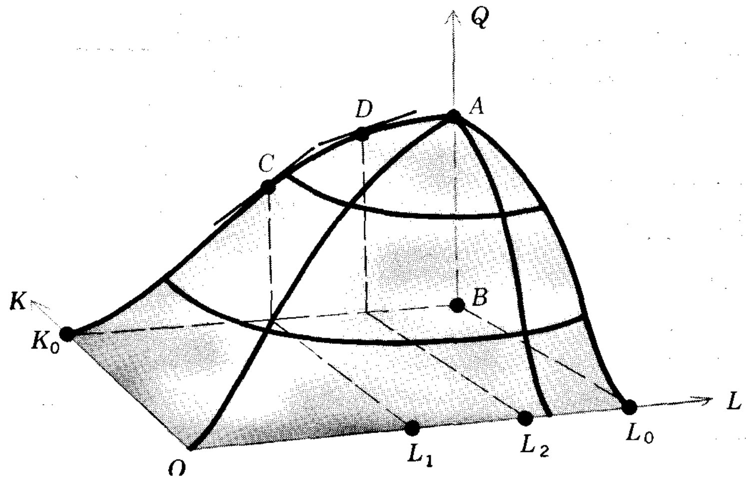

Figure 7.4

curve for a fixed amount of capital $K = K_0$; thus we may read from its slope the rate of change of Q with respect to changes in L while K is held constant. It is clear, therefore, that the slope of a curve such as K_0CDA represents the geometric counterpart of the partial derivative Q_L. Once again, we note that the slope of a total (TPP_L) curve is its corresponding marginal ($\text{MPP}_L \equiv Q_L$) curve.

It was mentioned earlier that a partial derivative is a function of all the independent variables of the primitive function. That Q_L is a function of L is immediately obvious from the K_0CDA curve itself. When $L = L_1$, the value of Q_L is equal to the slope of the curve at point C; but when $L = L_2$, the relevant slope is the one at point D. Why is Q_L also a function of K? The answer is that K can be fixed at various levels, and for each fixed level of K, there will result a different TPP_L curve (a different cross section of the production surface), with inevitable repercussions on the derivative Q_L. Hence Q_L is also a function of K.

An analogous interpretation can be given to the partial derivative Q_K. If the labor input is held constant instead of K (say, at the level of L_0), the line segment L_0B will be the relevant subset of the domain, and the curve L_0A will indicate the relevant subset of the production surface. The partial derivative Q_K can then be interpreted as the slope of the curve L_0A—bearing in mind that the K axis extends from southeast to northwest in Fig. 7.4. It should be noted that Q_K is again a function of both the variables L and K.

EXERCISE 7.4

1 Find $\partial y/\partial x_1$ and $\partial y/\partial x_2$ for each of the following functions:

(*a*) $y = 2x_1^3 - 11x_1^2x_2 + 3x_2^2$ (*c*) $y = (2x_1 + 3)(x_2 - 2)$

(*b*) $y = 7x_1 + 5x_1x_2^2 - 9x_2^3$ (*d*) $y = (4x_1 + 3)/(x_2 - 2)$

2 Find f_x and f_y from the following:

(*a*) $f(x, y) = x^2 + 5xy - y^3$ (*c*) $f(x, y) = \dfrac{2x - 3y}{x + y}$

(*b*) $f(x, y) = (x^2 - 3y)(x - 2)$ (*d*) $f(x, y) = \dfrac{x^2 - 1}{xy}$

3 From the answers to the preceding problem, find $f_x(1, 2)$—the value of the partial derivative f_x when $x = 1$ and $y = 2$—for each function.

4 Given the production function $Q = 96K^{0.3}L^{0.7}$, find the MPP_K and MPP_L functions. Is MPP_K a function of K alone, or of both K and L? What about MPP_L?

5 If the utility function of an individual takes the form

$$U = U(x_1, x_2) = (x_1 + 2)^2(x_2 + 3)^3$$

where U is total utility, and x_1 and x_2 are the quantities of two commodities consumed:

(*a*) Find the marginal-utility function of each of the two commodities.

(*b*) Find the value of the marginal utility of the first commodity when 3 units of each commodity are consumed.

7.5 APPLICATIONS TO COMPARATIVE-STATIC ANALYSIS

Equipped with the knowledge of the various rules of differentiation, we can at last tackle the problem posed in comparative-static analysis: namely, how the equilibrium value of an endogenous variable will change when there is a change in any of the exogenous variables or parameters.

Market Model

First let us consider again the simple one-commodity market model of (3.1). That model can be written in the form of two equations:

$$Q = a - bP \qquad (a, b > 0) \qquad \text{[demand]}$$

$$Q = -c + dP \qquad (c, d > 0) \qquad \text{[supply]}$$

with solutions

$$\bar{P} = \frac{a + c}{b + d} \tag{7.14}$$

$$\bar{Q} = \frac{ad - bc}{b + d} \tag{7.15}$$

These solutions will be referred to as being in the *reduced form*: the two endogenous variables have been reduced to explicit expressions of the four mutually independent parameters a, b, c, and d.

To find how an infinitesimal change in one of the parameters will affect the value of $\bar{P}$, one has only to differentiate (7.14) partially with respect to each of the parameters. If the *sign* of a partial derivative, say, $\partial\bar{P}/\partial a$, can be determined from the given information about the parameters, we shall know the direction in which $\bar{P}$ will move when the parameter a changes; this constitutes a qualitative conclusion. If the magnitude of $\partial\bar{P}/\partial a$ can be ascertained, it will constitute a quantitative conclusion.

Similarly, we can draw qualitative or quantitative conclusions from the partial derivatives of $\bar{Q}$ with respect to each parameter, such as $\partial\bar{Q}/\partial a$. To avoid misunderstanding, however, a clear distinction should be made between the two derivatives $\partial\bar{Q}/\partial a$ and $\partial Q/\partial a$. The latter derivative is a concept appropriate to the demand function taken alone, and without regard to the supply function. The derivative $\partial\bar{Q}/\partial a$ pertains, on the other hand, to the equilibrium quantity in (7.15) which, being in the nature of a solution of the model, takes into account the interaction of demand and supply together. To emphasize this distinction, we shall refer to the partial derivatives of $\bar{P}$ and $\bar{Q}$ with respect to the parameters as *comparative-static derivatives*.

Concentrating on $\bar{P}$ for the time being, we can get the following four partial derivatives from (7.14):

$$\frac{\partial\bar{P}}{\partial a} = \frac{1}{b+d} \qquad \left[\text{parameter } a \text{ has the coefficient } \frac{1}{b+d}\right]$$

$$\frac{\partial\bar{P}}{\partial b} = \frac{0(b+d) - 1(a+c)}{(b+d)^2} = \frac{-(a+c)}{(b+d)^2} \qquad [\text{quotient rule}]$$

$$\frac{\partial\bar{P}}{\partial c} = \frac{1}{b+d}\left(= \frac{\partial\bar{P}}{\partial a}\right)$$

$$\frac{\partial\bar{P}}{\partial d} = \frac{0(b+d) - 1(a+c)}{(b+d)^2} = \frac{-(a+c)}{(b+d)^2}\left(= \frac{\partial\bar{P}}{\partial b}\right)$$

Since all the parameters are restricted to being positive in the present model, we can conclude that

$$\text{(7.16)} \qquad \frac{\partial\bar{P}}{\partial a} = \frac{\partial\bar{P}}{\partial c} > 0 \qquad \text{and} \qquad \frac{\partial\bar{P}}{\partial b} = \frac{\partial\bar{P}}{\partial d} < 0$$

For a fuller appreciation of the results in (7.16), let us look at Fig. 7.5, where each diagram shows a change in *one* of the parameters. As before, we are plotting Q (rather than P) on the vertical axis.

Figure 7.5a pictures an increase in the parameter a (to a'). This means a higher vertical intercept for the demand curve, and inasmuch as the parameter b (the slope parameter) is unchanged, the increase in a results in a parallel upward shift of the demand curve from D to D'. The intersection of D' and the supply curve S determines an equilibrium price $\bar{P}'$, which is greater than the old equilibrium price $\bar{P}$. This corroborates the result that $\partial\bar{P}/\partial a > 0$, although for the sake of exposition we have shown in Fig. 7.5a a much larger change in the parameter a than what the concept of derivative implies.

The situation in Fig. 7.5c has a similar interpretation; but since the increase takes place in the parameter c, the result is a parallel shift of the supply curve instead. Note that this shift is downward because the supply curve has a vertical intercept of $-c$; thus an increase in c would mean a change in the intercept, say, from -2 to -4. The graphical comparative-static result, that $\bar{P}'$ exceeds $\bar{P}$, again conforms to what the positive sign of the derivative $\partial\bar{P}/\partial c$ would lead us to expect.

Figures 7.5*b* and 7.5*d* illustrate the effects of changes in the slope parameters b and d of the two functions in the model. An increase in b means that the slope of the demand curve will assume a larger numerical (absolute) value; i.e., it will become steeper. In accordance with the result $\partial\bar{P}/\partial b < 0$, we find a decrease in $\bar{P}$ in this diagram. The increase in d that makes the supply curve steeper also results in a decrease in the equilibrium price. This is, of course, again in line with the negative sign of the comparative-static derivative $\partial\bar{P}/\partial d$.

Thus far, all the results in (7.16) seem to have been obtainable graphically. If so, why should we bother to learn differentiation at all? The answer is that the differentiation approach has at least two major advantages. First, the graphical technique is subject to a dimensional restriction, but differentiation is not. Even

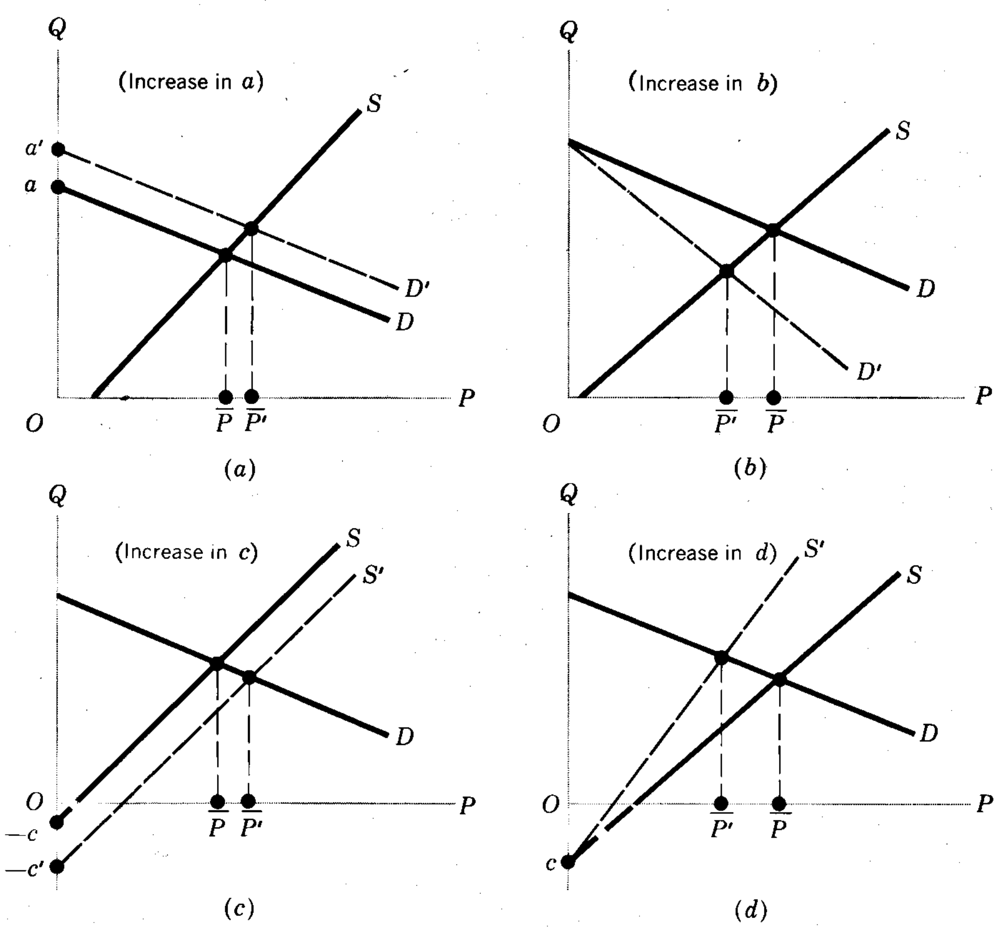

Figure 7.5

when the number of endogenous variables and parameters is such that the equilibrium state cannot be shown graphically, we can nevertheless apply the differentiation techniques to the problem. Second, the differentiation method can yield results that are on a higher level of generality. The results in (7.16) will remain valid, regardless of the specific values that the parameters a, b, c, and d take, as long as they satisfy the sign restrictions. So the comparative-static conclusions of this model are, in effect, applicable to an infinite number of combinations of (linear) demand and supply functions. In contrast, the graphical approach deals only with some specific members of the family of demand and supply curves, and the analytical result derived therefrom is applicable, strictly speaking, only to the specific functions depicted.

The above serves to illustrate the application of partial differentiation to comparative-static analysis of the simple market model, but only half of the task has actually been accomplished, for we can also find the comparative-static derivatives pertaining to $\bar{Q}$. This we shall leave to you as an exercise.

National-Income Model

In place of the simple national-income model discussed in Chap. 3, let us study a slightly enlarged model with three endogenous variables, Y (national income), C (consumption), and T (taxes):

$$\begin{aligned} & Y = C + I_0 + G_0 \\ (7.17)\qquad & C = \alpha + \beta(Y - T) \qquad (\alpha > 0;\quad 0 < \beta < 1) \\ & T = \gamma + \delta Y \qquad (\gamma > 0;\quad 0 < \delta < 1) \end{aligned}$$

The first equation in this system gives the equilibrium condition for national income, while the second and third equations show, respectively, how C and T are determined in the model.

The restrictions on the values of the parameters α, β, γ, and δ can be explained thus: α is positive because consumption is positive even if disposable income $(Y - T)$ is zero; β is a positive fraction because it represents the marginal propensity to consume; γ is positive because even if Y is zero the government will still have a positive tax revenue (from tax bases other than income); and finally, δ is a positive fraction because it represents an income tax rate, and as such it cannot exceed 100 percent. The exogenous variables I_0 (investment) and G_0 (government expenditure) are, of course, nonnegative. All the parameters and exogenous variables are assumed to be independent of one another, so that any one of them can be assigned a new value without affecting the others.

This model can be solved for $\bar{Y}$ by substituting the third equation of (7.17) into the second and then substituting the resulting equation into the first. The equilibrium income (in reduced form) is

$$(7.18)\qquad \bar{Y} = \frac{\alpha - \beta\gamma + I_0 + G_0}{1 - \beta + \beta\delta}$$

Similar equilibrium values can also be found for the endogenous variables C and T, but we shall concentrate on the equilibrium income.

From (7.18), there can be obtained six comparative-static derivatives. Among these, the following three have special policy significance:

$$\frac{\partial \overline{Y}}{\partial G_0} = \frac{1}{1-\beta+\beta\delta} > 0 \tag{7.19}$$

$$\frac{\partial \overline{Y}}{\partial \gamma} = \frac{-\beta}{1-\beta+\beta\delta} < 0 \tag{7.20}$$

$$\frac{\partial \overline{Y}}{\partial \delta} = \frac{-\beta(\alpha-\beta\gamma+I_0+G_0)}{(1-\beta+\beta\delta)^2} = \frac{-\beta\overline{Y}}{1-\beta+\beta\delta} < 0 \qquad \text{[by (7.18)]} \tag{7.21}$$

The partial derivative in (7.19) gives us the *government-expenditure multiplier*. It has a positive sign here because β is less than 1, and $\beta\delta$ is greater than zero. If numerical values are given for the parameters β and δ, we can also find the numerical value of this multiplier from (7.19). The derivative in (7.20) may be called the *nonincome-tax multiplier*, because it shows how a change in γ, the government revenue from nonincome-tax sources, will affect the equilibrium income. This multiplier is negative in the present model because the denominator in (7.20) is positive and the numerator is negative. Lastly, the partial derivative in (7.21) represents an *income-tax-rate multiplier*. For any positive equilibrium income, this multiplier is also negative in the model.

Again, note the difference between the two derivatives $\partial\overline{Y}/\partial G_0$ and $\partial Y/\partial G_0$. The former is derived from (7.18), the expression for the equilibrium income. The latter, obtainable from the first equation in (7.17), is $\partial Y/\partial G_0 = 1$, which is altogether different in magnitude and in concept.

Input-Output Model

The solution of an open input-output model appears as a matrix equation $\overline{x} = (I-A)^{-1}d$. If we denote the inverse matrix $(I-A)^{-1}$ by $B = [b_{ij}]$, then, for instance, the solution for a three-industry economy can be written as $\overline{x} = Bd$, or

$$\begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{x}_3 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \tag{7.22}$$

What will be the rates of change of the solution values $\overline{x}_j$ with respect to the exogenous final demands d_1, d_2, and d_3? The general answer is that

$$\frac{\partial \overline{x}_j}{\partial d_k} = b_{jk} \qquad (j, k = 1, 2, 3) \tag{7.23}$$

To see this, let us multiply out Bd in (7.22) and express the solution as

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} b_{11}d_1 + b_{12}d_2 + b_{13}d_3 \\ b_{21}d_1 + b_{22}d_2 + b_{23}d_3 \\ b_{31}d_1 + b_{32}d_2 + b_{33}d_3 \end{bmatrix}$$

In this system of three equations, each one gives a particular solution value as a function of the exogenous final demands. Partial differentiation of these will produce a total of nine comparative-static derivatives:

$$\begin{array}{lll} \dfrac{\partial \bar{x}_1}{\partial d_1} = b_{11} & \dfrac{\partial \bar{x}_1}{\partial d_2} = b_{12} & \dfrac{\partial \bar{x}_1}{\partial d_3} = b_{13} \\[2ex] \dfrac{\partial \bar{x}_2}{\partial d_1} = b_{21} & \dfrac{\partial \bar{x}_2}{\partial d_2} = b_{22} & \dfrac{\partial \bar{x}_2}{\partial d_3} = b_{23} \\[2ex] \dfrac{\partial \bar{x}_3}{\partial d_1} = b_{31} & \dfrac{\partial \bar{x}_3}{\partial d_2} = b_{32} & \dfrac{\partial \bar{x}_3}{\partial d_3} = b_{33} \end{array} \qquad (7.23')$$

This is simply the expanded version of (7.23).

Reading (7.23′) as three distinct columns, we may combine the three derivatives in each column into a matrix (vector) derivative:

$$\frac{\partial \bar{x}}{\partial d_1} \equiv \frac{\partial}{\partial d_1}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} \qquad \frac{\partial \bar{x}}{\partial d_2} = \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} \qquad \frac{\partial \bar{x}}{\partial d_3} = \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} \qquad (7.23'')$$

Since the three column vectors in (7.23″) are merely the columns of the matrix B, by further consolidation we can summarize the nine derivatives in a single matrix derivative $\partial \bar{x}/\partial d$. Given $\bar{x} = Bd$, we can simply write

$$\frac{\partial \bar{x}}{\partial d} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = B$$

This is a compact way of denoting all the comparative-static derivatives of our open input-output model. Obviously, this matrix derivative can easily be extended from the present three-industry model to the general n-industry case.

Comparative-static derivatives of the input-output model are useful as tools of economic planning, for they provide the answer to the question: If the planning targets, as reflected in $(d_1, d_2, \ldots, d_n)$, are revised, and if we wish to take care of all direct and indirect requirements in the economy so as to be completely free of bottlenecks, how must we change the output goals of the n industries?

EXERCISE 7.5

1 Examine the comparative-static properties of the equilibrium quantity in (7.15), and check your results by graphic analysis.

2 On the basis of (7.18), find the partial derivatives $\partial\overline{Y}/\partial I_0$, $\partial\overline{Y}/\partial\alpha$, and $\partial\overline{Y}/\partial\beta$. Interpret their meanings and determine their signs.

3 The numerical input-output model (5.21) was solved in Sec. 5.7.
(*a*) How many comparative-static derivatives can be derived?
(*b*) Write out these derivatives in the form of (7.23′) and (7.23″).

7.6 NOTE ON JACOBIAN DETERMINANTS

The study of partial derivatives above was motivated solely by comparative-static considerations. But partial derivatives also provide a means of testing whether there exists functional (linear *or* nonlinear) dependence among a set of n functions in n variables. This is related to the notion of Jacobian determinants (named after Jacobi).

Consider the two functions

$$(7.24)\qquad \begin{aligned} y_1 &= 2x_1 + 3x_2 \\ y_2 &= 4x_1^2 + 12x_1x_2 + 9x_2^2 \end{aligned}$$

If we get all the four partial derivatives

$$\frac{\partial y_1}{\partial x_1} = 2 \qquad \frac{\partial y_1}{\partial x_2} = 3 \qquad \frac{\partial y_2}{\partial x_1} = 8x_1 + 12x_2 \qquad \frac{\partial y_2}{\partial x_2} = 12x_1 + 18x_2$$

and arrange them into a square matrix in a prescribed order, called a Jacobian matrix and denoted by J, and then take its determinant, the result will be what is known as a *Jacobian determinant* (or a *Jacobian*, for short), denoted by $|J|$:

$$(7.25)\qquad |J| \equiv \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ (8x_1 + 12x_2) & (12x_1 + 18x_2) \end{vmatrix}$$

For economy of space, this Jacobian is sometimes also expressed as

$$|J| \equiv \left| \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \right|$$

More generally, if we have n differentiable functions in n variables, not necessarily linear,

$$\begin{aligned} y_1 &= f^1(x_1, x_2, \ldots, x_n) \\ y_2 &= f^2(x_1, x_2, \ldots, x_n) \\ &\cdots\cdots\cdots\cdots \\ y_n &= f^n(x_1, x_2, \ldots, x_n) \end{aligned} \tag{7.26}$$

where the symbol f^n denotes the nth function (and *not* the function raised to the nth power), we can derive a total of n^2 partial derivatives. Together, they will give rise to the Jacobian

$$|J| \equiv \left| \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} \right| \equiv \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \cdots & \dfrac{\partial y_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial y_n}{\partial x_1} & \dfrac{\partial y_n}{\partial x_2} & \cdots & \dfrac{\partial y_n}{\partial x_n} \end{vmatrix} \tag{7.27}$$

A Jacobian test for the existence of functional dependence among a set of n functions is provided by the following theorem: The Jacobian $|J|$ defined in (7.27) will be identically zero for all values of $x_1, \ldots, x_n$ if and only if the n functions $f^1, \ldots, f^n$ in (7.26) are functionally (linearly or nonlinearly) dependent.

As an example, for the two functions in (7.24) the Jacobian as given in (7.25) has the value

$$|J| = (24x_1 + 36x_2) - (24x_1 + 36x_2) = 0$$

That is, the Jacobian vanishes for all values of x_1 and x_2. Therefore, according to the theorem, the two functions in (7.24) must be dependent. You can verify that y_2 is simply y_1 squared; thus they are indeed functionally dependent—here *non*linearly dependent.

Let us now consider the special case of *linear* functions. We have earlier shown that the rows of the coefficient matrix A of a linear-equation system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= d_n \end{aligned} \tag{7.28}$$

are linearly dependent if and only if the determinant $|A| = 0$. This result can now be interpreted as a special application of the Jacobian criterion of functional dependence.

Take the left side of each equation in (7.28) as a separate function of the n variables $x_1, \ldots, x_n$, and denote these functions by $y_1, \ldots, y_n$. The partial derivatives of these functions will turn out to be $\partial y_1/\partial x_1 = a_{11}$, $\partial y_1/\partial x_2 = a_{12}$, etc., so that we may write, in general, $\partial y_i/\partial x_j = a_{ij}$. In view of this, the elements of the Jacobian of these n functions will be precisely the elements of the coefficient matrix A, already arranged in the correct order. That is, we have $|J| = |A|$, and thus the Jacobian criterion of functional dependence among $y_1, \ldots, y_n$—or, what amounts to the same thing, functional dependence among the rows of the coefficient matrix A—is equivalent to the criterion $|A| = 0$ in the present linear case.

In the above, the Jacobian was discussed in the context of a system of n functions in n variables. It should be pointed out, however, that the Jacobian in (7.27) is defined even if each function in (7.26) contains more than n variables, say, $n + 2$ variables:

$$y_i = f^i(x_1, \ldots, x_n, x_{n+1}, x_{n+2}) \qquad (i = 1, 2, \ldots, n)$$

In such a case, if we hold any two of the variables (say, x_{n+1} and x_{n+2}) constant, or treat them as parameters, we will again have n functions in exactly n variables and can form a Jacobian. Moreover, by holding a different pair of the x variables constant, we can form a different Jacobian. Such a situation will indeed be encountered in Chap. 8 in connection with the discussion of the implicit-function theorem.

EXERCISE 7.6

1 Use Jacobian determinants to test the existence of functional dependence between the functions paired below:

(*a*) $y_1 = 3x_1^2 + x_2$
$y_2 = 9x_1^4 + 6x_1^2(x_2 + 4) + x_2(x_2 + 8) + 12$

(*b*) $y_1 = 3x_1^2 + 2x_2^2$
$y_2 = 5x_1 + 1$

2 Consider (7.22) as a set of three functions $\bar{x}_i = f^i(d_1, d_2, d_3)$ (with $i = 1, 2, 3$).

(*a*) Write out the 3×3 Jacobian. Does it have some relation to (7.23′)? Can we write $|J| = |B|$?

(*b*) Since $B \equiv (I - A)^{-1}$, can we conclude that $|B| \neq 0$? What can we infer from this about the three equations in (7.22)?

CHAPTER
EIGHT
COMPARATIVE-STATIC ANALYSIS OF GENERAL-FUNCTION MODELS

The study of partial derivatives has enabled us, in the preceding chapter, to handle the simpler type of comparative-static problems, in which the equilibrium solution of the model can be explicitly stated in the reduced form. In that case, partial differentiation of the solution will directly yield the desired comparative-static information. You will recall that the definition of the partial derivative requires the absence of any functional relationship among the independent variables (say, x_i), so that x_1 can vary without affecting the values of $x_2, x_3, \ldots, x_n$. As applied to comparative-static analysis, this means that the parameters and/or exogenous variables which appear in the reduced-form solution must be mutually independent. Since these are indeed defined as predetermined data for purposes of the model, the possibility of their mutually affecting one another is inherently ruled out. The procedure of partial differentiation adopted in the last chapter is therefore fully justifiable.

However, no such expediency should be expected when, owing to the inclusion of general functions in a model, no explicit reduced-form solution can be obtained. In such cases, we will have to find the comparative-static derivatives directly from the originally given equations in the model. Take, for instance, a simple national-income model with two endogenous variables Y and C:

$$Y = C + I_0 + G_0$$
$$C = C(Y, T_0) \qquad [T_0\text{: exogenous taxes}]$$

which is reducible to a single equation (an equilibrium condition)

$$Y = C(Y, T_0) + I_0 + G_0$$

to be solved for $\overline{Y}$. Because of the general form of the C function, however, no explicit solution is available. We must, therefore, find the comparative-static derivatives directly from this equation. How might we approach the problem? What special difficulty might we encounter?

Let us suppose that an equilibrium solution $\overline{Y}$ does exist. Then, under certain rather general conditions (to be discussed later), we may take $\overline{Y}$ to be a differentiable function of the exogenous variables I_0, G_0, and T_0. Hence we may write the equation

$$\overline{Y} = \overline{Y}(I_0, G_0, T_0)$$

even though we are unable to determine explicitly the form which this function takes. Furthermore, in some neighborhood of the equilibrium value $\overline{Y}$, the following identical equality will hold:

$$\overline{Y} \equiv C(\overline{Y}, T_0) + I_0 + G_0$$

This type of identity will be referred to as an *equilibrium identity* because it is nothing but the equilibrium condition with the Y variable replaced by its equilibrium value $\overline{Y}$. Now that $\overline{Y}$ has entered into the picture, it may seem at first blush that simple partial differentiation of this identity will yield any desired comparative-static derivative, say, $\partial\overline{Y}/\partial T_0$. This, unfortunately, is not the case. Since $\overline{Y}$ is a function of T_0, the two arguments of the C function are *not* independent. Specifically, T_0 can in this case affect C not only *directly*, but also *indirectly* via $\overline{Y}$. Consequently, partial differentiation is no longer appropriate for our purposes. How, then, do we tackle this situation?

The answer is that we must resort to *total differentiation* (as against partial differentiation). Based on the notion of *total differentials*, the process of total differentiation can lead us to the related concept of *total derivative*, which measures the rate of change of a function such as $C(\overline{Y}, T_0)$ with respect to the argument T_0, when T_0 also affects the other argument, $\overline{Y}$. Thus, once we become familiar with these concepts, we shall be able to deal with functions whose arguments are not all independent, and that would remove the major stumbling block we have so far encountered in our study of the comparative statics of a general-function model. As a prelude to the discussion of these concepts, however, we should first introduce the notion of *differentials*.

8.1 DIFFERENTIALS

The symbol dy/dx, for the derivative of the function $y = f(x)$, has hitherto been regarded as a single entity. We shall now reinterpret it as a ratio of two quantities, dy and dx.

Differentials and Derivatives

Given a function $y = f(x)$, a specific Δx will call forth a corresponding Δy, and we can use the difference quotient $\Delta y/\Delta x$ to represent the rate of change of y with respect to x. Since it is true that

$$(8.1) \qquad \Delta y \equiv \left(\frac{\Delta y}{\Delta x}\right)\Delta x$$

the magnitude of Δy can be found, once the rate of change $\Delta y/\Delta x$ and the variation in x are known.

When Δx is infinitesimal, Δy will also be infinitesimal, and the difference quotient $\Delta y/\Delta x$ will turn into the derivative dy/dx. Then, if we denote the infinitesimal changes in x and y, respectively, by dx and dy (in place of Δx and Δy), the identity (8.1) will become

$$(8.2) \qquad dy \equiv \left(\frac{dy}{dx}\right) dx \qquad \text{or} \qquad dy \equiv f'(x)\, dx$$

The symbols dy and dx are called the *differentials* of y and x, respectively.

Dividing the two identities in (8.2) throughout by dx, we have

$$(8.2') \qquad \frac{(dy)}{(dx)} \equiv \left(\frac{dy}{dx}\right) \qquad \text{or} \qquad \frac{(dy)}{(dx)} \equiv f'(x)$$

This result shows that the derivative $(dy/dx) \equiv f'(x)$ may be interpreted as the quotient of two separate differentials dy and dx.

On the basis of (8.2), once we are given the derivative of a function $y = f(x)$, dy can immediately be written as $f'(x)\,dx$. The derivative $f'(x)$ may thus be viewed as a "converter" that serves to convert an infinitesimal change dx into a corresponding change dy.

Example 1 Given $y = 3x^2 + 7x - 5$, find dy. The derivative of the function is $dy/dx = 6x + 7$; thus the desired differential is

$$(8.3) \qquad dy = (6x + 7)\, dx$$

This result can be used to calculate the change in y resulting from a given change in x. It should be remembered, however, that the differentials dy and dx refer to infinitesimal changes only; hence, if we put an x change of substantial magnitude (Δx) into (8.3), the resulting dy can only serve as an approximation to the exact value of the corresponding y change (Δy). Let us calculate dy from (8.3), assuming that x is to change from 5 to 5.01. To do this, we set $x = 5$ and $dx = 0.01$ and substitute these into (8.3). The result is $dy = 37(0.01) = 0.37$. How does this figure compare with the *actual* change in y? When $x = 5$ (before change), we can compute from the given function that $y = 105$, but when $x = 5.01$ (after change), we get $y = 105.3703$. The true change in y is therefore $\Delta y = 0.3703$, for which our answer $dy = 0.37$ constitutes an approximation with an error of 0.0003.

The source of error in the approximation can be illustrated in general by means of Fig. 8.1. For the given Δx depicted in the figure (distance AC), the true change in y, or Δy, is the distance CB. Had we used the slope of line AB $(= \Delta y/\Delta x = CB/AC)$ as the relevant rate of change and applied (8.1) to find Δy, we would have obtained the correct answer:

$$\Delta y = \left(\frac{\Delta y}{\Delta x}\right)\Delta x = \frac{CB}{AC}AC = CB$$

But, in using (8.3)—a specific version of (8.2)—we actually employed the derivative dy/dx in lieu of $\Delta y/\Delta x$; that is, we used the slope of the tangent line AD $(= CD/AC)$ instead of the slope of line AB in the calculation. Thus we obtained the answer

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{CD}{AC}AC = CD$$

which differs from the true change CB by an error of DB. This error can, of course, be expected to become smaller, the smaller is the Δx, that is, the closer the point B moves toward point A.

The process of finding the differential dy is called *differentiation*. Recall that we have been using this term as a synonym for derivation, without having given an adequate explanation. In the light of our interpretation of a derivative as a quotient of two differentials, however, the rationale of the term becomes self-evident. It is still somewhat ambiguous, though, to use the single term "differentiation" to refer to the process of finding the differential dy as well as to that of finding the derivative dy/dx. To avoid confusion, the usual practice is to qualify the word "differentiation" with the phrase "with respect to x" when we take the derivative dy/dx. It should be clear from (8.2) that, given a function $y = f(x)$, we

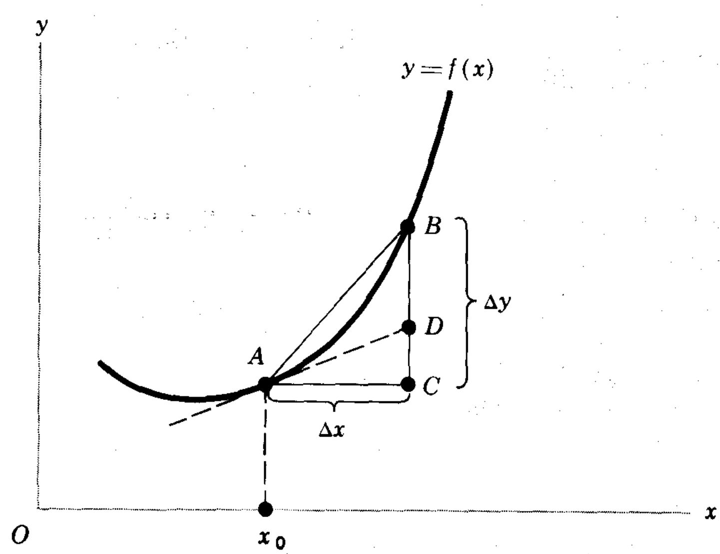

Figure 8.1

can always (1) transform a known differential dy into the derivative dy/dx by dividing it by dx and (2) transform a known derivative dy/dx into the differential dy by multiplying it by dx.

Differentials and Point Elasticity

As an illustration of the application of differentials in economics, let us consider the notion of the elasticity of a function. For a demand function $Q = f(P)$, for instance, the elasticity is defined as $(\Delta Q/Q)/(\Delta P/P)$. Now, if the change in P is infinitesimal, the expressions ΔP and ΔQ will reduce to the differentials dP and dQ, and the elasticity measure will then assume the sense of the *point elasticity* of demand, denoted by ε_d (the Greek letter epsilon, for "elasticity"):*

(8.4) $$\varepsilon_d \equiv \frac{dQ/Q}{dP/P} = \frac{dQ/dP}{Q/P}$$

Observe that in the expression on the extreme right we have rearranged the differentials dQ and dP into a ratio dQ/dP, which can be construed as the derivative, or the *marginal* function, of the demand function $Q = f(P)$. Since we can interpret similarly the ratio Q/P in the denominator as the *average* function of the demand function, the point elasticity of demand ε_d in (8.4) is seen to be the ratio of the marginal function to the average function of the demand function.

Indeed, this last-described relationship is valid not only for the demand function but also for any other function, because for any given *total* function $y = f(x)$ we can write the formula for the point elasticity of y with respect to x as

(8.5) $$\varepsilon_{yx} = \frac{dy/dx}{y/x} = \frac{\text{marginal function}}{\text{average function}}$$

As a matter of convention, the *absolute* value of the elasticity measure is used in deciding whether the function is elastic at a particular point. In the case of a demand function, for instance, we stipulate:

$$\text{The demand is } \begin{Bmatrix} \text{elastic} \\ \text{of unit elasticity} \\ \text{inelastic} \end{Bmatrix} \text{ at a point when } |\varepsilon_d| \gtreqless 1.$$

Example 2 Find ε_d if the demand function is $Q = 100 - 2P$. The marginal function and the average function of the given demand are

$$\frac{dQ}{dP} = -2 \qquad \text{and} \qquad \frac{Q}{P} = \frac{100 - 2P}{P}$$

so their ratio will give us

$$\varepsilon_d = \frac{-P}{50 - P}$$

* The point-elasticity measure can alternatively be interpreted as the limit of $\frac{\Delta Q/Q}{\Delta P/P} = \frac{\Delta Q/\Delta P}{Q/P}$ as $\Delta P \to 0$, which gives the same result as (8.4).

As written, the elasticity is shown as a function of P. As soon as a specific price is chosen, however, the point elasticity will be determinate in magnitude. When $P = 25$, for instance, we have $\varepsilon_d = -1$, or $|\varepsilon_d| = 1$, so that the demand elasticity is unitary at that point. When $P = 30$, in contrast, we have $|\varepsilon_d| = 1.5$; hence, demand is elastic at that price. More generally, it may be verified that we have $|\varepsilon_d| > 1$ for $25 < P < 50$ and $|\varepsilon_d| < 1$ for $0 < P < 25$ in the present example. (Can a price $P > 50$ be considered meaningful here?)

At the risk of digressing a trifle, it may also be added here that the interpretation of the ratio of two differentials as a derivative—and the consequent transformation of the elasticity formula of a function into a ratio of its marginal to its average—makes possible a quick way of determining the point elasticity graphically. The two diagrams in Fig. 8.2 illustrate the cases, respectively, of a negatively sloped curve and a positively sloped curve. In each case, the value of the marginal function at point A on the curve, or at $x = x_0$ in the domain, is measured by the slope of the tangent line AB. The value of the average function, on the other hand, is in each case measured by the slope of line OA (the line joining the point of origin with the given point A on the curve, like a radius vector), because at point A we have $y = x_0A$ and $x = Ox_0$, so that the average is $y/x = x_0A/Ox_0 =$ slope of OA. The elasticity at point A can thus be readily ascertained by comparing the *numerical* values of the two slopes involved: If AB is steeper than OA, the function is elastic at point A; in the opposite case, it is inelastic at A. Accordingly, the function pictured in Fig. 8.2a is inelastic at A (or at $x = x_0$), whereas the one in diagram b is elastic at A.

Moreover, the two slopes under comparison are directly dependent on the respective sizes of the two angles θ_m and θ_a (Greek letter theta; the subscripts m and a indicate marginal and average, respectively). Thus we may, alternatively, compare these two angles instead of the two corresponding slopes. Referring to Fig. 8.2 again, you can see that $\theta_m < \theta_a$ at point A in diagram a, indicating that

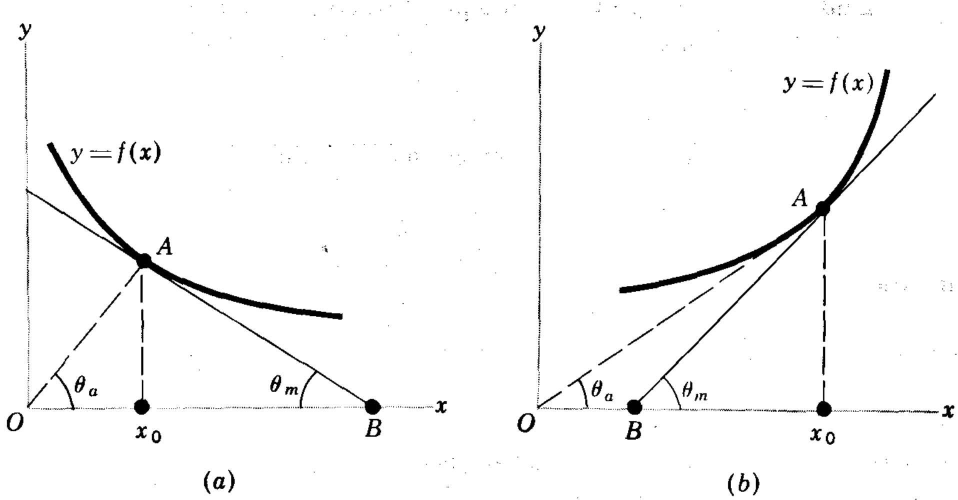

Figure 8.2

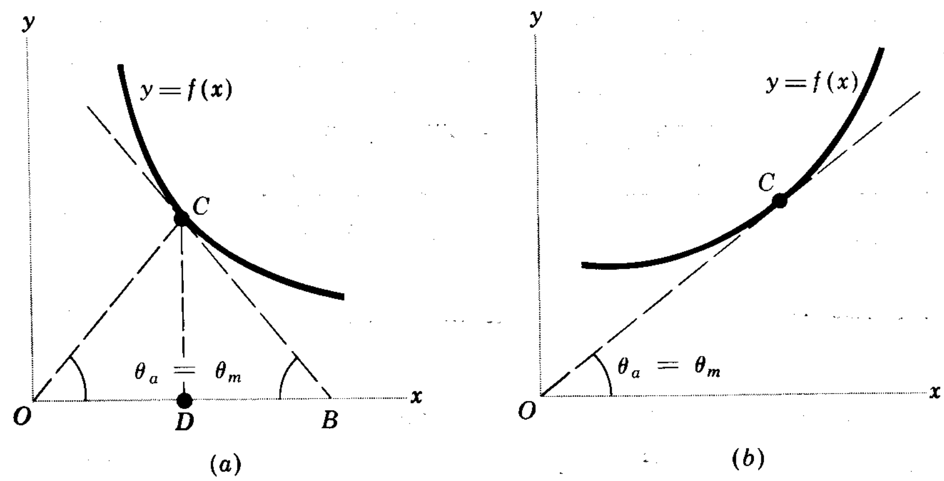

Figure 8.3

the marginal falls short of the average in numerical value; thus the function is inelastic at point A. The exact opposite is true in diagram b.

Sometimes, we are interested in locating a point of unitary elasticity on a given curve. This can now be done easily. If the curve is negatively sloped, as in Fig. 8.3a, we should find a point C such that the line OC and the tangent BC will make the same-sized angle with the x axis, though in the opposite direction. In the case of a positively sloped curve, as in Fig. 8.3b, one has only to find a point C such that the tangent line at C, when properly extended, passes through the point of origin.

We must warn you that the graphical method just described is based on the assumption that the function $y = f(x)$ is plotted with the dependent variable y on the vertical axis. In particular, in applying the method to a demand curve, we should make sure that Q is on the vertical axis. (Suppose that Q is actually plotted on the horizontal axis. How should our method of reading the point elasticity be modified?)

EXERCISE 8.1

1 Find the differential dy, given:

(a) $y = -x(x^2 + 3)$ (b) $y = (x - 8)(7x + 5)$ (c) $y = \dfrac{x}{x^2 + 1}$

2 Given the import function $M = f(Y)$, where M is imports and Y is national income, express the income elasticity of imports ε_{MY} in terms of the propensities to import.

3 Given the consumption function $C = a + bY$ (with $a > 0$; $0 < b < 1$):

(a) Find its marginal function and its average function.

(b) Find the income elasticity of consumption ε_{CY}, and determine its sign, assuming $Y > 0$.

(c) Show that this consumption function is inelastic at all positive income levels.

4 Find the point elasticity of demand, given $Q = k/P^n$, where k and n are positive constants.

(a) Does the elasticity depend on the price in this case?

(b) In the special case where $n = 1$, what is the shape of the demand curve? What is the point elasticity of demand?

8.2 TOTAL DIFFERENTIALS

The concept of differentials can easily be extended to a function of two or more independent variables. Consider a saving function

$$S = S(Y, i) \tag{8.6}$$

where S is savings, Y is national income, and i is interest rate. This function is assumed—as all the functions we shall use here will be assumed—to be continuous and to possess continuous (partial) derivatives, which is another way of saying that it is smooth and differentiable everywhere. We know that the partial derivative $\partial S/\partial Y$ (or S_Y) measures the rate of change of S with respect to an infinitesimal change in Y or, in short, that it signifies the marginal propensity to save. As a result, the change in S due to that change in Y may be represented by the expression $(\partial S/\partial Y)\, dY$, which is comparable to the right-hand expression in (8.2). By the same token, the change in S resulting from an infinitesimal change in i can be denoted as $(\partial S/\partial i)\, di$. The total change in S will then be equal to

$$dS = \frac{\partial S}{\partial Y} dY + \frac{\partial S}{\partial i} di$$

or, in an alternative notation,

$$dS = S_Y\, dY + S_i\, di$$

Note that the two partial derivatives S_Y and S_i again play the role of "converters" that serve to convert the infinitesimal changes dY and di, respectively, into a corresponding change dS. The expression dS, being the *sum* of the changes from both sources, is called the *total differential* of the saving function. And the process of finding such a total differential is called *total differentiation*.

It is possible, of course, that Y may change while i remains constant. In that case, $di = 0$, and the total differential will reduce to a *partial differential*: $dS = (\partial S/\partial Y)\, dY$. Dividing both sides by dY, we get

$$\frac{\partial S}{\partial Y} = \left(\frac{dS}{dY}\right)_{i \text{ constant}}$$

Thus it is clear that the partial derivative $\partial S/\partial Y$ can also be interpreted as the ratio of two differentials dS and dY, with the proviso that i, the other independent variable in the function, is held constant. In a wholly analogous manner, we can form another partial differential $dS = (\partial S/\partial i)\, di$ when $dY = 0$, and can then interpret the partial derivative $\partial S/\partial i$ as the ratio of the differential dS (with Y

held constant) to the differential di. Note that although dS and di can now each stand alone as a differential, the expression $\partial S/\partial i$ remains as a single entity.

The more general case of a function of n independent variables can be exemplified by, say, a utility function in the general form

(8.7) $U = U(x_1, x_2, \ldots, x_n)$

The total differential of this function can be written as

$$dU = \frac{\partial U}{\partial x_1}dx_1 + \frac{\partial U}{\partial x_2}dx_2 + \cdots + \frac{\partial U}{\partial x_n}dx_n$$

or $$dU = U_1\,dx_1 + U_2\,dx_2 + \cdots + U_n\,dx_n = \sum_{i=1}^{n} U_i\,dx_i$$

in which each term on the right side indicates the amount of change in U resulting from an infinitesimal change in one of the independent variables. Economically, the first term, $U_1\,dx_1$, means the marginal utility of the first commodity times the increment in consumption of that commodity, and similarly for the other terms. The sum of these thus represents the total change in utility originating from all possible sources of change.

Like any other function, the saving function (8.6) and the utility function (8.7) can both be expected to give rise to elasticity measures similar to that defined in (8.5). But each elasticity measure must in these instances be defined in terms of the change in *one* of the independent variables only; there will thus be *two* such elasticity measures to the saving function, and n of them to the utility function. These are accordingly called ***partial elasticities***. For the saving function, the partial elasticities may be written as

$$\varepsilon_{SY} = \frac{\partial S/\partial Y}{S/Y} = \frac{\partial S}{\partial Y}\frac{Y}{S} \qquad \text{and} \qquad \varepsilon_{Si} = \frac{\partial S/\partial i}{S/i} = \frac{\partial S}{\partial i}\frac{i}{S}$$

Know
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For the utility function, the n partial elasticities can be concisely denoted as follows:

$$\varepsilon_{Ux_i} = \frac{\partial U}{\partial x_i}\frac{x_i}{U} \qquad (i = 1, 2, \ldots, n)$$

EXERCISE 8.2

1 Find the total differential, given:
(a) $z = 3x^2 + xy - 2y^3$ (b) $U = 2x_1 + 9x_1x_2 + x_2^2$

2 Find the total differential, given:
(a) $y = \dfrac{x_1}{x_1 + x_2}$ (b) $y = \dfrac{2x_1x_2}{x_1 + x_2}$

3 The supply function of a certain commodity is:

$$Q = a + bP^2 + R^{1/2} \qquad (a < 0,\quad b > 0) \qquad [R\text{: rainfall}]$$

Find the price elasticity of supply ε_{QP}, and the rainfall elasticity of supply ε_{QR}.

4 How do the two partial elasticicities in the last problem vary with P and R? In a monotonic fashion (assuming positive P and R)?

5 The foreign demand for our exports X depends on the foreign income Y_f and our price level P: $X = Y_f^{1/2} + P^{-2}$. Find the partial elasticity of foreign demand for our exports with respect to our price level.

8.3 RULES OF DIFFERENTIALS

A straightforward way of finding the total differential dy, given a function

$$y = f(x_1, x_2)$$

is to find the partial derivatives f_1 and f_2 and substitute these into the equation

$$dy = f_1\,dx_1 + f_2\,dx_2$$

But sometimes it may be more convenient to apply certain rules of differentials which, in view of their striking resemblance to the derivative formulas studied before, are exceedingly easy to remember.

Let k be a constant and u and v be two functions of the variables x_1 and x_2. Then the following rules are valid:*

Rule I $\quad dk = 0 \qquad$ (cf. constant-function rule)

Rule II $\quad d(cu^n) = cnu^{n-1}\,du \qquad$ (cf. power-function rule)

Rule III $\quad d(u \pm v) = du \pm dv \qquad$ (cf. sum-difference rule)

Rule IV $\quad d(uv) = v\,du + u\,dv \qquad$ (cf. product rule)

Rule V $\quad d\left(\dfrac{u}{v}\right) = \dfrac{1}{v^2}(v\,du - u\,dv) \qquad$ (cf. quotient rule)

Instead of proving these rules here, we shall merely illustrate their practical application.

Example 1 Find the total differential dy of the function

$$y = 5x_1^2 + 3x_2$$

The straightfoward method calls for the evaluation of the partial derivatives $f_1 = 10x_1$ and $f_2 = 3$, which will then enable us to write

$$dy = f_1\,dx_1 + f_2\,dx_2 = 10x_1\,dx_1 + 3\,dx_2$$

* All the rules of differentials discussed in this section are also applicable when u and v are themselves the independent variables (rather than functions of some other variables x_1 and x_2).

We may, however, let $u = 5x_1^2$ and $v = 3x_2$ and apply the above-mentioned rules to get the identical answer as follows:

$$\begin{aligned} dy &= d(5x_1^2) + d(3x_2) && \text{[by Rule III]} \\ &= 10x_1\,dx_1 + 3\,dx_2 && \text{[by Rule II]} \end{aligned}$$

Example 2 Find the total differential of the function

$$y = 3x_1^2 + x_1x_2^2$$

Since $f_1 = 6x_1 + x_2^2$ and $f_2 = 2x_1x_2$, the desired differential is

$$dy = (6x_1 + x_2^2)\,dx_1 + 2x_1x_2\,dx_2$$

By applying the given rules, the same result can be arrived at thus:

$$\begin{aligned} dy &= d(3x_1^2) + d(x_1x_2^2) && \text{[by Rule III]} \\ &= 6x_1\,dx_1 + x_2^2\,dx_1 + x_1\,d(x_2^2) && \text{[by Rules II and IV]} \\ &= (6x_1 + x_2^2)\,dx_1 + 2x_1x_2\,dx_2 && \text{[by Rule II]} \end{aligned}$$

Example 3 Find the total differential of the function

$$y = \frac{x_1 + x_2}{2x_1^2}$$

In view of the fact that the partial derivatives in this case are

$$f_1 = \frac{-(x_1 + 2x_2)}{2x_1^3} \qquad \text{and} \qquad f_2 = \frac{1}{2x_1^2}$$

(check these as an exercise), the desired differential is

$$dy = \frac{-(x_1 + 2x_2)}{2x_1^3}\,dx_1 + \frac{1}{2x_1^2}\,dx_2$$

However, the same result may also be obtained by application of the rules as follows:

$$\begin{aligned} dy &= \frac{1}{4x_1^4}\left[2x_1^2\,d(x_1 + x_2) - (x_1 + x_2)\,d(2x_1^2)\right] && \text{[by Rule V]} \\ &= \frac{1}{4x_1^4}\left[2x_1^2(dx_1 + dx_2) - (x_1 + x_2)4x_1\,dx_1\right] && \text{[by Rules III and II]} \\ &= \frac{1}{4x_1^4}\left[-2x_1(x_1 + 2x_2)\,dx_1 + 2x_1^2\,dx_2\right] \\ &= \frac{-(x_1 + 2x_2)}{2x_1^3}\,dx_1 + \frac{1}{2x_1^2}\,dx_2 \end{aligned}$$

These rules can naturally be extended to cases where more than two functions of x_1 and x_2 are involved. In particular, we can add the following two rules to the previous collection:

Rule VI $d(u \pm v \pm w) = du \pm dv \pm dw$

Rule VII $d(uvw) = vw\,du + uw\,dv + uv\,dw$

To derive Rule VII, we can employ the familiar trick of first letting $z = vw$, so that

$$d(uvw) = d(uz) = z\,du + u\,dz \qquad \text{[by Rule IV]}$$

Then, by applying Rule IV again to dz, we get the intermediate result

$$dz = d(vw) = w\,dv + v\,dw$$

which, when substituted into the preceding equation, will yield

$$d(uvw) = vw\,du + u(w\,dv + v\,dw) = vw\,du + uw\,dv + uv\,dw$$

as the desired final result. A similar procedure can be employed to derive Rule VI.

EXERCISE 8.3

1 Use the rules of differentials to find (*a*) dz from $z = 3x^2 + xy - 2y^3$ and (*b*) dU from $U = 2x_1 + 9x_1x_2 + x_2^2$. Check your answers against those obtained for Exercise 8.2-1.

2 Use the rules of differentials to find dy from the following functions:

(*a*) $y = \dfrac{x_1}{x_1 + x_2}$ (*b*) $y = \dfrac{2x_1x_2}{x_1 + x_2}$

Check your answers against those obtained for Exercise 8.2-2.

3 Given $y = 3x_1(2x_2 - 1)(x_3 + 5)$

(*a*) Find dy by Rule VII.

(*b*) Find the partial differential of y, if $dx_2 = dx_3 = 0$.

(*c*) Derive from the above result the partial derivative $\partial y/\partial x_1$.

4 Prove Rules II, III, IV, and V, assuming u and v to be the independent variables (rather than functions of some other variables).

8.4 TOTAL DERIVATIVES

With the notion of differentials at our disposal, we are now equipped to answer the question posed at the beginning of the chapter, namely, how we find the rate of change of the function $C(\bar{Y}, T_0)$ with respect to T_0, when $\bar{Y}$ and T_0 are related. As previously mentioned, the answer lies in the concept of total derivative. Unlike a *partial* derivative, a *total* derivative does not require the argument $\bar{Y}$ to remain constant as T_0 varies, and can thus allow for the postulated relationship between the two arguments.

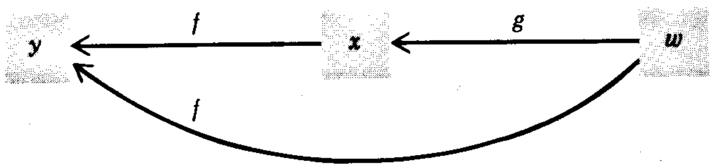


Figure 8.4

Finding the Total Derivative

To carry on the discussion in a more general framework, let us consider any function

(8.8) $\quad y = f(x, w) \qquad \text{where} \qquad x = g(w)$

with the three variables y, x, and w related to one another as in Fig. 8.4. In this figure, which we shall refer to as a *channel map*, it is clearly seen that w—the ultimate source of change in this case—can affect y through two channels: (1) *indirectly*, via the function g and then f (the straight arrows), and (2) *directly*, via the function f (the curved arrow). Whereas the partial derivative f_w is adequate for expressing the direct effect alone, a total derivative is needed to express both effects jointly.

To obtain this total derivative, we first differentiate y totally, to get the total differential $dy = f_x\,dx + f_w\,dw$. When both sides of this equation are divided by the differential dw, the result is

(8.9)
$$\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w \frac{dw}{dw}$$
$$= \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w} \qquad \left[\frac{dw}{dw} = 1\right]$$

Since the ratio of two differentials may be interpreted as a derivative, the expression dy/dw on the left may be regarded as some measure of the rate of change of y with respect to w. Moreover, if the two terms on the right side of (8.9) can be identified, respectively, as the indirect and direct effects of w on y, then dy/dw will indeed be the total derivative we are seeking. Now, the second term $(\partial y/\partial w)$ is already known to measure the direct effect, and it thus corresponds to the curved arrow in Fig. 8.4. That the first term $\left(\frac{\partial y}{\partial x}\frac{dx}{dw}\right)$ measures the indirect effect will also become evident when we analyze it with the help of some arrows as follows:*

$$\left(\frac{\partial y}{\partial x} \leftarrow \frac{dx}{dw}\right)$$

* The expression $\frac{\partial y}{\partial x}\frac{dx}{dw}$ is reminiscent of the chain rule (composite-function rule) discussed earlier, except that here a partial derivative appears because f happens to be a function of more than one variable.

The change in w (namely, dw) is in the first instance transmitted to the variable x, and through the resulting change in x (namely, dx) it is relayed to the variable y. But this is precisely the indirect effect, as depicted by the sequence of straight arrows in Fig. 8.4. Hence, the expression in (8.9) does indeed represent the desired total derivative. The process of finding the total derivative dy/dw is referred to as *total differentiation of y with respect to w.*

Example 1 Find the total derivative dy/dw, given the function

$$y = f(x, w) = 3x - w^2 \qquad \text{where} \qquad x = g(w) = 2w^2 + w + 4$$

By virtue of (8.9), the total derivative should be

$$\frac{dy}{dw} = 3(4w + 1) + (-2w) = 10w + 3$$

As a check, we may substitute the function g into the function f, to get

$$y = 3(2w^2 + w + 4) - w^2 = 5w^2 + 3w + 12$$

which is now a function of w alone. The derivative dy/dw is then easily found to be $10w + 3$, the identical answer.

Example 2 If we have a utility function $U = U(c, s)$, where c is the amount of coffee consumed and s is the amount of sugar consumed, and another function $s = g(c)$ indicating the complementarity between these two goods, then we can simply write

$$U = U[c, g(c)]$$

from which it follows that

$$\frac{dU}{dc} = \frac{\partial U}{\partial c} + \frac{\partial U}{\partial g(c)} g'(c)$$

A Variation on the Theme

The situation is only slightly more complicated when we have

$$(8.10) \qquad y = f(x_1, x_2, w) \qquad \text{where} \begin{cases} x_1 = g(w) \\ x_2 = h(w) \end{cases}$$

The channel map will now appear as in Fig. 8.5. This time, the variable w can affect y through three channels: (1) indirectly, via the function g and then f, (2) again indirectly, via the function h and then f, and (3) directly via f. From our previous experience, these three effects are expected to be expressible, respectively, as $\frac{\partial y}{\partial x_1}\frac{dx_1}{dw}$, $\frac{\partial y}{\partial x_2}\frac{dx_2}{dw}$, and $\frac{\partial y}{\partial w}$. This expectation is indeed correct, for when we take the total differential of y, and then divide both sides by dw, we do

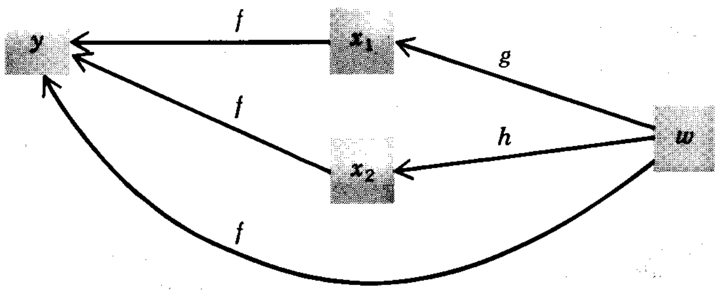


Figure 8.5

get

$$\text{(8.11)} \qquad \frac{dy}{dw} = f_1 \frac{dx_1}{dw} + f_2 \frac{dx_2}{dw} + f_w \frac{dw}{dw}$$

$$= \frac{\partial y}{\partial x_1}\frac{dx_1}{dw} + \frac{\partial y}{\partial x_2}\frac{dx_2}{dw} + \frac{\partial y}{\partial w}$$

which is comparable to (8.9) above.

Example 3 Let the production function be

$$Q = Q(K, L, t)$$

where, aside from the two inputs K and L, there is a third argument t, denoting time. The presence of the t argument indicates that the production function can shift over time in reflection of technological changes. Thus this is a dynamic rather than a static production function. Since capital and labor, too, can change over time, we may write

$$K = K(t) \qquad \text{and} \qquad L = L(t)$$

Then the rate of change of output with respect to time can be expressed, in line with the total-derivative formula (8.11), as

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial K}\frac{dK}{dt} + \frac{\partial Q}{\partial L}\frac{dL}{dt} + \frac{\partial Q}{\partial t}$$

or, in an alternative symbolism,

$$\frac{dQ}{dt} = Q_K K'(t) + Q_L L'(t) + Q_t$$

Another Variation on the Theme

When the ultimate source of change, w in (8.10), is replaced by two coexisting sources, u and v, the situation becomes the following:

$$\text{(8.12)} \qquad y = f(x_1, x_2, u, v) \qquad \text{where} \begin{cases} x_1 = g(u, v) \\ x_2 = h(u, v) \end{cases}$$

While the channel map will now contain more arrows, the principle of its construction remains the same; we shall, therefore, leave it to you to draw. To find the total derivative of y with respect to u (while v is held constant), we may once again resort to taking the total differential of y, and then dividing through by the differential du, with the result:

$$\begin{aligned}\frac{dy}{du} &= \frac{\partial y}{\partial x_1}\frac{dx_1}{du} + \frac{\partial y}{\partial x_2}\frac{dx_2}{du} + \frac{\partial y}{\partial u}\frac{du}{du} + \frac{\partial y}{\partial v}\frac{dv}{du} \\ &= \frac{\partial y}{\partial x_1}\frac{dx_1}{du} + \frac{\partial y}{\partial x_2}\frac{dx_2}{du} + \frac{\partial y}{\partial u} \qquad \left[\frac{dv}{du} = 0 \text{ since } v \text{ is held constant}\right]\end{aligned}$$

In view of the fact that we are varying u while holding v constant (as a single derivative cannot handle changes in u and v both), however, the above result must be modified in two ways: (1) the derivatives dx_1/du and dx_2/du on the right should be rewritten with the partial sign as $\partial x_1/\partial u$ and $\partial x_2/\partial u$, which is in line with the functions g and h in (8.12); and (2) the ratio dy/du on the left should also be interpreted as a *partial* derivative, even though—being derived through the process of total differentiation of y—it is actually in the nature of a *total* derivative. For this reason, we shall refer to it by the explicit name of *partial total derivative*, and denote it by $\S y/\S u$ (with § rather than ∂), in order to distinguish it from the simple partial derivative $\partial y/\partial u$ which, as the above result shows, is but one of three component terms that add up to the partial total derivative.*

With these modifications, our result becomes

$$\frac{\S y}{\S u} = \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial u} + \frac{\partial y}{\partial x_2}\frac{\partial x_2}{\partial u} + \frac{\partial y}{\partial u} \tag{8.13}$$

which is comparable to (8.11). Note the appearance of the symbol $\partial y/\partial u$ on the right, which necessitates the adoption of the new symbol $\S y/\S u$ on the left to indicate the broader concept of a partial total derivative. In a perfectly analogous manner, we can derive the other partial total derivative, $\S y/\S v$. Inasmuch as the roles of u and v are symmetrical in (8.12), however, a simpler alternative is available to us. All we have to do to obtain $\S y/\S v$ is to replace the symbol u in (8.13) by the symbol v throughout.

The use of the new symbols $\S y/\S u$ and $\S y/\S v$ for the partial total derivatives, if unconventional, serves the good purpose of avoiding confusion with the simple partial derivatives $\partial y/\partial u$ and $\partial y/\partial v$ that can arise from the function f alone in (8.12). However, in the special case where the f function takes the form of $y = f(x_1, x_2)$ without the arguments u and v, the simple partial derivatives $\partial y/\partial u$ and $\partial y/\partial v$ are not defined. Hence, it may not be inappropriate in such a

* An alternative way of denoting this partial total derivative is:

$$\left.\frac{dy}{du}\right|_{v \text{ constant}} \quad \text{or} \quad \left.\frac{dy}{du}\right|_{dv=0}$$

case to use the latter symbols for the partial total derivatives of y with respect to u and v, since no confusion is likely to arise. Even in that event, though, the use of a special symbol is advisable for the sake of greater clarity.

Some General Remarks

To conclude this section, we offer three general remarks regarding total derivative and total differentiation:

1. In the cases we have discussed, the situation involves without exception a variable that is functionally dependent on a second variable, which is in turn dependent functionally on a third variable. As a consequence, the notion of a *chain* inevitably enters the picture, as evidenced by the appearance of a product (or products) of two derivative expressions as the component(s) of a total derivative. For this reason, the total-derivative formulas in (8.9), (8.11), and (8.13) can also be regarded as expressions of the chain rule, or the composite-function rule—a more sophisticated version of the chain rule introduced in Sec. 7.3.
2. The chain of derivatives does not have to be limited to only two "links" (two derivatives being multiplied); the concept of total derivative should be extendible to cases where there are three or more links in the composite function.
3. In all cases discussed, total derivatives—including those which have been called *partial total derivatives*—measure rates of change with respect to some *ultimate* variables in the chain or, in other words, with respect to certain variables which are in a sense *exogenous* and which are *not* expressed as functions of some other variables. The essence of the total derivative and of the process of total differentiation is to make due allowance for *all* the channels, indirect as well as direct, through which the effects of a change in an *ultimate* variable can possibly be carried to the particular dependent variable under study.

EXERCISE 8.4

1 Find the total derivative dz/dy, given:
(*a*) $z = f(x, y) = 2x + xy - y^2$, where $x = g(y) = 3y^2$
(*b*) $z = 6x^2 - 3xy + 2y^2$, where $x = 1/y$
(*c*) $z = (x + y)(x - 2y)$, where $x = 2 - 7y$

2 Find the total derivative dz/dt, given:
(*a*) $z = x^2 - 8xy - y^3$, where $x = 3t$ and $y = 1 - t$
(*b*) $z = 3u + vt$, where $u = 2t^2$ and $v = t + 1$
(*c*) $z = f(x, y, t)$, where $x = a + bt$ and $y = c + dt$

3 Find the rate of change of output with respect to time, if the production function is $Q = A(t)K^{\alpha}L^{\beta}$, where $A(t)$ is an increasing function of t, and $K = K_0 + at$, and $L = L_0 + bt$.

4 Find the partial total derivatives $\S W/\S u$ and $\S W/\S v$ if:
(*a*) $W = ax^2 + bxy + cu$, where $x = \alpha u + \beta v$ and $y = \gamma u$
(*b*) $W = f(x_1, x_2)$, where $x_1 = 5u^2 + 3v$ and $x_2 = u - 4v^3$

5 Draw a channel map appropriate to the case of (8.12).

6 Derive the expression for $\S y/\S v$ formally from (8.12) by taking the total differential of y and then dividing through by dv.

8.5 DERIVATIVES OF IMPLICIT FUNCTIONS

The concept of total differentials can also enable us to find the derivatives of so-called "implicit functions."

Implicit Functions

A function given in the form of $y = f(x)$, say,

$$(8.14) \qquad y = f(x) = 3x^4$$

is called an *explicit function*, because the variable y is explicitly expressed as a function of x. If this function is written alternatively in the equivalent form

$$(8.14') \qquad y - 3x^4 = 0$$

however, we no longer have an explicit function. Rather, the function (8.14) is then only *implicitly* defined by the equation (8.14′). When we are (only) given an equation in the form of (8.14′), therefore, the function $y = f(x)$ which it implies, and whose specific form may not even be known to us, is referred to as an *implicit function*.

An equation in the form of (8.14′) can be denoted in general by $F(y, x) = 0$, because its left side is a function of the two variables y and x. Note that we are using the capital letter F here to distinguish it from the function f; the function F, representing the left-side expression in (8.14′), has two arguments, y and x, whereas the function f, representing the implicit function, has only one argument, x. There may, of course, be more than two arguments in the F function. For instance, we may encounter an equation $F(y, x_1, \ldots, x_m) = 0$. Such an equation *may* also define an implicit function $y = f(x_1, \ldots, x_m)$.

The equivocal word "may" in the last sentence was used advisedly. For, whereas an explicit function, say, $y = f(x)$, can always be transformed into an equation $F(y, x) = 0$ by simply transposing the $f(x)$ expression to the left side of the equals sign, the reverse transformation is not always possible. Indeed, in certain cases, a given equation in the form of $F(y, x) = 0$ may not implicitly define a function $y = f(x)$. For instance, the equation $x^2 + y^2 = 0$ is satisfied only at the point of origin (0, 0), and hence yields no meaningful function to speak of. As another example, the equation

$$F(y, x) = x^2 + y^2 - 9 = 0 \tag{8.15}$$

implies not a function, but a relation, because (8.15) plots as a circle, as shown in Fig. 8.6, so that no unique value of y corresponds to each value of x. Note, however, that if we restrict y to nonnegative values, then we will have the upper half of the circle only, and that does constitute a function, namely, $y = +\sqrt{9 - x^2}$. Similarly, the lower half of the circle, with y values nonpositive, constitutes another function, $y = -\sqrt{9 - x^2}$. In contrast, neither the left half nor the right half of the circle can qualify as a function.

In view of this uncertainty, it becomes of interest to ask whether there are known general conditions under which we can be sure that a given equation in the form of

$$F(y, x_1, \ldots, x_m) = 0 \tag{8.16}$$

does indeed define an implicit function

$$y = f(x_1, \ldots, x_m) \tag{8.17}$$

The answer to this lies in the so-called "implicit-function theorem," which states

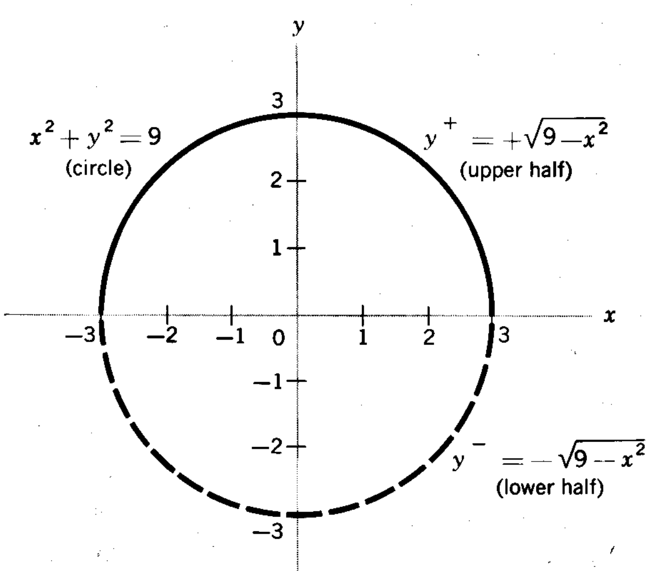


Figure 8.6

that:

> Given (8.16), if (a) the function F has continuous partial derivatives $F_y, F_1, \ldots, F_m$, and if (b) at a point $(y_0, x_{10}, \ldots, x_{m0})$ satisfying the equation (8.16), F_y is nonzero, then there exists an m-dimensional neighborhood of $(x_{10}, \ldots, x_{m0})$, N, in which y is an implicitly defined function of the variables $x_1, \ldots, x_m$, in the form of (8.17). This implicit function satisfies $y_0 = f(x_{10}, \ldots, x_{m0})$. It also satisfies the equation (8.16) for *every* m-tuple $(x_1, \ldots, x_m)$ in the neighborhood N—thereby giving (8.16) the status of an *identity* in that neighborhood. Moreover, the implicit function f is continuous, and has continuous partial derivatives $f_1, \ldots, f_m$.

Let us apply this theorem to the equation of the circle, (8.15), which contains only one x variable. First, we can duly verify that $F_y = 2y$ and $F_x = 2x$ are continuous, as required. Then we note that F_y is nonzero except when $y = 0$, that is, except at the leftmost point $(-3, 0)$ and the rightmost point $(3, 0)$ on the circle. Thus, around any point on the circle except $(-3, 0)$ and $(3, 0)$, we can construct a neighborhood in which the equation (8.15) defines an implicit function $y = f(x)$. This is easily verifiable in Fig. 8.6, where it is indeed possible to draw, say, a rectangle around any point on the circle—except $(-3, 0)$ and $(3, 0)$—such that the portion of the circle enclosed therein will constitute the graph of a function, with a unique y value for each value of x in that rectangle.

Several things should be noted about the implicit-function theorem. First, the conditions cited in the theorem are in the nature of sufficient (but not necessary) conditions. This means that if we happen to find $F_y = 0$ at a point satisfying (8.16), we cannot use the theorem to deny the existence of an implicit function around that point. For such a function may in fact exist (see Exercise 8.5-4).* Second, even if an implicit function f is assured to exist, the theorem gives no clue as to the specific form the function f takes. Nor, for that matter, does it tell us the exact size of the neighborhood N in which the implicit function is defined. However, despite these limitations, this theorem is one of great importance. For whenever the conditions of the theorem are satisfied, it now becomes meaningful to talk about and make use of a function such as (8.17), even if our model may contain an equation (8.16) which is difficult or impossible to solve explicitly for y in terms of the x variables. Moreover, since the theorem also guarantees the existence of the partial derivatives $f_1, \ldots, f_m$, it is now also meaningful to talk about these derivatives of the implicit function.

Derivatives of Implicit Functions

If the equation $F(y, x_1, \ldots, x_m) = 0$ can be solved for y, we can explicitly write out the function $y = f(x_1, \ldots, x_m)$, and find its derivatives by the methods

* On the other hand, if $F = 0$ in an entire neighborhood, then it can be concluded that no implicit function is defined in that neighborhood. By the same token if $F_y = 0$ identically, then no implicit function exists anywhere.

learned before. For instance, (8.15) can be solved to yield two separate functions

$$(8.15') \qquad \begin{aligned} y^+ &= +\sqrt{9 - x^2} \qquad \text{[upper half of circle]} \\ y^- &= -\sqrt{9 - x^2} \qquad \text{[lower half of circle]} \end{aligned}$$

and their derivatives can be found as follows:

$$(8.18) \qquad \begin{aligned} \frac{dy^+}{dx} &= \frac{d}{dx}(9 - x^2)^{1/2} = \tfrac{1}{2}(9 - x^2)^{-1/2}(-2x) \\ &= \frac{-x}{\sqrt{9 - x^2}} = \frac{-x}{y^+} \qquad (y^+ \neq 0) \\ \frac{dy^-}{dx} &= \frac{d}{dx}\left[-(9 - x^2)^{1/2}\right] = -\tfrac{1}{2}(9 - x^2)^{-1/2}(-2x) \\ &= \frac{x}{\sqrt{9 - x^2}} = \frac{-x}{y^-} \qquad (y^- \neq 0) \end{aligned}$$

But what if the given equation, $F(y, x_1, \ldots, x_m) = 0$, cannot be solved for y explicitly? In this case, if under the terms of the implicit-function theorem an implicit function is known to exist, we can still obtain the desired derivatives without having to solve for y first. To do this, we make use of the so-called "implicit-function rule"—a rule that can give us the derivatives of *every* implicit function defined by the given equation. The development of this rule depends on the following basic facts: (1) if two expressions are *identically* equal, their respective total differentials must be equal;* (2) differentiation of an expression that involves $y, x_1, \ldots, x_m$ will yield an expression involving the differentials $dy, dx_1, \ldots, dx_m$; and (3) if we divide dy by dx_1, and let all the other differentials $(dx_2, \ldots, dx_m)$ be zero, the quotient can be interpreted as the partial derivative

* Take, for example, the identity

$$x^2 - y^2 \equiv (x + y)(x - y)$$

This is an identity because the two sides are equal for *any* values of x and y that one may assign. Taking the total differential of each side, we have

$$\begin{aligned} d(\text{left side}) &= 2x\,dx - 2y\,dy \\ d(\text{right side}) &= (x - y)\,d(x + y) + (x + y)\,d(x - y) \\ &= (x - y)(dx + dy) + (x + y)(dx - dy) \\ &= 2x\,dx - 2y\,dy \end{aligned}$$

The two results are indeed equal. If two expressions are *not* identically equal, but are equal only for certain specific values of the variables, however, their total differentials will *not* be equal. The equation

$$x^2 - y^2 = x^2 + y^2 - 2$$

for instance, is valid only for $y = \pm 1$. The total differentials of the two sides are

$$\begin{aligned} d(\text{left side}) &= 2x\,dx - 2y\,dy \\ d(\text{right side}) &= 2x\,dx + 2y\,dy \end{aligned}$$

which are not equal. Note, in particular, that they are not even equal at $y = \pm 1$.

$\partial y/\partial x_1$; similar derivatives can be obtained if we divide dy by dx_2, etc. Applying these facts to the equation $F(y, x_1, \ldots, x_m) = 0$—which, we recall, has the status of an *identity* in the neighborhood N in which the implicit function is defined—we can write $dF = d0$, or

$$F_y\,dy + F_1\,dx_1 + \cdots + F_m\,dx_m = 0$$

Suppose that only y and x_1 are allowed to vary (only dy and dx_1 are *not* set equal to zero). Then the above equation reduces to $F_y\,dy + F_1\,dx_1 = 0$. Upon dividing through by dx_1, and solving for dy/dx_1, we then get

$$\left.\frac{dy}{dx_1}\right|_{\text{other variables constant}} \equiv \frac{\partial y}{\partial x_1} = -\frac{F_1}{F_y}$$

By similar means, we can derive all the other partial derivatives of the implicit function f. These may conveniently be summarized in a general rule—the *implicit-function rule*—as follows: Given $F(y, x_1, \ldots, x_m) = 0$, if an implicit function $y = f(x_1, \ldots, x_m)$ exists, then the partial derivatives of f are

$$\text{(8.19)} \qquad \frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y} \qquad (i = 1, 2, \ldots, m)$$

In the simple case where the given equation is $F(y, x) = 0$, the rule gives

$$\text{(8.19')} \qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

What this rule states is that, even if the specific form of the implicit function is not known to us, we can nevertheless find its derivative(s) by taking the *negative* of the ratio of a pair of partial derivatives of the F function which appears in the given equation that defines the implicit function. Observe that F_y always appears in the denominator of the ratio. This being the case, it is not admissible to have $F_y = 0$. Since the implicit-function theorem specifies that $F_y \neq 0$ at the point around which the implicit function is defined, the problem of a zero denominator is automatically taken care of in the relevant neighborhood of that point.

Example 1 Find dy/dx for the implicit function defined by (8.14'). Since $F(y, x)$ takes the form of $y - 3x^4$, we have, by (8.19'),

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-12x^3}{1} = 12x^3$$

In this particular case, we can easily solve the given equation for y, to get $y = 3x^4$. Thus the correctness of the above derivative is easily verified.

Example 2 Find dy/dx for the implicit functions defined by the equation of the circle (8.15). This time we have $F(y, x) = x^2 + y^2 - 9$; thus $F_y = 2y$ and $F_x = 2x$.

By (8.19′), the desired derivative is

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y} \qquad (y \neq 0)^*$$

Earlier, it was asserted that the implicit-function rule gives us the derivative of *every* implicit function defined by a given equation. Let us verify this with the two functions in (8.15′) and their derivatives in (8.18). If we substitute y^+ for y in the implicit-function-rule result $dy/dx = -x/y$, we will indeed obtain the derivative dy^+/dx as shown in (8.18); similarly, the substitution of y^- for y will yield the other derivative in (8.18). Thus our earlier assertion is duly verified.

Example 3 Find $\partial y/\partial x$ for any implicit function(s) that may be defined by the equation $F(y, x, w) = y^3x^2 + w^3 + yxw - 3 = 0$. This equation is not easily solved for y. But since F_y, F_x, and F_w are all obviously continuous, and since $F_y = 3y^2x^2 + xw$ is indeed nonzero at a point such as (1, 1, 1) which satisfies the given equation, an implicit function $y = f(x, w)$ assuredly exists around that point at least. It is thus meaningful to talk about the derivative $\partial y/\partial x$. By (8.19), moreover, we can immediately write

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{2y^3x + yw}{3y^2x^2 + xw}$$

At the point (1, 1, 1), this derivative has the value $-\frac{3}{4}$.

Example 4 Assume that the equation $F(Q, K, L) = 0$ implicitly defines a production function $Q = f(K, L)$. Let us find a way of expressing the marginal physical products MPP_K and MPP_L in relation to the function F. Since the marginal products are simply the partial derivatives $\partial Q/\partial K$ and $\partial Q/\partial L$, we can apply the implicit-function rule and write

$$\text{MPP}_K \equiv \frac{\partial Q}{\partial K} = -\frac{F_K}{F_Q} \qquad \text{and} \qquad \text{MPP}_L \equiv \frac{\partial Q}{\partial L} = -\frac{F_L}{F_Q}$$

Aside from these, we can obtain yet another partial derivative,

$$\frac{\partial K}{\partial L} = -\frac{F_L}{F_K}$$

from the equation $F(Q, K, L) = 0$. What is the economic meaning of $\partial K/\partial L$? The partial sign implies that the other variable, Q, is being held constant; it follows that the changes in K and L described by this derivative are in the nature of "compensatory" changes designed to keep the output Q constant at a specified level. These are therefore the type of changes pertaining to movements *along* a production *isoquant*. As a matter of fact, the derivative $\partial K/\partial L$ is the measure of the slope of such an isoquant, which is negative in the normal case. The absolute

* The restriction $y \neq 0$ is of course perfectly consistent with our earlier discussion of the equation (8.15) that follows the statement of the implicit-function theorem.

value of $\partial K/\partial L$, on the other hand, is the measure of the *marginal rate of technical substitution* between the two inputs.

Extension to the Simultaneous-Equation Case

The implicit-function theorem also comes in a more general and powerful version that deals with the conditions under which a set of simultaneous equations

$$\begin{aligned} &F^1(y_1, \ldots, y_n; x_1, \ldots, x_m) = 0 \\ &F^2(y_1, \ldots, y_n; x_1, \ldots, x_m) = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &F^n(y_1, \ldots, y_n; x_1, \ldots, x_m) = 0 \end{aligned} \tag{8.20}$$

will assuredly define a set of implicit functions*

$$\begin{aligned} &y_1 = f^1(x_1, \ldots, x_m) \\ &y_2 = f^2(x_1, \ldots, x_m) \\ &\cdots\cdots\cdots\cdots\cdots \\ &y_n = f^n(x_1, \ldots, x_m) \end{aligned} \tag{8.21}$$

The generalized version of the theorem states that:

> Given the equation system (8.20), if (a) the functions $F^1, \ldots, F^n$ all have continuous partial derivatives with respect to all the y and x variables, and if (b) at a point $(y_{10}, \ldots, y_{n0}; x_{10}, \ldots, x_{m0})$ satisfying (8.20), the following Jacobian determinant is nonzero:
>
> $$|J| \equiv \left| \frac{\partial(F^1, \ldots, F^n)}{\partial(y_1, \ldots, y_n)} \right| \equiv \begin{vmatrix} \dfrac{\partial F^1}{\partial y_1} & \dfrac{\partial F^1}{\partial y_2} & \cdots & \dfrac{\partial F^1}{\partial y_n} \\ \dfrac{\partial F^2}{\partial y_1} & \dfrac{\partial F^2}{\partial y_2} & \cdots & \dfrac{\partial F^2}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial F^n}{\partial y_1} & \dfrac{\partial F^n}{\partial y_2} & \cdots & \dfrac{\partial F^n}{\partial y_n} \end{vmatrix} \neq 0$$
>
> then there exists an m-dimensional neighborhood of $(x_{10}, \ldots, x_{m0})$, N, in which the variables $y_1, \ldots, y_n$ are functions of the variables $x_1, \ldots, x_m$ in the form of (8.21). These implicit functions satisfy
>
> $$\begin{aligned} &y_{10} = f^1(x_{10}, \ldots, x_{m0}) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &y_{n0} = f^n(x_{10}, \ldots, x_{m0}) \end{aligned}$$

* To view it another way, what these conditions serve to do is to assure us that the n equations in (8.20) can *in principle* be solved for the n variables—$y_1, \ldots, y_n$—even if we may not be able to obtain the solution (8.21) in an explicit form.

They also satisfy (8.20) for *every* m-tuple $(x_1, \ldots, x_m)$ in the neighborhood N —thereby giving (8.20) the status of a set of *identities* as far as this neighborhood is concerned. Moreover, the implicit functions $f^1, \ldots, f^n$ are continuous and have continuous partial derivatives with respect to all the x variables.

As in the single-equation case, it is possible to find the partial derivatives of the implicit functions directly from the n equations in (8.20), without having to solve them for the y variables. Taking advantage of the fact that, in the neighborhood N, (8.20) has the status of identities, we can take the total differential of each of these, and write $dF^j = 0$ $(j = 1, 2, \ldots, n)$. The result is a set of equations involving the differentials $dy_1, \ldots, dy_n$ and $dx_1, \ldots, dx_m$. Specifically, after transposing the dx_i terms to the right of the equals signs, we have

$$
\begin{aligned}
&\frac{\partial F^1}{\partial y_1} dy_1 + \frac{\partial F^1}{\partial y_2} dy_2 + \cdots + \frac{\partial F^1}{\partial y_n} dy_n \\
&\qquad\qquad = -\left(\frac{\partial F^1}{\partial x_1} dx_1 + \cdots + \frac{\partial F^1}{\partial x_m} dx_m \right) \\
&\frac{\partial F^2}{\partial y_1} dy_1 + \frac{\partial F^2}{\partial y_2} dy_2 + \cdots + \frac{\partial F^2}{\partial y_n} dy_n \\
&\qquad\qquad = -\left(\frac{\partial F^2}{\partial x_1} dx_1 + \cdots + \frac{\partial F^2}{\partial x_m} dx_m \right) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\frac{\partial F^n}{\partial y_1} dy_1 + \frac{\partial F^n}{\partial y_2} dy_2 + \cdots + \frac{\partial F^n}{\partial y_n} dy_n \\
&\qquad\qquad = -\left(\frac{\partial F^n}{\partial x_1} dx_1 + \cdots + \frac{\partial F^n}{\partial x_m} dx_m \right)
\end{aligned}
\tag{8.22}
$$

Since all the partial derivatives appearing in (8.22) will take specific (constant) values when evaluated at the point $(y_{10}, \ldots, y_{n0}; x_{10}, \ldots, x_{m0})$—the point around which the implicit functions (8.21) are defined—we have here a system of n *linear* equations, in which the differentials dy_j (considered to be endogenous) are expressed in terms of the differentials dx_i (considered to be exogenous). Now, suppose that we let all the differentials dx_i be zero except dx_1 (that is, only x_1 is allowed to vary); then all the terms involving $dx_2, \ldots, dx_m$ will drop out of the system. Suppose, further, that we divide each remaining term by dx_1; then there will emerge the expressions $dy_1/dx_1, \ldots, dy_n/dx_1$. These, however, should be interpreted as *partial* derivatives of (8.21) because all the x variables have been held constant except x_1. Thus, by taking the steps just described, we are led to the desired partial derivatives of the implicit functions. Note that, in fact, we can obtain in one fell swoop a total of n of these (here, they are $\partial y_1/\partial x_1, \ldots, \partial y_n/\partial x_1$).

What results from the above-cited steps is the following linear system:

$$
(8.23)\qquad
\begin{aligned}
\frac{\partial F^1}{\partial y_1}\left(\frac{\partial y_1}{\partial x_1}\right) + \frac{\partial F^1}{\partial y_2}\left(\frac{\partial y_2}{\partial x_1}\right) + \cdots + \frac{\partial F^1}{\partial y_n}\left(\frac{\partial y_n}{\partial x_1}\right) &= -\frac{\partial F^1}{\partial x_1} \\
\frac{\partial F^2}{\partial y_1}\left(\frac{\partial y_1}{\partial x_1}\right) + \frac{\partial F^2}{\partial y_2}\left(\frac{\partial y_2}{\partial x_1}\right) + \cdots + \frac{\partial F^2}{\partial y_n}\left(\frac{\partial y_n}{\partial x_1}\right) &= -\frac{\partial F^2}{\partial x_1} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial F^n}{\partial y_1}\left(\frac{\partial y_1}{\partial x_1}\right) + \frac{\partial F^n}{\partial y_2}\left(\frac{\partial y_2}{\partial x_1}\right) + \cdots + \frac{\partial F^n}{\partial y_n}\left(\frac{\partial y_n}{\partial x_1}\right) &= -\frac{\partial F^n}{\partial x_1}
\end{aligned}
$$

where, for visual clarity, we have placed parentheses around those derivatives for which we are seeking a solution, to distinguish them from the other derivatives that are now considered to be constants. In matrix notation, this system can be written as

$$
(8.23')\qquad
\begin{bmatrix}
\frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\
\frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n}
\end{bmatrix}
\begin{bmatrix}
\left(\frac{\partial y_1}{\partial x_1}\right) \\
\left(\frac{\partial y_2}{\partial x_1}\right) \\
\vdots \\
\left(\frac{\partial y_n}{\partial x_1}\right)
\end{bmatrix}
=
\begin{bmatrix}
-\frac{\partial F^1}{\partial x_1} \\
-\frac{\partial F^2}{\partial x_1} \\
\vdots \\
-\frac{\partial F^n}{\partial x_1}
\end{bmatrix}
$$

Since the determinant of the coefficient matrix in (8.23′) is nothing but the particular Jacobian determinant $|J|$ which is known to be nonzero under conditions of the implicit-function theorem, and since the system must be nonhomogeneous (why?), there should be a unique solution to (8.23′). By Cramer's rule, this solution may be expressed analytically as follows:

$$
(8.24)\qquad \left(\frac{\partial y_j}{\partial x_1}\right) = \frac{|J_j|}{|J|} \qquad (j = 1, 2, \ldots, n) \qquad [\text{see } (5.15)]
$$

By a suitable adaptation of this procedure, the partial derivatives of the implicit functions with respect to the other variables, $x_2, \ldots, x_m$, can also be obtained.

Similar to the implicit-function rule (8.19) for the single-equation case, the procedure just described calls only for the use of the partial derivatives of the F functions—evaluated at the point $(y_{10}, \ldots, y_{n0};\ x_{10}, \ldots, x_{m0})$—in the calculation of the partial derivatives of the implicit functions $f^1, \ldots, f^n$. Thus the matrix equation (8.23′) and its analytical solution (8.24) are in effect a statement of the simultaneous-equation version of the implicit-function rule.

Note that the requirement $|J| \neq 0$ rules out a zero denominator in (8.24), just as the requirement $F_y \neq 0$ did in the implicit-function rule (8.19) and (8.19′). Also, the role played by the condition $|J| \neq 0$ in guaranteeing a unique (albeit implicit) solution (8.21) to the general (possibly *nonlinear*) system (8.20) is very similar to the role of the nonsingularity condition $|A| \neq 0$ in a *linear* system $Ax = d$.

Example 5 Let the national-income model (7.17) be rewritten in the form

$$
\begin{aligned}
Y - C - I_0 - G_0 &= 0 \\
C - \alpha - \beta(Y - T) &= 0 \qquad (8.25)\\
T - \gamma - \delta Y &= 0
\end{aligned}
$$

If we take the endogenous variables (Y, C, T) to be (y_1, y_2, y_3), and take the exogenous variables and parameters $(I_0, G_0, \alpha, \beta, \gamma, \delta)$ to be $(x_1, x_2, \ldots, x_6)$, then the left-side expression in each equation can be regarded as a specific F function, in the form of $F^j(Y, C, T;\ I_0, G_0, \alpha, \beta, \gamma, \delta)$. Thus (8.25) is a specific case of (8.20), with $n = 3$ and $m = 6$. Since the functions F^1, F^2, and F^3 do have continuous partial derivatives, and since the relevant Jacobian determinant (the one involving only the endogenous variables),

$$
(8.26) \qquad |J| = \begin{vmatrix} \dfrac{\partial F^1}{\partial Y} & \dfrac{\partial F^1}{\partial C} & \dfrac{\partial F^1}{\partial T} \\[2ex] \dfrac{\partial F^2}{\partial Y} & \dfrac{\partial F^2}{\partial C} & \dfrac{\partial F^2}{\partial T} \\[2ex] \dfrac{\partial F^3}{\partial Y} & \dfrac{\partial F^3}{\partial C} & \dfrac{\partial F^3}{\partial T} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{vmatrix} = 1 - \beta + \beta\delta
$$

is always nonzero (both β and δ being restricted to be positive fractions), we can take Y, C and T to be implicit functions of $(I_0, G_0, \alpha, \beta, \gamma, \delta)$ *at* and *around* any point that satisfies (8.25). But a point that satisfies (8.25) would be an equilibrium solution, relating to $\overline{Y}$, $\overline{C}$ and $\overline{T}$. Hence, what the implicit-function theorem tells us is that we are justified in writing

$$
\begin{aligned}
\overline{Y} &= f^1(I_0, G_0, \alpha, \beta, \gamma, \delta) \\
\overline{C} &= f^2(I_0, G_0, \alpha, \beta, \gamma, \delta) \\
\overline{T} &= f^3(I_0, G_0, \alpha, \beta, \gamma, \delta)
\end{aligned}
$$

indicating that the equilibrium values of the endogenous variables are implicit functions of the exogenous variables and the parameters.

The partial derivatives of the implicit functions, such as $\partial\overline{Y}/\partial I_0$ and $\partial\overline{Y}/\partial G_0$, are in the nature of comparative-static derivatives. To find these, we need only the partial derivatives of the F functions, evaluated at the equilibrium state of the model. Moreover, since $n = 3$, three of these can be found in one operation. Suppose we now hold all exogenous variables and parameters fixed except G_0. Then, by adapting the result in (8.23′), we may write the equation

$$
\begin{bmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{bmatrix} \begin{bmatrix} \partial\overline{Y}/\partial G_0 \\ \partial\overline{C}/\partial G_0 \\ \partial\overline{T}/\partial G_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

from which three comparative-static derivatives (all with respect to G_0) can be calculated. The first one, representing the government-expenditure multiplier, will for instance come out to be

$$\frac{\partial \overline{Y}}{\partial G_0} = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{vmatrix}}{|J|} = \frac{1}{1 - \beta + \beta\delta} \qquad \text{[by (8.26)]}$$

This is, of course, nothing but the result obtained earlier in (7.19). Note, however. that in the present approach we have worked only with implicit functions, and have completely bypassed the step of solving the system (8.25) explicitly for $\overline{Y}$. $\overline{C}$. and $\overline{T}$. It is this particular feature of the method that will now enable us to tackle the comparative statics of general-function models which, by their very nature. can yield no explicit solution.

EXERCISE 8.5

1 Assuming that the equation $F(U, x_1, x_2, \ldots, x_n) = 0$ implicitly defines a utility function $U = f(x_1, x_2, \ldots, x_n)$:

(a) Find the expressions for $\partial U/\partial x_2$, $\partial U/\partial x_n$, $\partial x_3/\partial x_2$, and $\partial x_4/\partial x_n$.

(b) Interpret their respective economic meanings.

2 Given the equation $F(y, x) = 0$ shown below, is an implicit function $y = f(x)$ defined around the point $(y = 3, x = 1)$?

(a) $x^3 - 2x^2y + 3xy^2 - 22 = 0$

(b) $2x^2 + 4xy - y^4 + 67 = 0$

If your answer is affirmative, find dy/dx by the implicit-function rule, and evaluate it at the said point.

3 Given $x^2 + 3xy + 2yz + y^2 + z^2 - 11 = 0$, is an implicit function $z = f(x, y)$ defined around the point $(x = 1, y = 2, z = 0)$? If so, find $\partial z/\partial x$ and $\partial z/\partial y$ by the implicit-function rule, and evaluate them at that point.

4 By considering the equation $F(y, x) = (x - y)^3 = 0$ in a neighborhood around the point of origin, prove that the conditions cited in the implicit-function theorem are *not* in the nature of *necessary* conditions.

5 If the equation $F(x, y, z) = 0$ implicitly defines each of the three variables as a function of the other two variables, and if all the derivatives in question exist, find the value of $\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}$.

6 Justify the assertion in the text that the equation system (8.23′) must be nonhomogeneous.

7 From the national-income model (8.25), find the nonincome-tax multiplier and the income-tax-rate multiplier by the implicit-function rule. Check your results against (7.20) and (7.21).

8.6 COMPARATIVE STATICS OF GENERAL-FUNCTION MODELS

When we first considered the problem of comparative-static analysis in Chap. 7, we dealt with the case where the equilibrium values of the endogenous variables of the model are expressible explicitly in terms of the exogenous variables and parameters. There, the technique of simple partial differentiation was all we needed. When a model contains functions expressed in the general form, however, that technique becomes inapplicable because of the unavailability of explicit solutions. Instead, a new technique must be employed that makes use of such concepts as total differentials, total derivatives, as well as the implicit-function theorem and the implicit-function rule. We shall illustrate this first with a market model, and then move on to a national-income model.

Market Model

Consider a single-commodity market, where the quantity demanded Q_d is a function not only of price P but also of an exogenously determined income Y_0. The quantity supplied Q_s, on the other hand, is a function of price alone. If these functions are not given in specific forms, our model may be written generally as follows:

$$\begin{aligned} &Q_d = Q_s \\ (8.27) \qquad &Q_d = D(P, Y_0) \qquad (\partial D/\partial P < 0;\ \partial D/\partial Y_0 > 0) \\ &Q_s = S(P) \qquad (dS/dP > 0) \end{aligned}$$

Both the D and S functions are assumed to possess continuous derivatives or, in other words, to have smooth graphs. Moreover, in order to ensure economic relevance, we have imposed definite restrictions on the signs of these derivatives. By the restriction $dS/dP > 0$, the supply function is stipulated to be monotonically increasing, although it is permitted to be either linear or nonlinear. Similarly, by the restrictions on the two partial derivatives of the demand function, we indicate that it is a decreasing function of price but an increasing function of income. These restrictions serve to confine our analysis to the "normal" case we expect to encounter.

In drawing the usual type of two-dimensional demand curve, the income level is assumed to be held fixed. When income changes, it will upset a given equilibrium by causing a shift of the demand curve. Similarly, in (8.27), Y_0 can cause a disequilibrating change through the demand function. Here, Y_0 is the only exogenous variable or parameter; thus the comparative-static analysis of this model will be concerned exclusively with how a change in Y_0 will affect the equilibrium position of the model.

The equilibrium position of the market is defined by the equilibrium condition $Q_d = Q_s$, which, upon substitution and rearrangement, can be expressed by

$$(8.28) \qquad D(P, Y_0) - S(P) = 0$$

Even though this equation cannot be solved explicitly for the equilibrium price $\bar{P}$. we shall assume that there does exist a static equilibrium—for otherwise there would be no point in even raising the question of comparative statics. From our experience with specific-function models, we have learned to expect $\bar{P}$ to be a function of the exogenous variable Y_0:

$$\bar{P} = \bar{P}(Y_0) \tag{8.29}$$

But now we can provide a rigorous foundation for this expectation by appealing to the implicit-function theorem. Inasmuch as (8.28) is in the form of $F(P, Y_0) = 0$. the satisfaction of the conditions of the implicit-function theorem will guarantee that every value of Y_0 will yield a unique value of $\bar{P}$ in the neighborhood of a point satisfying (8.28), that is, in the neighborhood of an (initial or "old") equilibrium solution. In that case, we can indeed write the implicit function $\bar{P} = \bar{P}(Y_0)$ and discuss its derivative, $d\bar{P}/dY_0$—the very comparative-static derivative we desire—which is known to exist. Let us, therefore, check those conditions. First, the function $F(P, Y_0)$ indeed possesses continuous derivatives; this is because, by assumption, its two additive components $D(P, Y_0)$ and $S(P)$ have continuous derivatives. Second, the partial derivative of F with respect to P. namely, $F_P = \partial D/\partial P - dS/dP$, is negative, and hence nonzero, no matter where it is evaluated. Thus the implicit-function theorem applies, and (8.29) is indeed legitimate.

According to the same theorem, the equilibrium condition (8.28) can now be taken to be an identity in some neighborhood of the equilibrium solution. Consequently, we may write the equilibrium identity

$$D(\bar{P}, Y_0) - S(\bar{P}) \equiv 0 \tag{8.30}$$

It then requires only a straight application of the implicit-function rule to produce the comparative-static derivative, $d\bar{P}/dY_0$, which, for visual clarity, we shall from here on enclose in parentheses to distinguish it from the regular derivative expressions that merely constitute part of the model specification. The result is

$$\left(\frac{d\bar{P}}{dY_0}\right) = -\frac{\partial F/\partial Y_0}{\partial F/\partial \bar{P}} = -\frac{\partial D/\partial Y_0}{\partial D/\partial \bar{P} - dS/d\bar{P}} > 0 \tag{8.31}$$

In this result, the expression $\partial D/\partial \bar{P}$ refers to the derivative $\partial D/\partial P$ evaluated at the initial equilibrium, i.e., at $P = \bar{P}$; a similar interpretation attaches to $dS/d\bar{P}$. In fact, $\partial D/\partial Y_0$ must be evaluated at the equilibrium point as well. By virtue of the sign specifications in (8.27), $(d\bar{P}/dY_0)$ is invariably positive. Thus our *qualitative* conclusion is that an increase (decrease) in the income level will always result in an increase (decrease) in the equilibrium price. If the values which the derivatives of the demand and supply functions take at the initial equilibrium are known, (8.31) will, of course, yield a *quantitative* conclusion also.

The above discussion is concerned with the effect of a change in Y_0 on $\bar{P}$. Is it possible also to find out the effect on the equilibrium quantity $\bar{Q}$ $(= \bar{Q}_d = \bar{Q}_s)$? The answer is yes. Since, in the equilibrium state, we have $\bar{Q} = S(\bar{P})$, and since

$\bar{P} = \bar{P}(Y_0)$, we may apply the chain rule to get the derivative

$$\left(\frac{d\bar{Q}}{dY_0}\right) = \frac{dS}{d\bar{P}}\left(\frac{d\bar{P}}{dY_0}\right) > 0 \qquad \left[\text{since } \frac{dS}{d\bar{P}} > 0\right] \tag{8.32}$$

Thus the equilibrium quantity is also positively related to Y_0 in this model. Again, (8.32) can supply a quantitative conclusion if the values which the various derivatives take at the equilibrium are known.

The results in (8.31) and (8.32), which exhaust the comparative-static contents of the model (since the latter contains only one exogenous and two endogenous variables), are not surprising. In fact, they convey no more than the proposition that an upward shift of the demand curve will result in a higher equilibrium price as well as a higher equilibrium quantity. This same proposition, it may seem, could have been arrived at in a flash from a simple graphic analysis! This sounds plausible, but one should not lose sight of the far, far more general character of the analytical procedure we have used here. The graphic analysis, let us reiterate, is by its very nature limited to a specific set of curves (the geometric counterpart of a specific set of functions); its conclusions are therefore, strictly speaking, relevant and applicable to only that set of curves. In sharp contrast, the formulation in (8.27), simplified as it is, covers the entire set of possible combinations of negatively sloped demand curves and positively sloped supply curves. Thus it is vastly more general. Also, the analytical procedure used here can handle many problems of greater complexity that would prove to be beyond the capabilities of the graphic approach.

Simultaneous-Equation Approach

The above analysis of model (8.27) was carried out on the basis of a single equation, namely, (8.30). Since only one endogenous variable can fruitfully be incorporated into one equation, the inclusion of $\bar{P}$ means the exclusion of $\bar{Q}$. As a result, we were compelled to find $(d\bar{P}/dY_0)$ first and then to infer $(d\bar{Q}/dY_0)$ in a subsequent step. Now we shall show how $\bar{P}$ and $\bar{Q}$ can be studied simultaneously. As there are two endogenous variables, we shall accordingly set up a two-equation system. First, letting $Q = Q_d = Q_s$ in (8.27) and rearranging, we can express our market model as

$$\begin{aligned} F^1(P, Q; Y_0) &= D(P, Y_0) - Q = 0 \\ F^2(P, Q; Y_0) &= S(P) - Q = 0 \end{aligned} \tag{8.33}$$

which is in the form of (8.20), with $n = 2$ and $m = 1$. It becomes of interest, once again, to check the conditions of the implicit-function theorem. First, since the demand and supply functions are both assumed to possess continuous derivatives, so must the functions F^1 and F^2. Second, the endogenous-variable Jacobian (the one involving P and Q) indeed turns out to be nonzero, regardless of where it is evaluated, because

$$(8.34)\qquad |J| = \begin{vmatrix} \dfrac{\partial F^1}{\partial P} & \dfrac{\partial F^1}{\partial Q} \\ \dfrac{\partial F^2}{\partial P} & \dfrac{\partial F^2}{\partial Q} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial D}{\partial P} & -1 \\ \dfrac{dS}{dP} & -1 \end{vmatrix} = \frac{dS}{dP} - \frac{\partial D}{\partial P} > 0$$

Hence, if an equilibrium solution $(\bar{P}, \bar{Q})$ exists (as we must assume in order to make it meaningful to talk about comparative statics), the implicit-function theorem tells us that we can write the implicit functions

$$(8.35)\qquad \bar{P} = \bar{P}(Y_0) \qquad \text{and} \qquad \bar{Q} = \bar{Q}(Y_0)$$

even though we cannot solve for $\bar{P}$ and $\bar{Q}$ explicitly. These functions are known to have continuous derivatives. Moreover, (8.33) will have the status of a pair of identities in some neighborhood of the equilibrium state, so that we may also write

$$(8.36)\qquad \begin{aligned} D(\bar{P}, Y_0) - \bar{Q} &\equiv 0 \\ S(\bar{P}) - \bar{Q} &\equiv 0 \end{aligned}$$

From these, $(d\bar{P}/dY_0)$ and $(d\bar{Q}/dY_0)$ can be found simultaneously.

These two derivatives have as their ingredients the differentials $d\bar{P}$, $d\bar{Q}$, and dY_0. To bring these differential expressions into the picture, we differentiate each identity in (8.36) in turn. The result, upon rearrangement, is a linear system in $d\bar{P}$ and $d\bar{Q}$:

$$\begin{aligned} \frac{\partial D}{\partial \bar{P}} d\bar{P} - d\bar{Q} &= -\frac{\partial D}{\partial Y_0} dY_0 \\ \frac{dS}{d\bar{P}} d\bar{P} - d\bar{Q} &= 0 \end{aligned}$$

This system is linear because $d\bar{P}$ and $d\bar{Q}$ (the variables) both appear in the first degree, and the coefficient derivatives (all to be evaluated at the initial equilibrium) and dY_0 (an arbitrary, nonzero change in the exogenous variable) all represent specific constants. Upon dividing through by dY_0 and interpreting the quotient of two differentials as a derivative, we have the matrix equation*

$$\begin{bmatrix} \dfrac{\partial D}{\partial \bar{P}} & -1 \\ \dfrac{dS}{d\bar{P}} & -1 \end{bmatrix} \begin{bmatrix} \left(\dfrac{d\bar{P}}{dY_0}\right) \\ \left(\dfrac{d\bar{Q}}{dY_0}\right) \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial D}{\partial Y_0} \\ 0 \end{bmatrix}$$

* Without going through the steps of total differentiation and division by dY_0, the same matrix equation can be obtained from an adaptation of the implicit-function rule (8.23′).

By Cramer's rule, and using (8.34), we then find the solution to be

(8.37)
$$\left(\frac{d\bar{P}}{dY_0}\right) = \frac{\begin{vmatrix} -\frac{\partial D}{\partial Y_0} & -1 \\ 0 & -1 \end{vmatrix}}{|J|} = \frac{\frac{\partial D}{\partial Y_0}}{|J|}$$

$$\left(\frac{d\bar{Q}}{dY_0}\right) = \frac{\begin{vmatrix} \frac{\partial D}{\partial \bar{P}} & -\frac{\partial D}{\partial Y_0} \\ \frac{dS}{d\bar{P}} & 0 \end{vmatrix}}{|J|} = \frac{\frac{dS}{d\bar{P}}\frac{\partial D}{\partial Y_0}}{|J|}$$

where all the derivatives of the demand and supply functions (including those appearing in the Jacobian) are to be evaluated at the initial equilibrium. You can check that the results just obtained are identical with those obtained earlier in (8.31) and (8.32), by means of the single-equation approach.

Use of Total Derivatives

Both the single-equation and the simultaneous-equation approaches illustrated above have one feature in common: we take the *total differentials* of both sides of an equilibrium identity and then equate the two results. Instead of taking the total differentials, however, it is possible to take, and equate, the *total derivatives* of the two sides of the equilibrium identity with respect to a particular exogenous variable or parameter.

In the single-equation approach, for instance, the equilibrium identity is

$$D(\bar{P}, Y_0) - S(\bar{P}) \equiv 0 \qquad [\text{from } (8.30)]$$
$$\text{where} \qquad \bar{P} = \bar{P}(Y_0) \qquad [\text{from } (8.29)]$$

Taking the total derivative of the equilibrium identity with respect to Y_0—which takes into account the indirect as well as the direct effects of a change in Y_0—will therefore give us the equation

$$\underbrace{\frac{\partial D}{\partial \bar{P}}\left(\frac{d\bar{P}}{dY_0}\right)}_{\text{(indirect effect of } Y_0 \text{ on } D)} + \underbrace{\frac{\partial D}{\partial Y_0}}_{\text{(direct effect of } Y_0 \text{ on } D)} - \underbrace{\frac{dS}{d\bar{P}}\left(\frac{d\bar{P}}{dY_0}\right)}_{\text{(indirect effect of } Y_0 \text{ on } S)} = 0$$

When this is solved for $(d\bar{P}/dY_0)$, the result is identical with the one in (8.31).

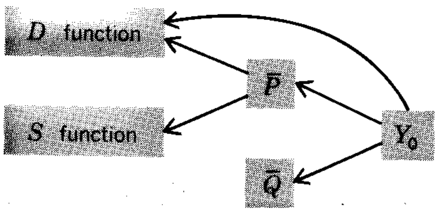


Figure 8.7

In the simultaneous-equation approach, on the other hand, there is a pair of equilibrium identities:

$$D(\bar{P}, Y_0) - \bar{Q} \equiv 0$$
$$S(\bar{P}) - \bar{Q} \equiv 0 \qquad \text{[from (8.36)]}$$
$$\text{where} \quad \bar{P} = \bar{P}(Y_0) \qquad \bar{Q} = \bar{Q}(Y_0) \qquad \text{[from (8.35)]}$$

The various effects of Y_0 are now harder to keep track of, but with the help of the channel map in Fig. 8.7, the pattern should become clear. This channel map tells us, for instance, that when differentiating the D function with respect to Y_0, we must allow for the indirect effect of Y_0 upon D through $\bar{P}$, as well as the direct effect of Y_0 (curved arrow). In differentiating the S function with respect to Y_0, on the other hand, there is only the indirect effect (through $\bar{P}$) to be taken into account. Thus the result of totally differentiating the two identities with respect to Y_0 is, upon rearrangement, the following pair of equations:

$$\frac{\partial D}{\partial \bar{P}}\left(\frac{d\bar{P}}{dY_0}\right) - \left(\frac{d\bar{Q}}{dY_0}\right) = -\frac{\partial D}{\partial Y_0}$$
$$\frac{dS}{d\bar{P}}\left(\frac{d\bar{P}}{dY_0}\right) - \left(\frac{d\bar{Q}}{dY_0}\right) = 0$$

These are, of course, identical with the equations obtained by the total-differential method, and they lead again to the comparative-static derivatives in (8.37).

National-Income Model

The procedure just illustrated will now be applied to a national-income model, also to be formulated in terms of general functions. This time, for the sake of variety, let us abstract from government expenditures and taxes and, instead, add foreign trade relations into the model. Furthermore, let us include the money market along with the market for goods.

More specifically, the *goods market* will be assumed to be characterized by the following four functions:

1. Investment expenditure I is a decreasing function of interest rate i:

$$I = I(i) \qquad (I' < 0)$$

where $I' \equiv dI/di$ is the derivative of the investment function.

2. Saving S is an increasing function of national income Y as well as interest rate i, with the marginal propensity to save being a positive fraction:

   $$S = S(Y, i) \qquad (0 < S_Y < 1; \quad S_i > 0)$$

   where $S_Y \equiv \partial S/\partial Y$ (marginal propensity to save) and $S_i \equiv \partial S/\partial i$ are the partial derivatives.
3. The expenditure on imports M is a function of national income, with the marginal propensity to import being another positive fraction:

   $$M = M(Y) \qquad (0 < M' < 1)$$

4. The level of exports X is exogenously determined:

   $$X = X_0$$

In the *money market*, we have two more functions as follows:

5. The quantity demanded of money M_d is an increasing function of national income (*transactions demand*) but a decreasing function of interest rate (*speculative demand*):

   $$M_d = L(Y, i) \qquad (L_Y > 0; \quad L_i < 0)$$

   The function symbol L is employed here because the money demand function is customarily referred to as the *liquidity function*. The symbol M_d, representing money demand, should be carefully distinguished from the symbol M, for imports.
6. The money supply is exogenously determined, as a matter of *monetary policy*:

   $$M_s = M_{s0}$$

Note that I, S, M, and X, representing *flow* concepts, are all measured *per period of time*, as is Y. On the other hand, M_d and M_s are *stock* concepts, and they indicate quantities in existence at some specific *point of time*. Whether stock or flow, all the above functions are assumed to have continuous derivatives.

The attainment of equilibrium in this model requires the simultaneous satisfaction of the equilibrium condition of the goods market (injections = leakages, or $I + X = S + M$) as well as that of the money market (demand for money = supply of money, or $M_d = M_s$). On the basis of the general functions cited above, the equilibrium state may be described by the following pair of conditions:

$$\begin{aligned} I(i) + X_0 &= S(Y, i) + M(Y) \\ L(Y, i) &= M_{s0} \end{aligned} \tag{8.38}$$

Since the symbols I, S, M, and L can be viewed as function symbols, we have in effect only two endogenous variables, income Y and interest rate i, plus two exogenous variables, exports X_0 (based on foreign decisions) and M_{s0} (determined by the monetary authorities). Thus (8.38) can be expressed in the form of (8.20), with $n = m = 2$:

$$(8.38') \qquad \begin{aligned} F^1(Y, i; X_0, M_{s0}) &= I(i) + X_0 - S(Y, i) - M(Y) = 0 \\ F^2(Y, i; X_0, M_{s0}) &= L(Y, i) - M_{s0} = 0 \end{aligned}$$

This system satisfies the conditions of the implicit-function theorem, because (1) F^1 and F^2 have continuous derivatives (since all the component functions therein have continuous derivatives by assumption) and (2) the endogenous-variable Jacobian is nonzero when evaluated at the initial equilibrium (which we assume to exist) as well as elsewhere:

$$(8.39) \qquad |J| = \begin{vmatrix} \partial F^1/\partial Y & \partial F^1/\partial i \\ \partial F^2/\partial Y & \partial F^2/\partial i \end{vmatrix} = \begin{vmatrix} -S_Y - M' & I' - S_i \\ L_Y & L_i \end{vmatrix}$$
$$= -L_i(S_Y + M') - L_Y(I' - S_i) > 0$$

Hence the implicit functions

$$(8.40) \qquad \bar{Y} = \bar{Y}(X_0, M_{s0}) \qquad \text{and} \qquad \bar{i} = \bar{i}(X_0, M_{s0})$$

can be written, even though we are unable to solve for $\bar{Y}$ and $\bar{i}$ explicitly. Furthermore, we may take (8.38′) to be a pair of identities in some neighborhood of the equilibrium, so that we may also write

$$(8.41) \qquad \begin{aligned} I(\bar{i}) + X_0 - S(\bar{Y}, \bar{i}) - M(\bar{Y}) &\equiv 0 \\ L(\bar{Y}, \bar{i}) - M_{s0} &\equiv 0 \end{aligned}$$

From these equilibrium identities, a total of four comparative-static derivatives will emerge, two relating to X_0 and the other two relating to M_{s0}. But we shall derive here only the former two, leaving the other two to be derived by you as an exercise.

Accordingly, after taking the total differential of each identity in (8.41), we set dM_{s0} equal to zero, so that dX_0 will be the sole disequilibrating factor. Next, dividing through by dX_0, and interpreting each quotient of two differentials as a partial derivative (partial, because the other exogenous variable M_{s0} is being held constant), we arrive at the matrix equation

$$(8.42) \qquad \begin{bmatrix} -S_Y - M' & I' - S_i \\ L_Y & L_i \end{bmatrix} \begin{bmatrix} (\partial\bar{Y}/\partial X_0) \\ (\partial\bar{i}/\partial X_0) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

The solution is, by Cramer's rule and using (8.39),

$$(8.43) \qquad \begin{aligned} \left(\frac{\partial\bar{Y}}{\partial X_0}\right) &= \frac{\begin{vmatrix} -1 & I' - S_i \\ 0 & L_i \end{vmatrix}}{|J|} = \frac{-L_i}{|J|} > 0 \\ \left(\frac{\partial\bar{i}}{\partial X_0}\right) &= \frac{\begin{vmatrix} -S_Y - M' & -1 \\ L_Y & 0 \end{vmatrix}}{|J|} = \frac{L_Y}{|J|} > 0 \end{aligned}$$

where all the derivatives on the right side of the equals sign (including those appearing in the Jacobian) are to be evaluated at the initial equilibrium, that is, at $Y = \bar{Y}$ and $i = \bar{i}$. When the specific values of these derivatives are known, (8.43) yields quantitative conclusions regarding the effect of a change in exports. Without the knowledge of those values, however, we must settle for the qualitative conclusions that both $\bar{Y}$ and $\bar{i}$ will increase with exports in the present model.

As in the market model, instead of using total differentials, the option is open to us to take the *total derivatives* of the equilibrium identities in (8.41) with respect to the particular exogenous variable under study, X_0. In doing so, we must, of course, bear in mind the implicit solutions (8.40). The various ways in which X_0 can affect the different components of the model—as given in (8.41) and (8.40)—are summarized in the channel map in Fig. 8.8. It should be noted, in particular, that in differentiating the saving function or the liquidity function with respect to X_0, we must allow for *two* indirect effects—one through $\bar{i}$ and the other through $\bar{Y}$. With the help of this channel map, we can differentiate the equilibrium identities totally with respect to X_0, to get the following pair of equations:

$$I'\left(\frac{\partial \bar{i}}{\partial X_0}\right) + 1 - S_Y\left(\frac{\partial \bar{Y}}{\partial X_0}\right) - S_i\left(\frac{\partial \bar{i}}{\partial X_0}\right) - M'\left(\frac{\partial \bar{Y}}{\partial X_0}\right) = 0$$

$$L_Y\left(\frac{\partial \bar{Y}}{\partial X_0}\right) + L_i\left(\frac{\partial \bar{i}}{\partial X_0}\right) = 0$$

Since the other exogenous variable, M_{s0}, is being held constant, the left side of each of these equations represents the *partial total* derivative of the left-side expression in the corresponding equilibrium identity. However, the comparative-static derivatives $(\partial\bar{Y}/\partial X_0)$ and $(\partial\bar{i}/\partial X_0)$, being derivatives of the implicit functions (8.40), are just plain *partial* derivatives. When properly condensed, these

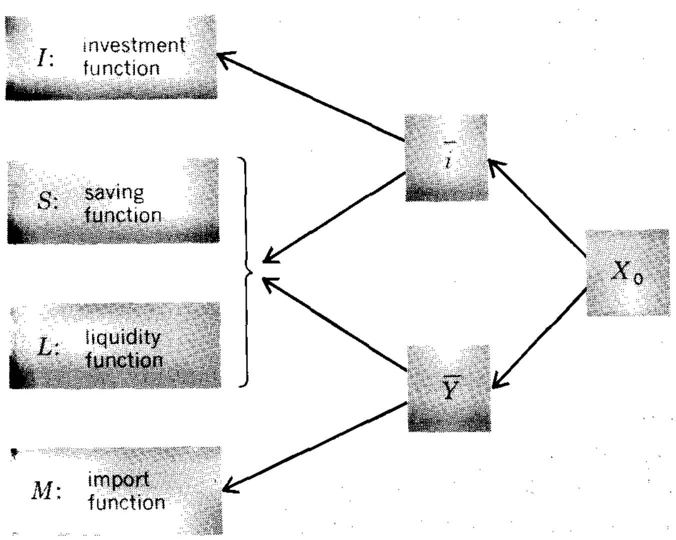


Figure 8.8

two equations reduce exactly to (8.42). So the total-differential method and the total-derivative method yield identical results.

You will observe that $(\partial\bar{Y}/\partial X_0)$ is in the nature of an export multiplier. Since the export-induced increase in the equilibrium income will, by virtue of the import function $M = M(Y)$, cause imports to rise as well, we can again apply the chain rule to find the (auxiliary) comparative-static derivative:

$$\left(\frac{\partial\bar{M}}{\partial X_0}\right) = M'\left(\frac{\partial\bar{Y}}{\partial X_0}\right) = \frac{-M'L_i}{|J|}$$

The sign of this derivative is positive because $M' > 0$. By a perfectly analogous procedure, we can also find the other auxiliary comparative-static derivatives. such as $(\partial\bar{I}/\partial X_0)$ and $(\partial\bar{S}/\partial X_0)$.

Summary of the Procedure

In the analysis of the general-function market model and national-income model, it is not possible to obtain explicit solution values of the endogenous variables. Instead, we rely on the implicit-function theorem to enable us to write the implicit solutions such as

$$\bar{P} = \bar{P}(Y_0) \qquad \text{and} \qquad \bar{i} = \bar{i}(X_0, M_{s0})$$

Our subsequent search for the comparative-static derivatives such as $(d\bar{P}/dY_0)$ and $(\partial\bar{i}/\partial X_0)$ then rests for its meaningfulness upon the known fact—thanks again to the implicit-function theorem—that the $\bar{P}$ and $\bar{i}$ functions do possess continuous derivatives.

To facilitate the application of that theorem, we make it a standard practice to write the equilibrium condition(s) of the model in the form of (8.16) or (8.20). We then check whether (1) the F function(s) have continuous derivatives and (2) the value of F_y or the endogenous-variable Jacobian determinant (as the case may be) is nonzero at the initial equilibrium of the model. However, as long as the individual functions in the model have continuous derivatives—an assumption which is often adopted as a matter of course in general-function models—the first condition above is automatically satisfied. As a practical matter, therefore, it is needed only to check the value of F_y or the endogenous-variable Jacobian. And if it is nonzero at the equilibrium, we may proceed at once to the task of finding the comparative-static derivatives.

To that end, the implicit-function rule is of help. For the single-equation case, simply set the endogenous variables equal to its equilibrium value (e.g., set $P = \bar{P}$) in the equilibrium condition, and then apply the rule as stated in (8.19) to the resulting equilibrium identity. For the simultaneous-equation case, we must also first set all endogenous variables equal to their respective equilibrium values in the equilibrium conditions. Then we can either apply the implicit-function rule as illustrated in (8.24) to the resulting equilibrium identities, or carry out the several steps outlined below:

1. Take the total differential of each equilibrium identity in turn.
2. Select one, and only one, exogenous variable (say, X_0) as the sole disequilibrating factor, and set the differentials of *all other* exogenous variables equal to zero. Then divide all remaining terms in each identity by dX_0, and interpret each quotient of two differentials as a comparative-static derivative —a *partial* one if the model contains two or more exogenous variables.*
3. Solve the resulting equation system for the comparative-static derivatives appearing therein, and interpret their economic implications. In this step, if Cramer's rule is used, we can take advantage of the fact that, earlier, in checking the condition $|J| \neq 0$, we have in fact already calculated the determinant of the coefficient matrix of the equation system now being solved.
4. For the analysis of another disequilibrating factor (another exogenous variable), if any, repeat steps 2 and 3. Although a different group of comparative-static derivatives will emerge in the new equation system, the coefficient matrix will be the same as before, and thus the known value of $|J|$ can again be put to use.

Given a model with m exogenous variables, it will take exactly m applications of the above-described procedure to catch all the comparative-static derivatives there are.

EXERCISE 8.6

1 Let the equilibrium condition for national income be

$$S(Y) + T(Y) = I(Y) + G_0 \qquad (S', T', I' > 0; \quad S' + T' > I')$$

where S, Y, T, I, and G stand for saving, national income, taxes, investment, and government expenditure, respectively. All derivatives are continuous.

(a) Interpret the economic meanings of the derivatives S', T', and I'.

(b) Check whether the conditions of the implicit-function theorem are satisfied. If so, write the equilibrium identity.

(c) Find $(d\overline{Y}/dG_0)$ and discuss its economic implications.

2 Let the demand and supply functions for a commodity be

$$Q_d = D(P, Y_0) \qquad (D_p < 0; \quad D_{Y_0} > 0)$$

$$Q_s = S(P, T_0) \qquad (S_p > 0; \quad S_{T_0} < 0)$$

* Instead of taking steps 1 and 2, we may equivalently resort to the total-derivative method by differentiating (both sides of) each equilibrium identity totally with respect to the selected exogenous variable. In so doing, a channel map will prove to be of help.

where Y_0 is income and T_0 is the tax on the commodity. All derivatives are continuous.

(a) Write the equilibrium condition in a single equation.

(b) Check whether the implicit-function theorem is applicable. If so, write the equilibrium identity.

(c) Find $(\partial \bar{P}/\partial Y_0)$ and $(\partial \bar{P}/\partial T_0)$, and discuss their economic implications.

(d) Using a procedure similar to (8.32), find $(\partial \bar{Q}/\partial Y_0)$ from the supply function and $(\partial \bar{Q}/\partial T_0)$ from the demand function. (Why not use the demand function for the former, and the supply function for the latter?)

3 Solve the preceding problem by the simultaneous-equation approach.

4 Let the demand and supply functions for a commodity be

$$Q_d = D(P, t_0) \qquad \left(\frac{\partial D}{\partial P} < 0; \quad \frac{\partial D}{\partial t_0} > 0\right) \qquad \text{and} \qquad Q_s = Q_{s0}$$

where t_0 is consumers' taste for the commodity, and where both partial derivatives are continuous.

(a) Write the equilibrium condition as a single equation.

(b) Is the implicit-function theorem applicable?

(c) How would the equilibrium price vary with consumers' taste?

5 From the national-income model in (8.38), find $(\partial \bar{Y}/\partial M_{s0})$ and $(\partial \bar{i}/\partial M_{s0})$, and interpret their economic meanings. Use *both* the total-differential method and the total-derivative method, and verify that the end results are the same.

6 Consider the following national-income model (with taxes ignored):

$$Y - C(Y) - I(i) - G_0 = 0 \qquad (0 < C' < 1;\, I' < 0)$$

$$kY + L(i) - M_{s0} = 0 \qquad (k = \text{positive constant};\; L' < 0)$$

(a) Is the first equation in the nature of an equilibrium condition?

(b) What is the total quantity demanded for money in this model?

(c) Analyze the comparative statics of the model when money supply changes (monetary policy) and when government expenditure changes (fiscal policy).

8.7 LIMITATIONS OF COMPARATIVE STATICS

Comparative statics is a useful area of study, because in economics we are often interested in finding out how a disequilibrating change in a parameter will affect the equilibrium state of a model. It is important to realize, however, that by its very nature comparative statics ignores the process of adjustment from the old equilibrium to the new and also neglects the time element involved in that adjustment process. As a consequence, it must of necessity also disregard the possibility that, because of the inherent instability of the model, the new equilibrium may not be attainable ever. The study of the process of adjustment per se belongs to the field of *economic dynamics*. When we come to that, particular attention will be directed toward the manner in which a variable will change over time, and explicit consideration will be given to the question of stability of equilibrium.

The important topic of dynamics, however, must wait its turn. Meanwhile, in the next part of the book, we shall undertake to study the problem of *optimization*, an exceedingly important special variety of equilibrium analysis with attendant comparative-static implications (and complications) of its own.

PART FOUR

OPTIMIZATION PROBLEMS

CHAPTER

NINE

OPTIMIZATION: A SPECIAL VARIETY OF EQUILIBRIUM ANALYSIS

When we first introduced the term equilibrium in Chap. 3, we made a broad distinction between goal and nongoal equilibrium. In the latter type, exemplified by our study of market and national-income models, the interplay of certain opposing forces in the model—e.g., the forces of demand and supply in the market models and the forces of leakages and injections in the income models—dictates an equilibrium state, if any, in which these opposing forces are just balanced against each other, thus obviating any further tendency to change. The attainment of this type of equilibrium is the outcome of the impersonal balancing of these forces and does not require the conscious effort on the part of anyone to accomplish a specified goal. True, the consuming households behind the forces of demand and the firms behind the forces of supply are each striving for an optimal position under the given circumstances, but as far as the market itself is concerned, no one is aiming at any particular equilibrium price or equilibrium quantity (unless, of course, the government happens to be trying to peg the price). Similarly, in national-income determination, the impersonal balancing of leakages and injections is what brings about an equilibrium state, and no conscious effort at reaching any particular goal (such as an attempt to alter an undesirable income level by means of monetary or fiscal policies) needs to be involved at all.

In the present part of the book, however, our attention will be turned to the study of *goal equilibrium*, in which the equilibrium state is defined as the optimum position for a given economic unit (a household, a business firm, or even an entire economy) and in which the said economic unit will be deliberately striving for attainment of that equilibrium. As a result, in this context—but only in this context—our earlier warning that equilibrium does not imply desirability will become irrelevant and immaterial. In this part of the book, our primary focus will be on the classical techniques for locating optimum positions—those using differential calculus. More modern developments, known as mathematical programming, will be discussed later.

9.1 OPTIMUM VALUES AND EXTREME VALUES

Economics is by and large a science of choice. When an economic project is to be carried out, such as the production of a specified level of output, there are normally a number of alternative ways of accomplishing it. One (or more) of these alternatives will, however, be more desirable than others from the standpoint of some criterion, and it is the essence of the optimization problem to choose, on the basis of that specified criterion, the best alternative available.

The most common criterion of choice among alternatives in economics is the goal of *maximizing* something (such as maximizing a firm's profit, a consumer's utility, or the rate of growth of a firm or of a country's economy) or of *minimizing* something (such as minimizing the cost of producing a given output). Economically, we may categorize such maximization and minimization problems under the general heading of *optimization*, meaning "the quest for the best." From a purely mathematical point of view, however, the terms "maximum" and "minimum" do not carry with them any connotation of optimality. Therefore, the collective term for maximum and minimum, as mathematical concepts, is the more matter-of-fact designation *extremum*, meaning an extreme value.

In formulating an optimization problem, the first order of business is to delineate an *objective function* in which the dependent variable represents the object of maximization or minimization and in which the set of independent variables indicates the objects whose magnitudes the economic unit in question can pick and choose, with a view to optimizing. We shall therefore refer to the independent variables as *choice variables*.* The essence of the optimization process is simply to find the set of values of the choice variables that will yield the desired extremum of the objective function.

For example, a business firm may seek to maximize profit π, that is, to maximize the difference between total revenue R and total cost C. Since, within the framework of a given state of technology and a given market demand for the firm's product, R and C are both functions of the output level Q, it follows that π

* They can also be called *decision variables*, or *policy variables*.

is also expressible as a function of Q:

$$\pi(Q) = R(Q) - C(Q)$$

This equation constitutes the relevant objective function, with π as the object of maximization and Q as the (only) choice variable. The optimization problem is then that of choosing the level of Q such that π will be a maximum. Note that the *optimal* level of π is by definition its *maximal* level, but the optimal level of the choice variable Q is itself not required to be either a maximum or a minimum.

To cast the problem into a more general mold for further discussion (though still confining ourselves to objective functions of one variable only), let us consider the general function

$$y = f(x)$$

and attempt to develop a procedure for finding the level of x that will maximize or minimize the value of y. It will be assumed in this discussion that the function f is continuously differentiable.

9.2 RELATIVE MAXIMUM AND MINIMUM: FIRST-DERIVATIVE TEST

Since the objective function $y = f(x)$ is stated in the general form, there is no restriction as to whether it is linear or nonlinear or whether it is monotonic or contains both increasing and decreasing parts. From among the many possible types of function compatible with the above objective-function form, we have selected three specific cases to be depicted in Fig. 9.1. Simple as they may be, the graphs in Fig. 9.1 should give us valuable insight into the problem of locating the maximum or minimum value of the function $y = f(x)$.

Relative versus Absolute Extremum

If the objective function is a constant function, as in Fig. 9.1a, all values of the choice variable x will result in the same value of y, and the height of each point

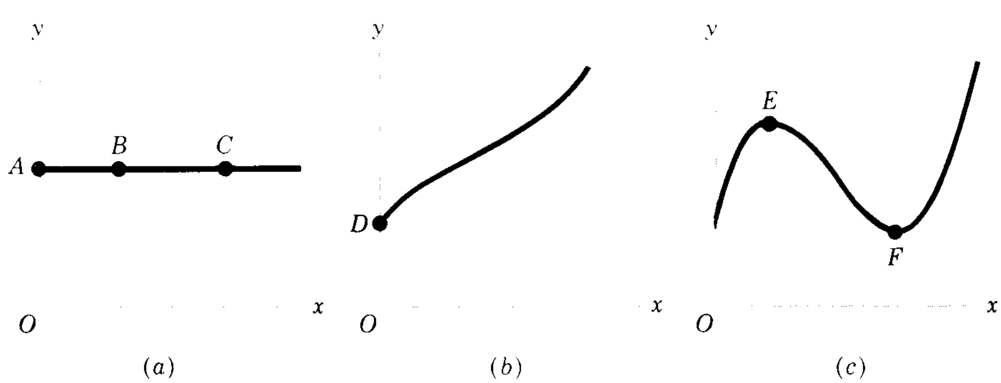


Figure 9.1

on the graph of the function (such as A or B or C) may be considered a maximum or, for that matter, a minimum—or, indeed, neither. In this case, there is in effect no significant choice to be made regarding the value of x for the maximization or minimization of y.

In Fig. 9.1*b*, the function is monotonically increasing, and there is no finite maximum if the set of nonnegative real numbers is taken to be its domain. However, we may consider the end point D on the left (the y intercept) as representing a minimum; in fact, it is in this case the *absolute* (or *global*) minimum in the range of the function.

The points E and F in Fig. 9.1*c*, on the other hand, are examples of a *relative* (or *local*) extremum, in the sense that each of these points represents an extremum in the immediate neighborhood of the point only. The fact that point F is a relative minimum is, of course, no guarantee that it is also the global minimum of the function, although this may happen to be the case. Similarly, a relative maximum point such as E may or may not be a global maximum. Note also that a function can very well have several relative extrema, some of which may be maxima while others are minima.

In most economic problems that we shall be dealing with, our primary, if not exclusive, concern will be with extreme values other than end-point values, for with most such problems the domain of the objective function is restricted to be the set of nonnegative numbers, and thus an end point (on the left) will represent the zero level of the choice variable, which is often of no practical interest. Actually, the type of function most frequently encountered in economic analysis is that shown in Fig. 9.1*c*, or some variant thereof which contains only a single bend in the curve. We shall therefore continue our discussion mainly with reference to the search for *relative* extrema such as points E and F. This will, however, by no means foreclose the knowledge of an absolute maximum if we want it, because an absolute maximum must be either a relative maximum or one of the end points of the function. Thus if we know all the relative maxima, it is necessary only to select the largest of these and compare it with the end points in order to determine the absolute maximum. The absolute minimum of a function can be found analogously. Hereafter, the extreme values considered will be *relative* or *local* ones, unless indicated otherwise.

First-Derivative Test

As a matter of terminology, from now on we shall refer to the derivative of a function alternatively as its *first* derivative (short for *first-order* derivative). The reason for this will become apparent shortly.

Given a function $y = f(x)$, the first derivative $f'(x)$ plays a major role in our search for its extreme values. This is due to the fact that, if a relative extremum of the function occurs at $x = x_0$, then either (1) we have $f'(x_0) = 0$, or (2) $f'(x_0)$ does not exist. The second eventuality is illustrated in Fig. 9.2*a*, where both points A and B depict relative extreme values of y, and yet no derivative is defined at either of these sharp points. Since in the present discussion we are assuming that $y = f(x)$ is continuous and possesses a continuous derivative, however, we are in effect ruling out sharp points. For smooth functions, relative extreme values can occur only where the first derivative has a zero value. This is illustrated by points C and D in Fig. 9.2b, both of which represent extreme values, and both of which are characterized by a zero slope—$f'(x_1) = 0$ and $f'(x_2) = 0$. It is also easy to see that when the slope is nonzero we cannot possibly have a relative minimum (the bottom of a valley) or a relative maximum (the peak of a hill). For this reason, we can, in the context of smooth functions, take the condition $f'(x) = 0$ as a *necessary* condition for a relative extremum (either maximum or minimum).

We must add, however, that a zero slope, while *necessary*, is *not sufficient* to establish a relative extremum. An example of the case where a zero slope is not associated with an extremum will be presented shortly. By appending a certain proviso to the zero-slope condition, however, we can obtain a decisive test for a relative extremum. This may be stated as follows:

First-derivative test for relative extremum If the first derivative of a function $f(x)$ at $x = x_0$ is $f'(x_0) = 0$, then the value of the function at x_0, $f(x_0)$, will be

a. A relative *maximum* if the derivative $f'(x)$ changes its sign from positive to negative from the immediate left of the point x_0 to its immediate right.
b. A relative *minimum* if $f'(x)$ changes its sign from negative to positive from the immediate left of x_0 to its immediate right.
c. Neither a relative maximum nor a relative minimum if $f'(x)$ has the same sign on both the immediate left and right of point x_0.

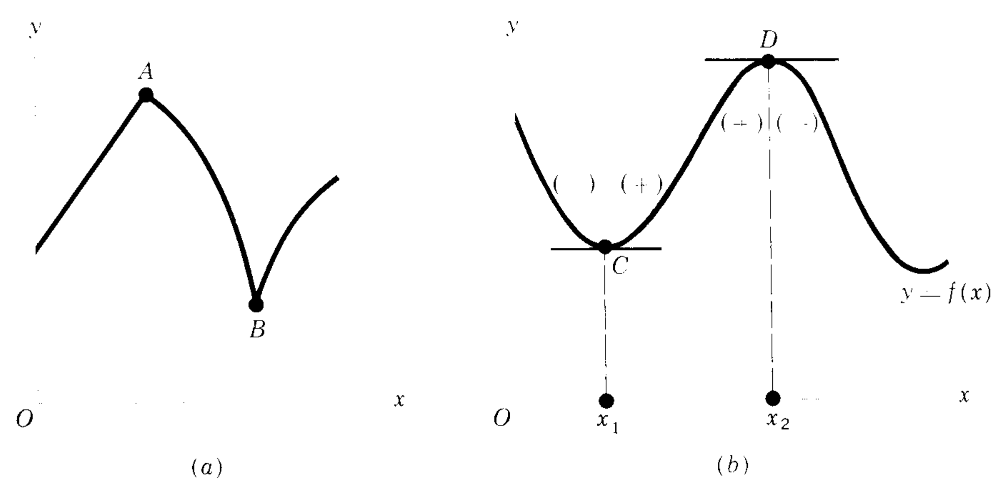


Figure 9.2

Let us call the value x_0 a *critical value* of x if $f'(x_0) = 0$, and refer to $f(x_0)$ as a *stationary value* of y (or of the function f). The point with coordinates x_0 and $f(x_0)$ can, accordingly, be called a *stationary point*. (The rationale for the word "stationary" should be self-evident—wherever the slope is zero, the point in question is never situated on an upward or downward incline, but is rather at a standstill position.) Then, graphically, the first possibility listed in this test will establish the stationary point as the peak of a hill, such as point D in Fig. 9.2*b*, whereas the second possibility will establish the stationary point as the bottom of a valley, such as point C in the same diagram. Note, however, that in view of the existence of a third possibility, yet to be discussed, we are unable to regard the condition $f'(x) = 0$ as a *sufficient condition* for a relative extremum. But we now see that, *if* the necessary condition $f'(x) = 0$ is satisfied, *then* the change-of-derivative-sign proviso can serve as a *sufficient condition* for a relative maximum or minimum, depending on the direction of the sign change.

Let us now explain the third possibility. In Fig. 9.3*a*, the function f is shown to attain a zero slope at point J (when $x = j$). Even though $f'(j)$ is zero—which makes $f(j)$ a stationary value—the derivative does not change its sign from one side of $x = j$ to the other; therefore, according to the test above, point J gives neither a maximum nor a minimum, as is duly confirmed by the graph of the function. Rather, it exemplifies what is known as an *inflection point*.

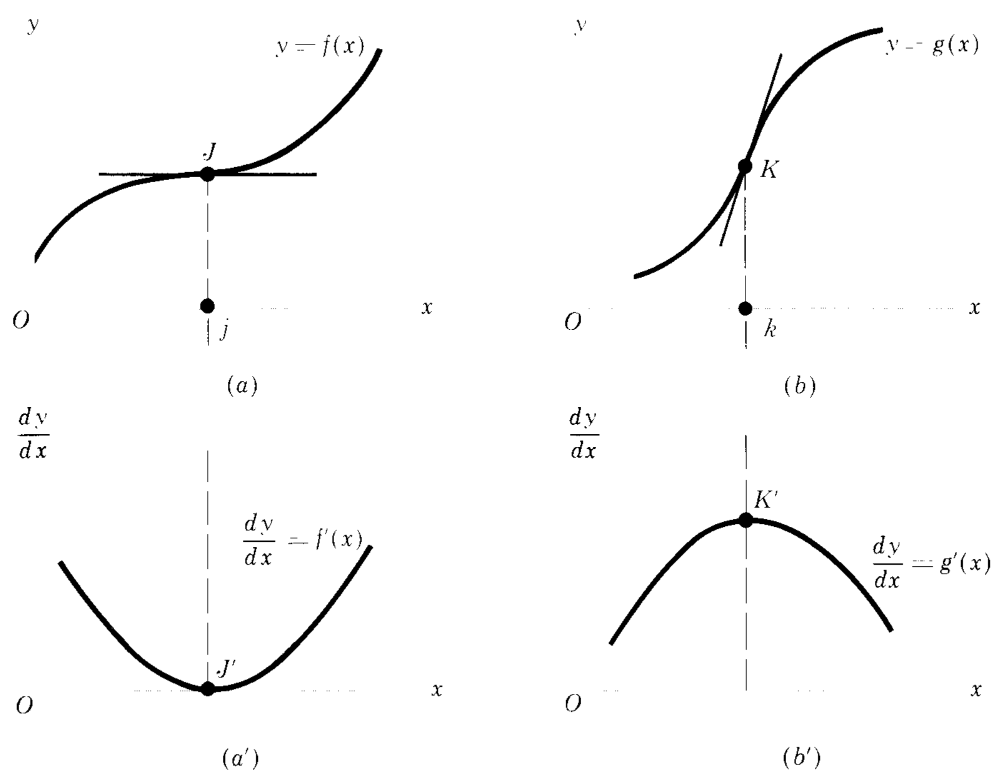


Figure 9.3

The characteristic feature of an inflection point is that, at that point, the derivative (as against the primitive) function reaches an extreme value. Since this extreme value can be either a maximum or a minimum, we have two types of inflection points. In Fig. 9.3a', where we have plotted the derivative $f'(x)$, we see that its value is zero when $x = j$ (see point J') but is positive on both sides of point J'; this makes J' a *minimum* point of the derivative function $f'(x)$.

The other type of inflection point is portrayed in Fig 9.3b, where the slope of the function $g(x)$ increases till the point k is reached and decreases thereafter. Consequently, the graph of the derivative function $g'(x)$ will assume the shape shown in diagram b', where point K' gives a *maximum* value of the derivative function $g'(x)$.*

To sum up: A relative extremum must be a stationary value, but a stationary value may be associated with either a relative extremum or an inflection point. To find the relative maximum or minimum of a given function, therefore, the procedure should be first to find the stationary values of the function where $f'(x) = 0$ and then to apply the first-derivative test to determine whether each of the stationary values is a relative maximum, a relative minimum, or neither.

Example 1 Find the relative extrema of the function

$$y = f(x) = x^3 - 12x^2 + 36x + 8$$

First, we find the derivative function to be

$$f'(x) = 3x^2 - 24x + 36$$

To get the critical values, i.e., the values of x satisfying the condition $f'(x) = 0$, we set the quadratic derivative function equal to zero and get the quadratic equation

$$3x^2 - 24x + 36 = 0$$

By factoring the polynomial or by applying the quadratic formula, we then obtain the following pair of roots (solutions):

$$\bar{x}_1 = 2 \quad [\text{at which we have } f'(2) = 0 \text{ and } f(2) = 40]$$

$$\bar{x}_2 = 6 \quad [\text{at which we have } f'(6) = 0 \text{ and } f(6) = 8]$$

Since $f'(2) = f'(6) = 0$, these two values of x are the critical values we desire.

It is easy to verify that $f'(x) > 0$ for $x < 2$, and $f'(x) < 0$ for $x > 2$, in the immediate neighborhood of $x = 2$; thus, the corresponding value of the function $f(2) = 40$ is established as a relative maximum. Similarly, since $f'(x) < 0$ for $x < 6$, and $f'(x) > 0$ for $x > 6$, in the immediate neighborhood of $x = 6$, the value of the function $f(6) = 8$ must be a relaive minimum.

* Note that a zero derivative value, while a necessary condition for a relative extremum, is *not* required for an inflection point; for the derivative $g'(x)$ has a positive value at $x = k$, and yet point K is an inflection point.

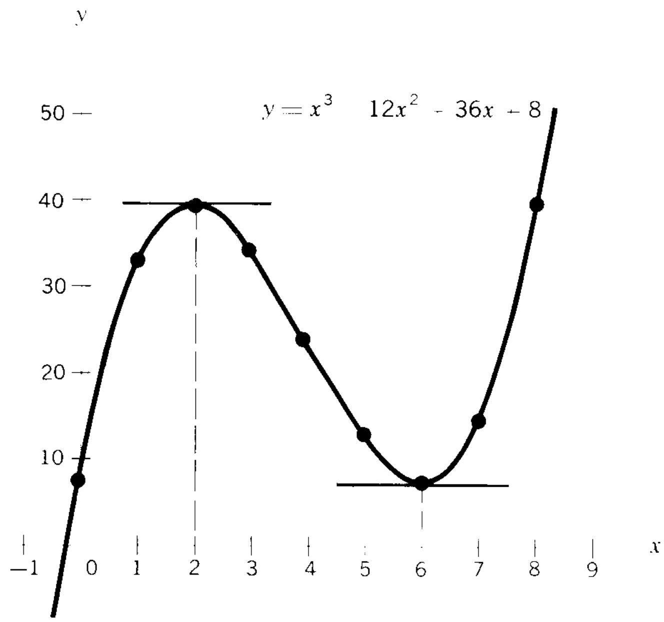


Figure 9.4

The graph of the function of this example is shown in Fig. 9.4. Such a graph may be used to verify the location of extreme values obtained through use of the first-derivative test. But, in reality, in most cases "helpfulness" flows in the opposite direction—the mathematically derived extreme values will help in plotting the graph. The accurate plotting of a graph ideally requires knowledge of the value of the function at every point in the domain; but as a matter of actual practice, only a few points in the domain are selected for purposes of plotting, and the rest of the points typically are filled in by interpolation. The pitfall of this practice is that, unless we hit upon the stationary point(s) by coincidence, we shall miss the exact location of the turning point(s) in the curve. Now, with the first-derivative test at our disposal, it becomes possible to determine these turning points precisely.

Example 2 Find the relative extremum of the average-cost function

$$\text{AC} = f(Q) = Q^2 - 5Q + 8$$

The derivative here is $f'(Q) = 2Q - 5$, a linear function. Setting $f'(Q)$ equal to zero, we get the linear equation $2Q - 5 = 0$, which has the single root $\bar{Q} = 2.5$. This is the only critical value in this case. To apply the first-derivative test, let us find the values of the derivative at, say, $Q = 2.4$ and $Q = 2.6$, respectively. Since $f'(2.4) = -0.2 < 0$ whereas $f'(2.6) = 0.2 > 0$, we can conclude that the stationary value $\text{AC} = f(2.5) = 1.75$ represents a relative minimum. The graph of the function of this example is actually a U-shaped curve, so that the relative minimum already found will also be the absolute minimum. Our knowledge of the exact location of this point should be of great help in plotting the AC curve.

EXERCISE 9.2

1 Find the stationary values of the following (check whether relative maxima or minima or inflection points), assuming the domain to be the set of all real numbers:

(*a*) $y = -2x^2 + 4x + 9$ (*c*) $y = x^2 + 3$

(*b*) $y = 5x^2 + x$ (*d*) $y = 3x^2 - 6x + 2$

2 Find the stationary values of the following (check whether relative maxima or minima or inflection points), assuming the domain to be the interval $[0, \infty)$:

(*a*) $y = x^3 - 3x + 5$

(*b*) $y = \frac{1}{3}x^3 - x^2 + x + 10$

(*c*) $y = -x^3 + 4.5x^2 - 6x + 6$

3 Show that the function $y = x + 1/x$ (with $x \neq 0$) has two relative extrema, one a maximum and the other a minimum. Is the "minimum" larger or smaller than the "maximum"? How is this paradoxical result possible?

4 Let $T = \phi(x)$ be a *total* function (e.g., total product or total cost):

(*a*) Write out the expressions for the *marginal* function M and the *average* function A.

(*b*) Show that, when A reaches a relative extremum, M and A must have the same value.

(*c*) What general principle does this suggest for the drawing of a marginal curve and an average curve in the same diagram?

(*d*) What can you conclude about the elasticity of the total function T at the point where A reaches an extreme value?

9.3 SECOND AND HIGHER DERIVATIVES

Hitherto we have considered only the first derivative $f'(x)$ of a function $y = f(x)$; now let us introduce the concept of *second derivative* (short for *second-order derivative*), and derivatives of even higher orders. These will enable us to develop alternative criteria for locating the relative extrema of a function.

Derivative of a Derivative

Since the first derivative $f'(x)$ is itself a function of x, it, too, should be differentiable with respect to x, provided that it is continuous and smooth. The result of this differentiation, known as the second derivative of the function f, is denoted by

$f''(x)$ where the double prime indicates that $f(x)$ has been differentiated with respect to x twice, and where the expression (x) following the double prime suggests that the second derivative is again a function of x

or

$\frac{d^2y}{dx^2}$ where the notation stems from the consideration that the second derivative means, in fact, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$; hence the d^2 in the numerator and dx^2 in the denominator of this symbol

If the second derivative $f''(x)$ exists for all x values in the domain, the function $f(x)$ is said to be *twice differentiable*; if, in addition, $f''(x)$ is continuous, the function $f(x)$ is said to be *twice continuously differentiable*.*

As a function of x the second derivative can be differentiated with respect to x again to produce a *third* derivative, which in turn can be the source of a *fourth* derivative, and so on ad infinitum, as long as the differentiability condition is met. These higher-order derivatives are symbolized along the same line as the second derivative:

$$f'''(x), f^{(4)}(x), \ldots, f^{(n)}(x) \qquad \text{[with superscripts enclosed in ( )]}$$

or

$$\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots, \frac{d^ny}{dx^n}$$

The last of these can also be written as $\frac{d^n}{dx^n}y$, where the $\frac{d^n}{dx^n}$ part serves as an operator symbol instructing us to take the nth derivative of (some function) with respect to x.

Almost all the *specific* functions we shall be working with possess continuous derivatives up to any order we desire; i.e., they are continuously differentiable any number of times. Whenever a *general* function is used, such as $f(x)$, we always assume that it has derivatives up to any order we need.

Example 1 Find the first through the fifth derivatives of the function

$$y = f(x) = 4x^4 - x^3 + 17x^2 + 3x - 1$$

The desired derivatives are as follows:

$$f'(x) = 16x^3 - 3x^2 + 34x + 3$$
$$f''(x) = 48x^2 - 6x + 34$$
$$f'''(x) = 96x - 6$$
$$f^{(4)}(x) = 96$$
$$f^{(5)}(x) = 0$$

* The following notations are often used to denote continuity and differentiability of a function:

$f \in C^{(0)}$ or $f \in C$:	f is a continuous function
$f \in C^{(1)}$ or $f \in C'$:	f is continuously differentiable
$f \in C^{(2)}$:	f is twice continuously differentiable

The symbol $C^{(n)}$ denotes the set of all functions that possess nth-order derivatives which are continuous in the domain.

In this particular (polynomial-function) example, each successive derivative emerges as a simpler expression than the one before, until we reach a fifth derivative, which is identically zero. This is not generally true, however, of all types of function, as the next example will show. It should be stressed here that the statement "the fifth derivative is zero" is not the same as the statement "the fifth derivative does not exist," which describes an altogether different situation. Note, also, that $f^{(5)}(x) = 0$ (zero at all values of x) is not the same as $f^{(5)}(x_0) = 0$ (zero at x_0 only).

Example 2 Find the first four derivatives of the rational function

$$y = g(x) = \frac{x}{1+x} \qquad (x \neq -1)$$

These derivatives can be found either by use of the quotient rule, or, after rewriting the function as $y = x(1+x)^{-1}$, by the product rule:

$$\left.\begin{aligned} g'(x) &= (1+x)^{-2} \\ g''(x) &= -2(1+x)^{-3} \\ g'''(x) &= 6(1+x)^{-4} \\ g^{(4)}(x) &= -24(1+x)^{-5} \end{aligned}\right\} \qquad (x \neq -1)$$

In this case, repeated derivation evidently does not tend to simplify the subsequent derivative expressions.

Note that, like the primitive function $g(x)$, all the successive derivatives obtained are themselves functions of x. Given specific values of x, these derivative functions will then take specific values. When $x = 2$, for instance, the second derivative in Example 2 can be evaluated as

$$g''(2) = -2(3)^{-3} = \frac{-2}{27}$$

and similarly for other values of x. It is of the utmost importance to realize that to evaluate this second derivative $g''(x)$ at $x = 2$, as we did, we must first obtain $g''(x)$ from $g'(x)$ and then substitute $x = 2$ into the equation for $g''(x)$. It is *incorrect* to substitute $x = 2$ into $g(x)$ or $g'(x)$ *prior* to the differentiation process leading to $g''(x)$.

Interpretation of the Second Derivative

The derivative function $f'(x)$ measures the rate of change of the function f. By the same token, the second-derivative function f'' is the measure of the rate of change of the first derivative f'; in other words, the second derivative measures the *rate of change* of the *rate of change* of the original function f. To put it differently, with a given infinitesimal increase in the independent variable x from a point $x = x_0$,

$$\left.\begin{aligned} f'(x_0) > 0 \\ f'(x_0) < 0 \end{aligned}\right\} \text{ means that the } \textit{value of the function} \text{ tends to } \begin{cases} \text{increase} \\ \text{decrease} \end{cases}$$

whereas, with regard to the second derivative,

$$\left.\begin{matrix} f''(x_0) > 0 \\ f''(x_0) < 0 \end{matrix}\right\} \text{ means that the } \textit{slope of the curve} \text{ tends to } \left\{\begin{matrix} \text{increase} \\ \text{decrease} \end{matrix}\right.$$

Thus a positive first derivative coupled with a positive second derivative at $x = x_0$ implies that the slope of the curve at that point is *positive and increasing*—the value of the function is increasing at an increasing rate. Likewise, a positive first derivative with a negative second derivative indicates that the slope of the curve is *positive but decreasing*—the value of the function is increasing at a decreasing rate. The case of a negative first derivative can be interpreted analogously, but a warning should accompany this case: When $f'(x_0) < 0$ and $f''(x_0) > 0$, the slope of the curve is *negative and increasing*, but this does *not* mean that the slope is changing, say, from (-10) to (-11); on the contrary, the change should be from (-11), a smaller number, to (-10), a larger number. In other words, the negative slope must tend to be *less* steep as x increases. Lastly, when $f'(x_0) < 0$ and $f''(x_0) < 0$, the slope of the curve must be *negative and decreasing*. This refers to a negative slope that tends to become *steeper* as x increases.

Since we have been talking about slopes, it may be useful to continue the discussion with a graphical illustration. In Fig. 9.5 we have marked out six points (A, B, C, D, E, and F) on the two parabolas shown; each of these points illustrates a different combination of first- and second-derivative signs, as follows:

If at	the derivative signs are		we can illustrate it by
$x = x_1$	$f'(x_1) > 0$	$f''(x_1) < 0$	point A
$x = x_2$	$f'(x_2) = 0$	$f''(x_2) < 0$	point B
$x = x_3$	$f'(x_3) < 0$	$f''(x_3) < 0$	point C
$x = x_4$	$g'(x_4) < 0$	$g''(x_4) > 0$	point D
$x = x_5$	$g'(x_5) = 0$	$g''(x_5) > 0$	point E
$x = x_6$	$g'(x_6) > 0$	$g''(x_6) > 0$	point F

From this, we see that a *negative* second derivative (the first three cases) is consistently reflected in an inverse U-shaped curve, or a portion thereof, because the curve in question is required to have a smaller and smaller slope as x increases. In contrast, a *positive* second derivative (the last three cases) consistently points to a U-shaped curve, or a portion thereof, since the curve in question must display a larger and larger slope as x increases. Viewing the two curves in Fig. 9.5 from the standpoint of the horizontal axis, we find the one in diagram a to be concave throughout, whereas the one in diagram b is convex throughout. Since concavity and convexity are descriptions of how the curve "bends," we may now expect the second derivative of a function to inform us about the *curvature* of its graph, just as the first derivative tells us about its *slope*.

Although the words "concave" and "convex" adequately convey the differing curvature of the two curves in Fig. 9.5, writers today would more specifically label them as *strictly concave* and *strictly convex*, respectively. In line with this terminol-

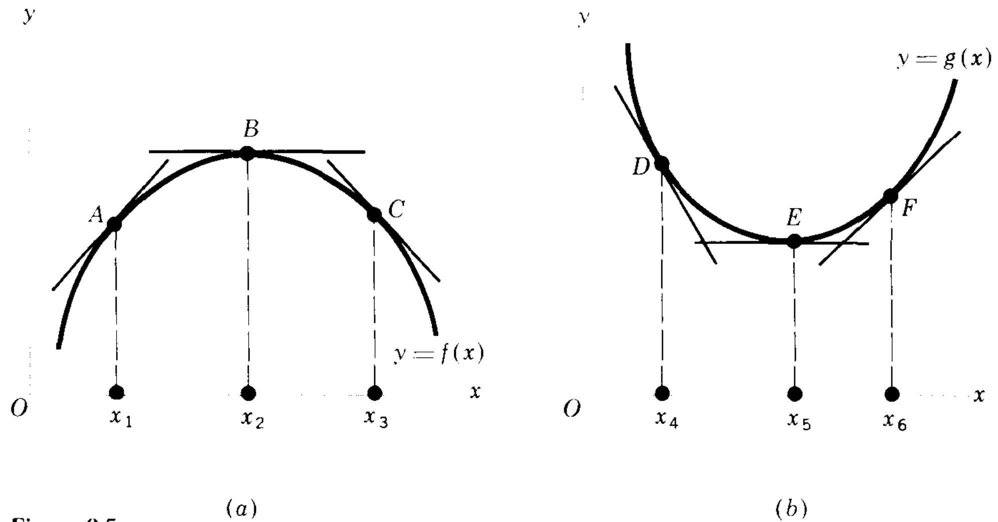


Figure 9.5

ogy, a function whose graph is strictly concave (strictly convex) is called a *strictly concave (strictly convex) function*. The precise geometric characterization of a strictly concave function is as follows. If we pick any pair of points M and N on its curve and join them by a straight line, the line segment MN must lie entirely *below* the curve, except at points M and N. The characterization of a strictly convex function can be obtained by substituting the word "above" for the word "below" in the last statement. Try this out in Fig. 9.5. If the characterizing condition is relaxed somewhat, so that the line segment MN is allowed to lie *either* below the curve, *or* along (coinciding with) the curve, then we will be describing instead a *concave function*, without the adverb "strictly." Similarly, if the line segment MN *either* lies above, *or* lies along the curve, then the function is *convex*, again without the adverb "strictly." Note that, since the line segment MN may coincide with a (nonstrictly) concave or convex curve, the latter may very well contain a linear segment. In contrast, a *strictly* concave or convex curve can never contain a linear segment anywhere. It follows that while a strictly concave (convex) function is automatically a concave (convex) function, the converse is not true.*

From our earlier discussion of the second derivative, we may now infer that if the second derivative $f''(x)$ is negative for all x, then the primitive function $f(x)$ must be a strictly concave function. Similarly, $f(x)$ must be strictly convex, if $f''(x)$ is positive for all x. Despite this, it is *not* valid to reverse the above inference and say that, if $f(x)$ is strictly concave (strictly convex), then $f''(x)$ must be negative (positive) for all x. This is because, in certain exceptional cases, the second derivative may have a *zero* value at a stationary point on such a curve. An example of this can be found in the function $y = f(x) = x^4$, which plots as a strictly convex curve, but whose derivatives

$$f'(x) = 4x^3 \qquad f''(x) = 12x^2$$

* We shall discuss these concepts further in Sec. 11.5 below.

indicate that, at the stationary point where $x = 0$, the value of the second derivative is $f''(0) = 0$. Note, however, that at any other point, with $x \neq 0$, the second derivative of this function does have the (expected) positive sign. Aside from the possibility of a zero value at a stationary point, therefore, the second derivative of a strictly concave or convex function may be expected in general to adhere to a single algebraic sign.

For other types of function, the second derivative may take both positive and negative values, depending on the value of x. In Fig. 9.3*a* and *b*, for instance, both $f(x)$ and $g(x)$ undergo a sign change in the second derivative at their respective inflection points J and K. According to Fig. 9.3*a*′, the slope of $f'(x)$—that is, the value of $f''(x)$—changes from negative to positive at $x = j$; the exact opposite occurs with the slope of $g'(x)$—that is, the value of $g''(x)$—on the basis of Fig. 9.3*b*′. Translated into curvature terms, this means that the graph of $f(x)$ turns from concave to convex at point J, whereas the graph of $g(x)$ has the reverse change at point K. Consequently, instead of characterizing an inflection point as a point where the first derivative reaches an extreme value, we may alternatively characterize it as a point where the function undergoes a change in curvature or a change in the sign of its second derivative.

An Application

The two curves in Fig. 9.5 exemplify the graphs of quadratic functions, which may be expressed generally in the form

$$y = ax^2 + bx + c \qquad (a \neq 0)$$

From our discussion of the second derivative, we can now derive a convenient way of determining whether a given quadratic function will have a strictly convex (U-shaped) or a strictly concave (inverse U-shaped) graph.

Since the second derivative of the quadratic function cited is $d^2y/dx^2 = 2a$, this derivative will always have the same algebraic sign as the coefficient a. Recalling that a positive second derivative implies a strictly convex curve, we can infer that a positive coefficient a in the above quadratic function gives rise to a U-shaped graph. In contrast, a negative coefficient a leads to a strictly concave curve, shaped like an inverted U.

As intimated at the end of Sec. 9.2, the relative extremum of this function will also prove to be its absolute extremum, because in a quadratic function there can be found only a single valley or peak, evident in a U or inverted U, respectively.

EXERCISE 9.3

1 Find the second and third derivatives of the following functions:

(*a*) $ax^2 + bx + c$ (*c*) $\dfrac{2x}{1-x} \quad (x \neq 1)$

(*b*) $6x^4 - 3x - 4$ (*d*) $\dfrac{1+x}{1-x} \quad (x \neq 1)$

2 Which of the following quadratic functions are strictly convex?
(a) $y = 9x^2 - 4x + 2$ (c) $u = 9 - x^2$
(b) $w = -3x^2 + 39$ (d) $v = 8 - 3x + x^2$

3 Draw (a) a concave curve which is *not* strictly concave, and (b) a curve which qualifies simultaneously as a concave curve and a convex curve.

4 Given the function $y = a - \dfrac{b}{c+x}$ ($a, b, c > 0$; $x \geq 0$), determine the general shape of its graph by examining (a) its first and second derivatives, (b) its vertical intercept, and (c) the limit of y as x tends to infinity. If this function is to be used as a consumption function, how should the parameters be restricted in order to make it economically sensible?

5 Draw the graph of a function $f(x)$ such that $f'(x) = 0$, and the graph of a function $g(x)$ such that $g'(3) = 0$. Summarize in one sentence the essential difference between $f(x)$ and $g(x)$ in terms of the concept of stationary point.

9.4 SECOND-DERIVATIVE TEST

Returning to the pair of extreme points B and E in Fig. 9.5 and remembering the newly established relationship between the second derivative and the curvature of a curve, we should be able to see the validity of the following criterion for a relative extremum:

Second-derivative test for relative extremum If the first derivative of a function f at $x = x_0$ is $f'(x_0) = 0$, then the value of the function at x_0, $f(x_0)$, will be

a. A relative *maximum* if the second-derivative value at x_0 is $f''(x_0) < 0$.
b. A relative *minimum* if the second-derivative value at x_0 is $f''(x_0) > 0$.

This test is in general more convenient to use than the first-derivative test, because it does not require us to check the derivative sign to both the left and the right of x_0. But it has the drawback that no unequivocal conclusion can be drawn in the event that $f''(x_0) = 0$. For then the stationary value $f(x_0)$ can be *either* a relative maximum, *or* a relative minimum, *or* even an inflectional value.* When the situation of $f''(x_0) = 0$ is encountered, we must either revert to the first-derivative test, or resort to another test, to be developed in Sec. 9.6, that involves

* To see that an inflection point is possible when $f''(x_0) = 0$, let us refer back to Fig. 9.3a and 9.3a'. Point J in the upper diagram is an inflection point, with $x = j$ as its critical value. Since the $f'(x)$ curve in the lower diagram attains a minimum at $x = j$, the slope of $f'(x)$ [i.e., $f''(x)$] must be zero at the critical value $x = j$. Thus point J illustrates an inflection point occurring when $f''(x_0) = 0$.

To see that a relative extremum is also consistent with $f''(x_0) = 0$, consider the function $y = x^4$. This function plots as a U-shaped curve and has a minimum, $y = 0$, attained at the critical value $x = 0$. Since the second derivative of this function is $f''(x) = 12x^2$, we again obtain a zero value for this derivative at the critical value $x = 0$. Thus this function illustrates a relative extremum occurring when $f''(x_0) = 0$.

the third or even higher derivatives. For most problems in economics, however, the second-derivative test should prove to be adequate for determining a relative maximum or minimum.

Example 1 Find the relative extremum of the function

$$y = f(x) = 4x^2 - x$$

The first and second derivatives are

$$f'(x) = 8x - 1 \qquad \text{and} \qquad f''(x) = 8$$

Setting $f'(x)$ equal to zero and solving the resulting equation, we find the (only) critical value to be $\bar{x} = \frac{1}{8}$, which yields the (only) stationary value $f(\frac{1}{8}) = -\frac{1}{16}$. Because the second derivative is positive (in this case it is indeed positive for any value of x), the extremum is established as a minimum. Indeed, since the given function plots as a U-shaped curve, the relative minimum is also the absolute minimum.

Example 2 Find the relative extrema of the function

$$y = g(x) = x^3 - 3x^2 + 2$$

The first two derivatives of this function are

$$g'(x) = 3x^2 - 6x \qquad \text{and} \qquad g''(x) = 6x - 6$$

Setting $g'(x)$ equal to zero and solving the resulting quadratic equation, $3x^2 - 6x = 0$, we obtain the critical values $\bar{x}_1 = 0$ and $\bar{x}_2 = 2$, which in turn yield the two stationary values:

$$g(0) = 2 \qquad [\text{a maximum because } g''(0) = -6 < 0]$$

$$g(2) = -2 \qquad [\text{a minimum because } g''(2) = 6 > 0]$$

Necessary versus Sufficient Conditions

As was the case with the first-derivative test, the zero-slope condition $f'(x) = 0$ plays the role of a *necessary* condition in the second-derivative test. Since this condition is based on the first-order derivative, it is often referred to as the *first-order condition*. Once we find the first-order condition satisfied at $x = x_0$, the negative (positive) sign of $f''(x_0)$ is *sufficient* to establish the stationary value in question as a relative maximum (minimum). These sufficient conditions, which are based on the second-order derivative, are often referred to as *second-order conditions*.

It bears repeating that the first-order condition is *necessary*, but *not sufficient*, for a relative maximum or minimum. (Remember inflection points?) In sharp contrast, while the second-order condition that $f''(x)$ be negative (positive) at the critical value x_0 is *sufficient* for a relative maximum (minimum), it is *not necessary*. [Remember the relative extremum that occurs when $f''(x_0) = 0$?] For this reason, one should carefully guard against the following line of argument: "Since the stationary value $f(x_0)$ is already known to be a minimum, we must have $f''(x_0) > 0$." The reasoning here is faulty because it incorrectly treats the positive sign of $f''(x_0)$ as a necessary condition for $f(x_0)$ to be a minimum.

This is not to say that second-order derivatives can never be used in stating *necessary* conditions for relative extrema. Indeed they can. But care must then be taken to allow for the fact that a relative maximum (minimum) can occur not only when $f''(x_0)$ is negative (positive), but also when $f''(x_0)$ is zero. Consequently, *second-order necessary conditions* must be couched in terms of weak inequalities: for a stationary value $f(x_0)$ to be a relative $\begin{Bmatrix} \text{maximum} \\ \text{minimum} \end{Bmatrix}$, it is necessary that $f''(x_0)\begin{Bmatrix} \leq \\ \geq \end{Bmatrix}0$.

Conditions for Profit Maximization

We shall now present some economic examples of extreme-value problems, i.e., problems of optimization.

One of the first things that a student of economics learns is that, in order to maximize profit, a firm must equate marginal cost and marginal revenue. Let us show the mathematical derivation of this condition. To keep the analysis on a general level, we shall work with the total-revenue function $R = R(Q)$ and total-cost function $C = C(Q)$, both of which are functions of a single variable Q. From these it follows that a profit function (the objective function) may also be formulated in terms of Q (the choice variable):

$$\pi = \pi(Q) = R(Q) - C(Q) \tag{9.1}$$

To find the profit-maximizing output level, we must satisfy the first-order necessary condition for a maximum: $d\pi/dQ = 0$. Accordingly, let us differentiate (9.1) with respect to Q and set the resulting derivative equal to zero. The result is

$$\begin{aligned} \frac{d\pi}{dQ} &\equiv \pi'(Q) = R'(Q) - C'(Q) \\ &= 0 \qquad \text{iff} \qquad R'(Q) = C'(Q) \end{aligned} \tag{9.2}$$

Thus the *optimum* output (*equilibrium* output) $\bar{Q}$ must satisfy the equation $R'(\bar{Q}) = C'(\bar{Q})$, or MR = MC. This condition constitutes the first-order condition for profit maximization.

However, the first-order condition may lead to a minimum rather than a maximum; thus we must check the second-order condition next. We can obtain the second derivative by differentiating the first derivative in (9.2) with respect to Q:

$$\begin{aligned} \frac{d^2\pi}{dQ^2} &\equiv \pi''(Q) = R''(Q) - C''(Q) \\ &< 0 \qquad \text{iff} \qquad R''(Q) < C''(Q) \end{aligned}$$

For an output level $\bar{Q}$ such that $R'(\bar{Q}) = C'(\bar{Q})$, the satisfaction of the second-order condition $R''(\bar{Q}) < C''(\bar{Q})$ is sufficient to establish it as a profit-maximizing output. Economically, this would mean that, if the rate of change of MR is less than the rate of change of MC at the output where MC = MR, then that output will maximize profit.

These conditions are illustrated in Fig. 9.6. In diagram *a* we have drawn a total-revenue and a total-cost curve, which are seen to intersect twice, at output levels of Q_2 and Q_4. In the open interval (Q_2, Q_4), total revenue R exceeds total cost C, and thus π is positive. But in the intervals $[0, Q_2)$ and $(Q_4, Q_5]$, where Q_5 represents the upper limit of the firm's productive capacity, π is negative. This fact is reflected in diagram *b*, where the profit curve—obtained by plotting the vertical distance between the R and C curves for each level of output—lies above the horizontal axis only in the interval (Q_2, Q_4).

When we set $d\pi/dQ = 0$, in line with the first-order condition, it is our intention to locate the peak point K on the profit curve, at output Q_3, where the slope of the curve is zero. However, the relative-minimum point M (output Q_1) will also offer itself as a candidate, because it, too, meets the zero-slope requirement. We shall later resort to the second-order condition to eliminate the "wrong" kind of extremum.

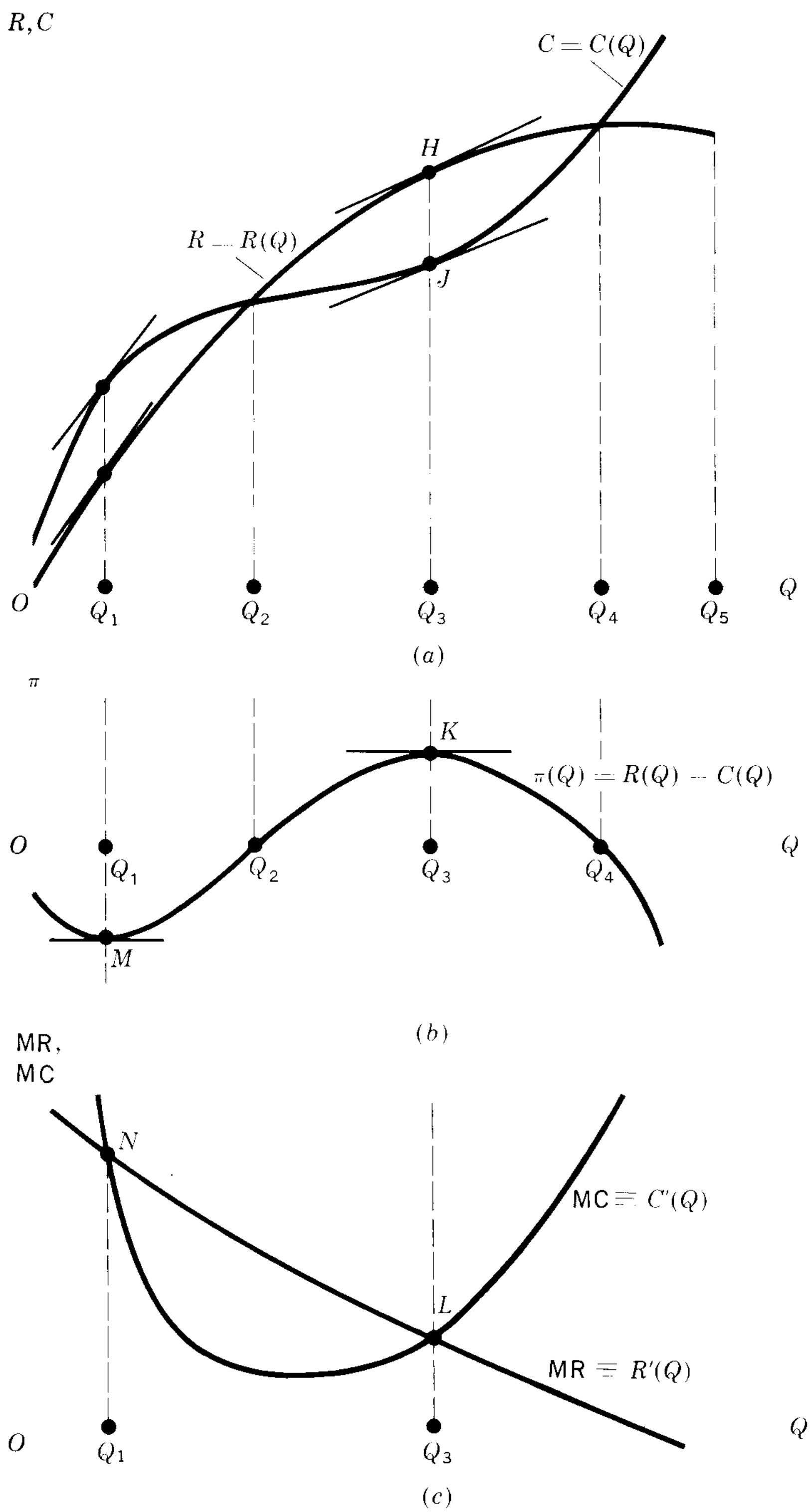
The first-order condition $d\pi/dQ = 0$ is equivalent to the condition $R'(Q) = C'(Q)$. In Fig. 9.6*a*, the output level Q_3 satisfies this, because the R and C curves do have the same slope at Q_3 (the tangent lines drawn to the two curves at H and J are parallel to each other). The same is true for output Q_1. Since the equality of the slopes of R and C means the equality of MR and MC, outputs Q_3 and Q_1 must obviously be where the MR and MC curves intersect, as illustrated in Fig. 9.6*c*.

How does the second-order condition enter into the picture? Let us first look at Fig. 9.6*b*. At point K, the second derivative of the π function will (barring the exceptional zero-value case) have a negative value, $\pi''(Q_3) < 0$, because the curve is inverse U-shaped around K; this means that Q_3 will maximize profit. At point M, on the other hand, we would expect that $\pi''(Q_1) > 0$; thus Q_1 provides a relative minimum for π instead. The second-order sufficient condition for a maximum can, of course, be stated alternatively as $R''(Q) < C''(Q)$, that is, that the slope of the MR curve be less than the slope of the MC curve. From Fig. 9.6*c*, it is immediately apparent that output Q_3 satisfies this condition, since the slope of MR is negative while that of MC is positive at point L. But output Q_1 violates this condition because both MC and MR have negative slopes, and that of MR is *numerically smaller* than that of MC at point N, which implies that $R''(Q_1)$ is *greater* than $C''(Q_1)$ instead. In fact, therefore, output Q_1 also violates the second-order *necessary* condition for a relative maximum, but satisfies the second-order *sufficient* condition for a relative minimum.

Example 3 Let the $R(Q)$ and $C(Q)$ functions be

$$R(Q) = 1200Q - 2Q^2$$

$$C(Q) = Q^3 - 61.25Q^2 + 1528.5Q + 2000$$



R, C
C = C(Q)
H
R = R(Q)
J
O
Q₁
Q₂
Q₃
Q₄
Q₅
Q
(a)
π
K
π(Q) = R(Q) − C(Q)
O
Q₁
Q₂
Q₃
Q₄
Q
M
(b)
MR,
MC
N
MC ≡ C′(Q)
L
MR ≡ R′(Q)
O
Q₁
Q₃
Q
(c)

Figure 9.6

Then the profit function is

$$\pi(Q) = -Q^3 + 59.25Q^2 - 328.5Q - 2000$$

where R, C, and π are all in dollar units and Q is in units of (say) tons per week. This profit function has two critical values, $Q = 3$ and $Q = 36.5$, because

$$\frac{d\pi}{dQ} = -3Q^2 + 118.5Q - 328.5 = 0 \qquad \text{when } Q = \begin{cases} 3 \\ 36.5 \end{cases}$$

But since the second derivative is

$$\frac{d^2\pi}{dQ^2} = -6Q + 118.5 \qquad \begin{cases} > 0 & \text{when } Q = 3 \\ < 0 & \text{when } Q = 36.5 \end{cases}$$

the profit-maximizing output is $\bar{Q} = 36.5$ (tons per week). (The other output minimizes profit.) By substituting $\bar{Q}$ into the profit function, we can find the maximized profit to be $\bar{\pi} = \pi(36.5) = 16{,}318.44$ (dollars per week).

As an alternative approach to the above, we can first find the MR and MC functions and then equate the two, i.e., find their intersection. Since

$$R'(Q) = 1200 - 4Q$$

$$C'(Q) = 3Q^2 - 122.5Q + 1528.5$$

equating the two functions will result in a quadratic equation identical with $d\pi/dQ = 0$ which has yielded the two critical values of Q cited above.

Coefficients of a Cubic Total-Cost Function

In Example 3 above, a cubic function is used to represent the total-cost function. The traditional total-cost curve $C = C(Q)$, as illustrated in Fig. 9.6*a*, is supposed to contain two wiggles that form a concave segment (decreasing marginal cost) and a subsequent convex segment (increasing marginal cost). Since the graph of a cubic function always contains exactly two wiggles, as illustrated in Fig. 9.4, it should suit that role well. However, Fig. 9.4 immediately alerts us to a problem: the cubic function can possibly produce a downward-sloping segment in its graph, whereas the total-cost function, to make economic sense, should be upward-sloping everywhere (a larger output always entails a higher total cost). If we wish to use a cubic total-cost function such as

$$C = C(Q) = aQ^3 + bQ^2 + cQ + d \tag{9.3}$$

therefore, it is essential to place appropriate restrictions on the parameters so as to prevent the C curve from ever bending downward.

An equivalent way of stating this requirement is that the MC function should be positive throughout, and this can be ensured only if the *absolute minimum* of the MC function turns out to be positive. Differentiating (9.3) with respect to Q, we obtain the MC function

$$\text{MC} = C'(Q) = 3aQ^2 + 2bQ + c \tag{9.4}$$

which, because it is a quadratic, plots as a parabola as in Fig. 9.6*c*. In order for the MC curve to stay positive (above the horizontal axis) everywhere, it is necessary that the parabola be U-shaped (otherwise, with an inverse U, the curve is bound to extend itself into the second quadrant). Hence the coefficient of the Q^2 term in (9.4) has to be positive; i.e., we must impose the restriction $a > 0$. This restriction, however, is by no means sufficient, because the minimum value of a U-shaped MC curve—call it $\text{MC}_{\min}$ (a relative minimum which also happens to be an absolute minimum)—may still occur below the horizontal axis. Thus we must next find $\text{MC}_{\min}$ and ascertain the parameter restrictions that would make it positive.

According to our knowledge of relative extremum, the minimum of MC will occur where

$$\frac{d}{dQ}\text{MC} = 6aQ + 2b = 0$$

The output level that satisfies this first-order condition is

$$Q^* = \frac{-2b}{6a} = \frac{-b}{3a}$$

This minimizes (rather than maximizes) MC because the second derivative $d^2(\text{MC})/dQ^2 = 6a$ is assuredly positive in view of the restriction $a > 0$. The knowledge of Q^* now enables us to calculate $\text{MC}_{\min}$, but we may first infer the sign of coefficient b from it. Inasmuch as negative output levels are ruled out, we see that b can never be positive (given $a > 0$). Moreover, since the law of diminishing returns is assumed to set in at a positive output level (that is, MC is assumed to have an initial declining segment), Q^* should be positive (rather than zero). Consequently, we must impose the restriction $b < 0$.

It is a simple matter now to substitute the MC-minimizing output Q^* into (9.4) to find that

$$\text{MC}_{\min} = 3a\left(\frac{-b}{3a}\right)^2 + 2b\frac{-b}{3a} + c = \frac{3ac - b^2}{3a}$$

Thus, to guarantee the positivity of $\text{MC}_{\min}$, we must impose the restriction* $b^2 < 3ac$. This last restriction, we may add, in effect also implies the restriction $c > 0$. (Why?)

* This restriction may also be obtained by the method of *completing the square*. The MC function can be successively transformed as follows:

$$\begin{aligned}\text{MC} &= 3aQ^2 + 2bQ + c \\ &= \left(3aQ^2 + 2bQ + \frac{b^2}{3a}\right) - \frac{b^2}{3a} + c \\ &= \left(\sqrt{3a}\,Q + \sqrt{\frac{b^2}{3a}}\right)^2 + \frac{-b^2 + 3ac}{3a}\end{aligned}$$

Since the squared expression can possibly be zero, the positivity of MC will be ensured—on the knowledge that $a > 0$—only if $b^2 < 3ac$.

The above discussion has involved the three parameters a, b, and c. What about the other parameter, d? The answer is that there is need for a restriction on d also, but that has nothing to do with the problem of keeping the MC positive. If we let $Q = 0$ in (9.3), we find that $C(0) = d$. The role of d is thus to determine the vertical intercept of the C curve only, with no bearing on its slope. Since the economic meaning of d is the fixed cost of a firm, the appropriate restriction (in the short-run context) would be $d > 0$.

In sum, the coefficients of the total-cost function (9.3) should be restricted as follows (assuming the short-run context):

$$a, c, d > 0 \qquad b < 0 \qquad b^2 < 3ac \tag{9.5}$$

As you can readily verify, the $C(Q)$ function in Example 3 does satisfy (9.5).

Upward-Sloping Marginal-Revenue Curve

The marginal-revenue curve in Fig. 9.6c is shown to be downward-sloping throughout. This, of course, is how the MR curve is traditionally drawn for a firm under imperfect competition. However, the possibility of the MR curve being partially, or even wholly, upward-sloping can by no means be ruled out a priori.*

Given an average-revenue function $\text{AR} = f(Q)$, the marginal-revenue function can be expressed by

$$\text{MR} = f(Q) + Qf'(Q) \qquad [\text{from (7.7)}]$$

The slope of the MR curve can thus be ascertained from the derivative

$$\frac{d}{dQ}\text{MR} = f'(Q) + f'(Q) + Qf''(Q) = 2f'(Q) + Qf''(Q)$$

As long as the AR curve is downward-sloping (as it would be under imperfect competition), the $2f'(Q)$ term is assuredly negative. But the $Qf''(Q)$ term can be either negative, zero, or positive, depending on the sign of the second derivative of the AR function, i.e., depending on whether the AR curve is strictly concave, linear, or strictly convex. If the AR curve is strictly convex either in its entirety (as illustrated in Fig. 7.2) or along a specific segment, the possibility will exist that the (positive) $Qf''(Q)$ term may dominate the (negative) $2f'(Q)$ term, thereby causing the MR curve to be wholly or partially upward-sloping.

Example 4 Let the average-revenue function be

$$\text{AR} = f(Q) = 8000 - 23Q + 1.1Q^2 - 0.018Q^3$$

As can be verified (see Exercise 9.4-7), this function gives rise to a downward-sloping AR curve, as is appropriate for a firm under imperfect competition. Since

$$\text{MR} = f(Q) + Qf'(Q) = 8000 - 46Q + 3.3Q^2 - 0.072Q^3$$

* This point is emphatically brought out in John P. Formby, Stephen Layson, and W. James Smith, "The Law of Demand, Positive Sloping Marginal Revenue, and Multiple Profit Equilibria," *Economic Inquiry*, April 1982, pp. 303–311.

it follows that the slope of MR is

$$\frac{d}{dQ}\text{MR} = -46 + 6.6Q - 0.216Q^2$$

Because this is a quadratic function and since the coefficient of Q^2 is negative, $d\text{MR}/dQ$ must plot as an inverse-U-shaped curve against Q, such as shown in Fig. 9.5*a*. If a segment of this curve happens to lie above the horizontal axis, therefore, the slope of MR will take positive values.

Setting $d\text{MR}/dQ = 0$, and applying the quadratic formula, we find the two zeros of the quadratic function to be $Q_1 = 10.76$ and $Q_2 = 19.79$ (approximately). This means that, for values of Q in the open interval (Q_1, Q_2), the $d\text{MR}/dQ$ curve does lie above the horizontal axis. Thus the marginal-revenue curve indeed is positively sloped for output levels between Q_1 and Q_2.

The presence of a positively sloped segment on the MR curve has interesting implications. With more bends in its configuration, such an MR curve may produce more than one intersection with the MC curve satisfying the second-order sufficient condition for profit maximization. While all such intersections constitute local optima, however, only one of them is the global optimum that the firm is seeking.

EXERCISE 9.4

1 Find the relative maxima and minima of y by the second-derivative test:

(*a*) $y = -2x^2 + 8x + 25$ (*c*) $y = \frac{1}{3}x^3 - 3x^2 + 5x + 3$

(*b*) $y = x^3 + 6x^2 + 7$ (*d*) $y = \frac{2x}{1 - 2x} \quad \left(x \neq \frac{1}{2}\right)$

2 Mr. Greenthumb wishes to mark out a rectangular flower bed along the side wall of his house. The other three sides are to be marked by wire netting, of which he has only 32 ft available. What are the length L and width W of the rectangle that would give him the largest possible planting area? How do you make sure that your answer gives the largest, not the smallest area?

3 A firm has the following total-cost and demand functions:

$$C = \tfrac{1}{3}Q^3 - 7Q^2 + 111Q + 50$$

$$Q = 100 - P$$

(*a*) Does the total-cost function satisfy the coefficient restrictions of (9.5)?
(*b*) Write out the total-revenue function R in terms of Q.
(*c*) Formulate the total-profit function π in terms of Q.
(*d*) Find the profit-maximizing level of output $\bar{Q}$.
(*e*) What is the maximum profit?

4 If coefficient b in (9.3) were to take a zero value, what would happen to the marginal-cost and total-cost curves?

5 A quadratic profit function $\pi(Q) = hQ^2 + jQ + k$ is to be used to reflect the following assumptions:

(*a*) If nothing is produced, the profit will be negative (because of fixed costs).

(*b*) The profit function is strictly concave.

(*c*) The maximum profit occurs at a positive output level $\bar{Q}$.

What parameter restrictions are called for?

6 A purely competitive firm has a single variable input L (labor), with the wage rate W per period. Its fixed inputs cost the firm a total of F dollars per period. The price of the product is P_0.

(*a*) Write the production function, revenue function, cost function, and profit function of the firm.

(*b*) What is the first-order condition for profit maximization? Interpret the condition economically.

(*c*) What economic circumstances would ensure that profit is maximized rather than minimized?

7 Use the following procedure to verify that the AR curve in Example 4 is negatively sloped:

(*a*) Denote the slope of AR by S. Write an expression for S.

(*b*) Find the maximum value of S, S_{max}, by using the second-derivative test.

(*c*) Then deduce from the value of S_{max} that the AR curve is negatively sloped.

9.5 DIGRESSION ON MACLAURIN AND TAYLOR SERIES

The time has now come for us to develop a test for relative extrema that can apply even when the second derivative turns out to have a zero value at the stationary point. Before we can do that, however, it will first be necessary to discuss the so-called "expansion" of a function $y = f(x)$ into what are known, respectively, as a *Maclaurin series* (expansion around the point $x = 0$) and a *Taylor series* (expansion around any point $x = x_0$).

To *expand* a function $y = f(x)$ around a point x_0 means, in the present context, to transform that function into a *polynomial* form, in which the coefficients of the various terms are expressed in terms of the derivative values $f'(x_0)$, $f''(x_0)$, etc.—all evaluated at the point of expansion x_0. In the Maclaurin series, these will be evaluated at $x = 0$; thus we have $f'(0)$, $f''(0)$, etc., in the coefficients. The result of expansion may be referred to as a *power series* because, being a polynomial, it consists of a sum of power functions.

Maclaurin Series of a Polynomial Function

Let us consider first the expansion of a *polynomial* function of the nth degree,

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_nx^n \qquad (9.6)$$

Since this involves the transformation of one polynomial into another, it may seem a sterile and purposeless exercise, but actually it will serve to shed much light on the whole idea of expansion.

Since the power series after expansion will involve the derivatives of various orders of the function f, let us first find these. By successive differentiation of (9.6), we can get the derivatives as follows:

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + na_nx^{n-1}$$
$$f''(x) = 2a_2 + 3(2)a_3x + 4(3)a_4x^2 + \cdots + n(n-1)a_nx^{n-2}$$
$$f'''(x) = 3(2)a_3 + 4(3)(2)a_4x + \cdots + n(n-1)(n-2)a_nx^{n-3}$$
$$f^{(4)}(x) = 4(3)(2)a_4 + 5(4)(3)(2)a_5x + \cdots + n(n-1)(n-2)(n-3)a_nx^{n-4}$$
$$\vdots$$
$$f^{(n)}(x) = n(n-1)(n-2)(n-3)\cdots(3)(2)(1)a_n$$

Note that each successive differentiation reduces the number of terms by one—the additive constant in front drops out—until, in the nth derivative, we are left with a single constant term (a product term). These derivatives can be evaluated at various values of x; here we shall evaluate them at $x = 0$, with the result that all terms involving x will drop out. We are then left with the following exceptionally neat derivative values:

$$f'(0) = a_1 \qquad f''(0) = 2a_2 \qquad f'''(0) = 3(2)a_3 \qquad f^{(4)}(0) = 4(3)(2)a_4 \qquad \cdots \qquad f^{(n)}(0) = n(n-1)(n-2)(n-3)\cdots(3)(2)(1)a_n \tag{9.7}$$

If we now adopt a shorthand symbol $n!$ (read: "n factorial"), defined as

$$n! \equiv n(n-1)(n-2)\cdots(3)(2)(1) \qquad (n = \text{a positive integer})$$

so that, for example, $2! = 2 \times 1 = 2$ and $3! = 3 \times 2 \times 1 = 6$, etc. (with $0!$ defined as equal to 1), then the result in (9.7) can be rewritten as

$$a_1 = \frac{f'(0)}{1!} \qquad a_2 = \frac{f''(0)}{2!} \qquad a_3 = \frac{f'''(0)}{3!} \qquad a_4 = \frac{f^{(4)}(0)}{4!} \qquad \cdots \qquad a_n = \frac{f^{(n)}(0)}{n!}$$

Substituting these into (9.6) and utilizing the obvious fact that $f(0) = a_0$, we can now express the given function $f(x)$ as a new polynomial in which the coefficients are expressed in terms of derivatives evaluated at $x = 0$:*

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \tag{9.8}$$

This new polynomial, the Maclaurin series of the polynomial function $f(x)$, represents the expansion of the function $f(x)$ around zero ($x = 0$).

* Since $0! = 1$ and $1! = 1$, the first two terms on the right of the equals sign in (9.8) can be written more simply as $f(0)$, and $f'(0)x$, respectively. We have included the denominators $0!$ and $1!$ here to call attention to the symmetry among the various terms in the expansion.

Example 1 Find the Maclaurin series for the function

$$f(x) = 2 + 4x + 3x^2 \tag{9.9}$$

This function has the derivatives

$$f'(x) = 4 + 6x \qquad f''(x) = 6 \qquad \text{so that} \begin{cases} f'(0) = 4 \\ f''(0) = 6 \end{cases}$$

Thus the Maclaurin series is

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \\ &= 2 + 4x + 3x^2 \end{aligned}$$

This verifies that the Maclaurin series does indeed correctly represent the given function.

Taylor Series of a Polynomial Function

More generally, the polynomial function in (9.6) can be expanded around any point x_0, not necessarily zero. In the interest of simplicity, we shall explain this by means of the specific quadratic function in (9.9) and generalize the result later.

For the purpose of expansion around a specific point x_0, we may first interpret any given value of x as a *deviation* from x_0. More specifically, we shall let $x = x_0 + \delta$, where δ represents the deviation from the value x_0. Upon such interpretation, the given function (9.9) and its derivatives will now become

$$\begin{aligned} f(x) &= 2 + 4(x_0 + \delta) + 3(x_0 + \delta)^2 \\ f'(x) &= 4 + 6(x_0 + \delta) \\ f''(x) &= 6 \end{aligned} \tag{9.10}$$

We know that the expression $(x_0 + \delta) = x$ is a variable in the function, but since x_0 in the present context is a *fixed* number, only δ can be properly regarded as a variable in (9.10). Consequently, $f(x)$ is in fact a function of δ, say, $g(\delta)$:

$$g(\delta) = 2 + 4(x_0 + \delta) + 3(x_0 + \delta)^2 \qquad [\equiv f(x)]$$

with derivatives

$$\begin{aligned} g'(\delta) &= 4 + 6(x_0 + \delta) \qquad && [\equiv f'(x)] \\ g''(\delta) &= 6 && [\equiv f''(x)] \end{aligned}$$

We already know how to expand $g(\delta)$ around zero ($\delta = 0$). According to (9.8), such an expansion will yield the following Maclaurin series:

$$g(\delta) = \frac{g(0)}{0!} + \frac{g'(0)}{1!}\delta + \frac{g''(0)}{2!}\delta^2 \tag{9.11}$$

But since we have let $x = x_0 + \delta$, the fact that $\delta = 0$ will imply that $x = x_0$; hence, on the basis of the identity $g(\delta) \equiv f(x)$, we can write for the case of $\delta = 0$:

$$g(0) = f(x_0) \qquad g'(0) = f'(x_0) \qquad g''(0) = f''(x_0)$$

Upon substituting these into (9.11), we find the result to represent the expansion of $f(x)$ around the point x_0, because the coefficients involve the derivatives $f'(x_0)$, $f''(x_0)$, etc., all evaluated at $x = x_0$:

$$(9.12) \qquad f(x)[= g(\delta)] = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$

You should compare this result—a Taylor series of $f(x)$—with the Maclaurin series of $g(\delta)$ in (9.11).

Since for the specific function under consideration, namely, (9.9), we have

$$f(x_0) = 2 + 4x_0 + 3x_0^2 \qquad f'(x_0) = 4 + 6x_0 \qquad f''(x_0) = 6$$

the Taylor-series formula (9.12) will yield

$$\begin{aligned} f(x) &= 2 + 4x_0 + 3x_0^2 + (4 + 6x_0)(x - x_0) + \tfrac{6}{2}(x - x_0)^2 \\ &= 2 + 4x + 3x^2 \end{aligned}$$

This verifies that the Taylor series does correctly represent the given function.

The expansion formula in (9.12) can be generalized to apply to the nth-degree polynomial of (9.6). The generalized Taylor-series formula is

$$(9.13) \qquad f(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

This differs from the Maclaurin series of (9.8) only in the replacement of zero by x_0 as the point of expansion and in the replacement of x by the expression $(x - x_0)$. What (9.13) tells us is that, given an nth-degree polynomial $f(x)$, if we let $x = 7$ (say) in the terms on the right of (9.13), select an arbitrary number x_0, then evaluate and add these terms, we will end up exactly with $f(7)$—the value of $f(x)$ at $x = 7$.

Example 2 Taking $x_0 = 3$ as the point of expansion, we can rewrite (9.6) equivalently as

$$f(x) = f(3) + f'(3)(x - 3) + \frac{f''(3)}{2}(x - 3)^2 + \cdots + \frac{f^{(n)}(3)}{n!}(x - 3)^n$$

Expansion of an Arbitrary Function

Heretofore, we have shown how an nth-degree polynomial function can be expressed in another nth-degree polynomial form. As it turns out, it is also possible to express any *arbitrary* function $\phi(x)$—one that is not even necessarily a polynomial—in a polynomial form similar to (9.13), provided $\phi(x)$ has finite, continuous derivatives up to the desired order at the expansion point x_0.

According to a mathematical proposition known as *Taylor's theorem*, given an arbitrary function $\phi(x)$, if we know the value of the function at $x = x_0$ [that is, $\phi(x_0)$] and the values of its derivatives at x_0 [that is, $\phi'(x_0)$, $\phi''(x_0)$, etc.], then this function can be expanded around the point x_0 as follows (n = a fixed positive integer arbitrarily chosen):

$$\begin{aligned}(9.14) \qquad \phi(x) &= \left[\frac{\phi(x_0)}{0!} + \frac{\phi'(x_0)}{1!}(x - x_0)\right.\\ &\quad \left. + \frac{\phi''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{\phi^{(n)}(x_0)}{n!}(x - x_0)^n\right] + R_n \\ &\equiv P_n + R_n\end{aligned}$$

where P_n represents the (bracketed) nth-degree polynomial [the first $(n + 1)$ terms on the right], and R_n denotes a *remainder*, to be explained below.* The presence of R_n distinguishes (9.14) from (9.13), and for this reason (9.14) is called a *Taylor series with remainder*. The form of the polynomial P_n and the size of the remainder R_n will depend on the value of n we choose. The larger the n, the more terms there will be in P_n; accordingly, R_n will in general assume a different value for each different n. This fact explains the need for the subscript n in these two symbols. As a memory aid, we can identify n as the order of the highest derivative in P_n. (In the special case of $n = 0$, no derivative will appear in P_n at all.)

The appearance of R_n in (9.14) is due to the fact that we are here dealing with an arbitrary function ϕ which cannot always be transformed *exactly* into the polynomial form shown in (9.13). Therefore, a remainder term is included as a supplement to the P_n part, in order to represent the difference between $\phi(x)$ and the polynomial P_n. Looked at differently, P_n may be considered a polynomial approximation to $\phi(x)$, with the term R_n as a measure of the error of approximation. If we choose $n = 1$, for example, we have

$$\phi(x) = [\phi(x_0) + \phi'(x_0)(x - x_0)] + R_1 = P_1 + R_1$$

where P_1 consists of $n + 1 = 2$ terms and constitutes a *linear* approximation to $\phi(x)$. If we choose $n = 2$, a second-power term will appear, so that

$$\phi(x) = \left[\phi(x_0) + \phi'(x_0)(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2\right] + R_2 = P_2 + R_2$$

where P_2, consisting of $n + 1 = 3$ terms, will be a *quadratic* approximation to $\phi(x)$. And so forth.

* The symbol R_n (remainder) is not to be confused with the symbol R^n (n-space).

We should mention, in passing, that the arbitrary function $\phi(x)$ could obviously encompass the nth-degree polynomial of (9.6) as a special case. For this latter case, if the expansion is into another nth-degree polynomial, the result of (9.13) will exactly apply; or in other words, we can use the result in (9.14), with $R_n \equiv 0$. However, if the given nth-degree polynomial $f(x)$ is to be expanded into a polynomial of a *lesser* degree, then the latter can only be considered an approximation to $f(x)$, and a remainder will appear; accordingly, the result in (9.14) can be applied with a nonzero remainder. Thus the Taylor series in the form of (9.14) is perfectly general.

Example 3 Expand the nonpolynomial function

$$\phi(x) = \frac{1}{1+x}$$

around the point $x_0 = 1$, with $n = 4$. We shall need the first four derivatives of $\phi(x)$, which are

$$\phi'(x) = -(1+x)^{-2} \qquad \text{so that} \qquad \phi'(1) = -(2)^{-2} = \frac{-1}{4}$$

$$\phi''(x) = 2(1+x)^{-3} \qquad \qquad \phi''(1) = 2(2)^{-3} = \frac{1}{4}$$

$$\phi'''(x) = -6(1+x)^{-4} \qquad \qquad \phi'''(1) = -6(2)^{-4} = \frac{-3}{8}$$

$$\phi^{(4)}(x) = 24(1+x)^{-5} \qquad \qquad \phi^{(4)}(1) = 24(2)^{-5} = \frac{3}{4}$$

Also, we see that $\phi(1) = \frac{1}{2}$. Thus, setting $x_0 = 1$ in (9.14) and utilizing the information derived above, we obtain the following Taylor series with remainder:

$$\begin{aligned}\phi(x) &= \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + \frac{1}{32}(x-1)^4 + R_4 \\ &= \frac{31}{32} - \frac{13}{16}x + \frac{1}{2}x^2 - \frac{3}{16}x^3 + \frac{1}{32}x^4 + R_4\end{aligned}$$

It is possible, of course, to choose $x_0 = 0$ as the point of expansion here, too. In that case, with x_0 set equal to zero in (9.14), the expansion will result in a *Maclaurin series with remainder*.

Example 4 Expand the quadratic function

$$\phi(x) = 5 + 2x + x^2$$

around $x_0 = 1$, with $n = 1$. This function is, like (9.9) in Example 1, a second-degree polynomial. But since our assigned task is to expand it into a *first*-degree polynomial ($n = 1$)—i.e., to find a linear approximation to the given quadratic function—a remainder term is bound to appear. For this reason, $\phi(x)$ is viewed as an "arbitrary" function for the purpose of the Taylor expansion.

To carry out this expansion, we need only the first derivative $\phi'(x) = 2 + 2x$. Evaluated at $x_0 = 1$, the given function and its derivative yield

$$\phi(x_0) = \phi(1) = 8 \qquad \phi'(x_0) = \phi'(1) = 4$$

Thus the Taylor series with remainder is

$$\begin{aligned}\phi(x) &= \phi(x_0) + \phi'(x_0)(x - x_0) + R_1 \\ &= 8 + 4(x - 1) + R_1 = 4 + 4x + R_1\end{aligned}$$

where the $(4 + 4x)$ term is a linear approximation and the R_1 term represents the error of approximation.

In Fig. 9.7, $\phi(x)$ plots as a parabola, and its linear approximation, a straight line tangent to the $\phi(x)$ curve at the point (1, 8). The occurrence of the point of tangency at $x = 1$ is not a matter of coincidence; rather, it is the direct consequence of the fact that the point of expansion is set at that particular value of x. This suggests that, when an arbitrary function $\phi(x)$ is approximated by a polynomial, the latter will give the exact value of $\phi(x)$ *at* (but only *at*) the point of expansion, with zero error of approximation ($R_1 = 0$). Elsewhere, R_1 is strictly nonzero and, in fact, shows increasingly larger errors of approximation as we try to approximate $\phi(x)$ for x values farther and farther away from the point of expansion x_0.

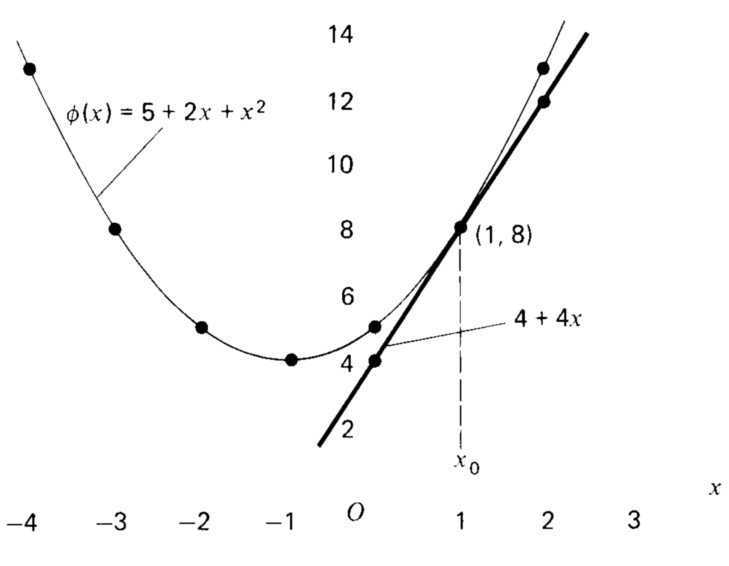


Figure 9.7

Lagrange Form of the Remainder

Now we must comment further on the remainder term. According to the *Lagrange form of the remainder*, we can express R_n as

$$R_n = \frac{\phi^{(n+1)}(p)}{(n+1)!}(x - x_0)^{n+1} \tag{9.15}$$

where p is some number between x (the point where we wish to evaluate the arbitrary function ϕ) and x_0 (the point where we expand the function ϕ). Note that this expression closely resembles the term which should logically follow the last term in P_n in (9.14), except that the derivative involved is here to be evaluated at a point p instead of x_0. Since the point p is not otherwise specified, this formula does not really enable us to calculate R_n; nevertheless, it does have great analytical significance. Let us therefore illustrate its meaning graphically, although we shall do it only for the simple case of $n = 0$.

When $n = 0$, no derivatives whatever will appear in the polynomial part P_0; therefore (9.14) reduces to

$$\phi(x) = P_0 + R_0 = \phi(x_0) + \phi'(p)(x - x_0)$$

or $$\phi(x) - \phi(x_0) = \phi'(p)(x - x_0)$$

This result, a simple version of the *mean-value theorem*, states that the difference between the value of the function ϕ at x_0 and at any other x value can be expressed as the product of the difference $(x - x_0)$ and the derivative ϕ' evaluated at p (with p being some point between x and x_0). Let us look at Fig. 9.8, where the function $\phi(x)$ is shown as a continuous curve with derivative values defined at all points. Let x_0 be the chosen point of expansion, and let x be *any* point on the horizontal axis. If we try to approximate $\phi(x)$, or distance xB, by $\phi(x_0)$, or distance x_0A, it will involve an error equal to $\phi(x) - \phi(x_0)$, or the distance CB.

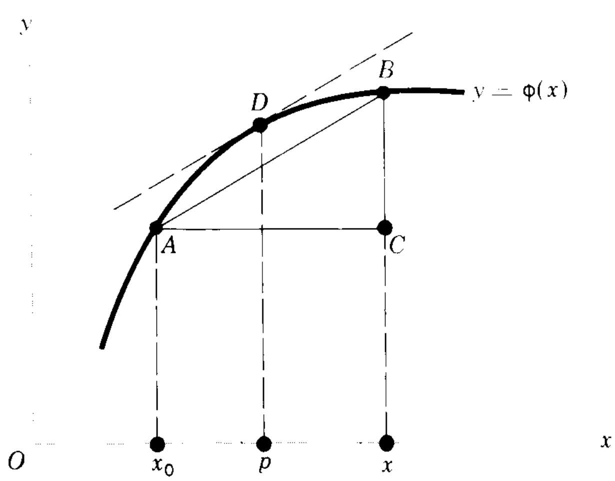


Figure 9.8

What the mean-value theorem says is that the error CB—which constitutes the value of the remainder term R_0 in the expansion—can be expressed as $\phi'(p)(x - x_0)$, where p is some point between x and x_0. First we locate, on the curve between points A and B, a point D such that the tangent line at D is parallel to line AB; such a point D must exist, since the curve passes from A to B in a continuous and smooth manner. Then, the remainder will be

$$\begin{aligned} R_0 = CB &= \frac{CB}{AC}AC = (\text{slope of } AB) \cdot AC \\ &= (\text{slope of tangent at } D) \cdot AC \\ &= (\text{slope of curve at } x = p) \cdot AC = \phi'(p)(x - x_0) \end{aligned}$$

where the point p is between x and x_0, as required. This demonstrates the rationale of the Lagrange form of the remainder for the case $n = 0$. We can always express R_0 as $\phi'(p)(x - x_0)$ because, even though p cannot be assigned a specific value, we can be sure that such a point exists.

Equation (9.15) provides a way of expressing the remainder term R_n, but it does not eliminate R_n as a source of discrepancy between $\phi(x)$ and the polynomial P_n. However, if it happens that

$$R_n \to 0 \text{ as } n \to \infty \qquad \text{so that} \qquad P_n \to \phi(x) \text{ as } n \to \infty$$

then it will be possible to make P_n as accurate an approximation to $\phi(x)$ as we desire by choosing a large enough value for n, that is, by including a large enough number of terms in the polynomial P_n.* In this (convenient) event, the Taylor series is said to be *convergent* to $\phi(x)$ at the point of expansion. An example of this will be discussed in Sec. 10.2 below.

EXERCISE 9.5

1 Find the value of the following factorial expressions:
(a) $5!$ (b) $7!$ (c) $\frac{4!}{3!}$ (d) $\frac{6!}{3!}$ (e) $\frac{(n+2)!}{n!}$

2 Find the first five terms of the Maclaurin series (i.e., choose $n = 4$ and let $x_0 = 0$) for:
(a) $\phi(x) = \frac{1}{1-x}$ (b) $\phi(x) = \frac{1-x}{1+x}$

3 Find the Taylor series, with $n = 4$ and $x_0 = -2$, for the two functions in the preceding problem.

4 On the basis of the Taylor series with the Lagrange form of the remainder [see (9.14) and (9.15)], show that at the point of expansion ($x = x_0$) the Taylor series will always give *exactly* the value of the function at that point, $\phi(x_0)$, not merely an approximation.

* This should be reminiscent of the method of finding the inverse matrix by approximation, as discussed in Sec. 5.7.

9.6 *N*TH-DERIVATIVE TEST FOR RELATIVE EXTREMUM OF A FUNCTION OF ONE VARIABLE

The expansion of a function into a Taylor (or Maclaurin) series is useful as an approximation device in the circumstance that $R_n \to 0$ as $n \to \infty$, but our present concern is with its application in the development of a general test for a relative extremum.

Taylor Expansion and Relative Extremum

As a preparatory step for that task, let us redefine a relative extremum as follows:

> A function $f(x)$ attains a relative maximum (minimum) value at x_0 if $f(x) - f(x_0)$ is negative (positive) for values of x in the immediate neighborhood of x_0, both to its left and to its right.

This can be made clear by reference to Fig. 9.9, where x_1 is a value of x to the left of x_0, and x_2 is a value of x to the right of x_0. In diagram *a*, $f(x_0)$ is a relative maximum; thus $f(x_0)$ exceeds both $f(x_1)$ and $f(x_2)$. In short, $f(x) - f(x_0)$ is negative for any value of x in the immediate neighborhood of x_0. The opposite is true of diagram *b*, where $f(x_0)$ is a relative minimum, and thus $f(x) - f(x_0) > 0$.

Assuming $f(x)$ to have finite, continuous derivatives up to the desired order at the point $x = x_0$, the function $f(x)$—not necessarily polynomial—can be expanded around the point x_0 as a Taylor series. On the basis of (9.14) (after duly changing ϕ to f), and using the Lagrange form of the remainder, we can write

$$(9.16) \qquad f(x) - f(x_0) = f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(p)}{(n+1)!}(x - x_0)^{n+1}$$

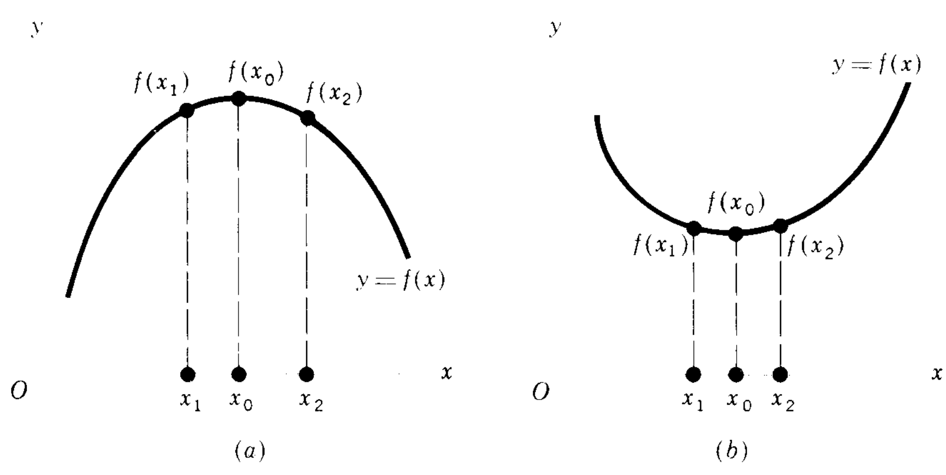


Figure 9.9

If we can determine the sign of the expression $f(x) - f(x_0)$ for values of x to the immediate left and right of x_0, we can readily come to a conclusion as to whether $f(x_0)$ is an extremum, and if so, whether it is a maximum or a minimum. For this, it is necessary to examine the right-hand sum of (9.16). Altogether, there are $(n + 1)$ terms in this sum—n terms from P_n, plus the remainder—and thus the actual number of terms is indefinite, being dependent upon the value of n we choose. However, by properly choosing n, we can always make sure that there will exist only a single term on the right, thereby drastically simplifying the task of evaluating the sign of $f(x) - f(x_0)$ and ascertaining whether $f(x_0)$ is an extremum, and if so, which kind.

Some Specific Cases

This will become clearer through some specific illustrations.

Case 1 $f'(x_0) \neq 0$

If the first derivative at x_0 is nonzero, let us choose $n = 0$; then there will be only $n + 1 = 1$ term on the right side, implying that only the remainder R_0 will be there. That is, we have

$$f(x) - f(x_0) = \frac{f'(p)}{1!}(x - x_0) = f'(p)(x - x_0)$$

where p is some number between x_0 and a value of x in the immediate neighborhood of x_0. Note that p must accordingly be very, very close to x_0.

What is the sign of the expression on the right? Because of the continuity of the derivative, $f'(p)$ will have the same sign as $f'(x_0)$ since, as mentioned above, p is very, very close to x_0. In the present case, $f'(p)$ must be nonzero; in fact, it must be a specific positive or negative number. But what about the $(x - x_0)$ part? When we go from the left of x_0 to its right, x shifts from a magnitude $x_1 < x_0$ to a magnitude $x_2 > x_0$ (see Fig. 9.9). Consequently, the expression $(x - x_0)$ must turn from negative to positive as we move, and $f(x) - f(x_0) = f'(p)(x - x_0)$ must also change sign from the left of x_0 to its right. However, this violates our new definition of a relative extremum; accordingly, there cannot exist a relative extremum at $f(x_0)$ when $f'(x_0) \neq 0$—a fact that is already well known to us.

Case 2 $f'(x_0) = 0$; $f''(x_0) \neq 0$

In this case, choose $n = 1$, so that initially there will be $n + 1 = 2$ terms on the right. But one of these terms will vanish because $f'(x_0) = 0$, and we shall again be left with only one term to evaluate:

$$\begin{aligned} f(x) - f(x_0) &= f'(x_0)(x - x_0) + \frac{f''(p)}{2!}(x - x_0)^2 \\ &= \tfrac{1}{2}f''(p)(x - x_0)^2 \qquad [\text{because } f'(x_0) = 0] \end{aligned}$$

As before, $f''(p)$ will have the same sign as $f''(x_0)$, a sign that is specified and unvarying; whereas the $(x - x_0)^2$ part, being a square, is invariably positive. Thus the expression $f(x) - f(x_0)$ must take the same sign as $f''(x_0)$ and, according to the above definition of relative extremum, will specify

$$\begin{aligned} &\text{A relative maximum of } f(x) \text{ if } f''(x_0) < 0 \\ &\text{A relative minimum of } f(x) \text{ if } f''(x_0) > 0 \end{aligned} \qquad [\text{with } f'(x_0) = 0]$$

You will recognize this as the second-derivative test introduced earlier.

Case 3 $f'(x_0) = f''(x_0) = 0$, but $f'''(x_0) \neq 0$

Here we are encountering a situation that the second-derivative test is incapable of handling, for $f''(x_0)$ is now zero. With the help of the Taylor series, however, a conclusive result can be established without difficulty.

Let us choose $n = 2$; then three terms will initially appear on the right. But two of these will drop out because $f'(x_0) = f''(x_0) = 0$, so that we again have only one term to evaluate:

$$\begin{aligned} f(x) - f(x_0) &= f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \frac{1}{3!}f'''(p)(x - x_0)^3 \\ &= \frac{1}{6}f'''(p)(x - x_0)^3 \end{aligned}$$

As previously, the sign of $f'''(p)$ is identical with that of $f'''(x_0)$ because of the continuity of the derivative and because p is very close to x_0. But the $(x - x_0)^3$ part has a varying sign. Specifically, since $(x - x_0)$ is negative to the left of x_0, so also will be $(x - x_0)^3$; yet, to the right of x_0, the $(x - x_0)^3$ part will be positive. Again there is a change in the sign of $f(x) - f(x_0)$ as we pass through x_0, which violates the definition of a relative extremum. However, we know that x_0 is a critical value $[f'(x_0) = 0]$, and thus it must give an inflection point, inasmuch as it does not give a relative extremum.

Case 4 $f'(x_0) = f''(x_0) = \cdots = f^{(N-1)}(x_0) = 0$, but $f^{(N)}(x_0) \neq 0$

This is a very general case, and we can therefore derive a general result from it. Note that here all the derivative values are zero until we arrive at the Nth one.

Analogously to the preceding three cases, the Taylor series for Case 4 will reduce to

$$f(x) - f(x_0) = \frac{1}{N!}f^{(N)}(p)(x - x_0)^N$$

Again, $f^{(N)}(p)$ takes the same sign as $f^{(N)}(x_0)$, which is unvarying. The sign of the $(x - x_0)^N$ part, on the other hand, will *vary* if N is *odd* (cf. Cases 1 and 3) and will *remain unchanged* (positive) if N is *even* (cf. Case 2). When N is odd, accordingly, $f(x) - f(x_0)$ will change sign as we pass through the point x_0, thereby violating the definition of a relative extremum (which means that x_0 must give us an inflection point). But when N is even, $f(x) - f(x_0)$ will not change sign from the left of x_0 to its right, and this will establish the stationary value $f(x_0)$ as a relative maximum or minimum, depending on whether $f^{(N)}(x_0)$ is negative or positive.

Nth-Derivative Test

At last, then, we may state the following general test.

Nth-Derivative test for relative extremum of a function of one variable If the first derivative of a function $f(x)$ at x_0 is $f'(x_0) = 0$ and if the first *nonzero* derivative value at x_0 encountered in successive derivation is that of the Nth derivative, $f^{(N)}(x_0) \neq 0$, then the stationary value $f(x_0)$ will be

a. A relative *maximum* if N is an even number and $f^{(N)}(x_0) < 0$.
b. A relative *minimum* if N is an even number but $f^{(N)}(x_0) > 0$.
c. An *inflection point* if N is odd.

It should be clear from the above statement that the Nth-derivative test can work if and only if the function $f(x)$ is capable of yielding, sooner or later, a nonzero derivative value at the critical value x_0. While there do exist exceptional functions that fail to satisfy this condition, most of the functions we are likely to encounter will indeed produce nonzero $f^{(N)}(x_0)$ in successive differentiation.* Thus the test should prove serviceable in most instances.

Example 1 Examine the function $y = (7 - x)^4$ for its relative extremum. Since $f'(x) = -4(7 - x)^3$ is zero when $x = 7$, we take $x = 7$ as the critical value for testing, with $y = 0$ as the stationary value of the function. By successive derivation (continued until we encounter a nonzero derivative value at the point $x = 7$),

* If $f(x)$ is a constant function, for instance, then obviously $f'(x) = f''(x) = \cdots = 0$, so that no nonzero derivative value can ever be found. This, however, is a trivial case, since a constant function requires no test for extremum anyway. As a nontrivial example, consider the function

$$y = \begin{cases} e^{-1/x^2} & (\text{for } x \neq 0) \\ 0 & (\text{for } x = 0) \end{cases}$$

where the function $y = e^{-1/x^2}$ is an exponential function, yet to be introduced (Chap. 10). By itself, $y = e^{-1/x^2}$ is discontinuous at $x = 0$, because $x = 0$ is not in the domain (division by zero is undefined). However, since $\lim_{x \to 0} y = 0$, we can, by appending the stipulation that $y = 0$ for $x = 0$, fill the gap in the domain and thereby obtain a continuous function. The graph of this function shows that it attains a minimum at $x = 0$. But it turns out that, at $x = 0$, all the derivatives (up to any order) have zero values. Thus we are unable to apply the Nth-derivative test to confirm the graphically ascertainable fact that the function has a minimum at $x = 0$. For further discussion of this exceptional case, see R. Courant, *Differential and Integral Calculus* (translated by E. J. McShane), Interscience, New York, vol. I, 2d ed., 1937, pp. 196, 197, and 336.

we get

$$f''(x) = 12(7 - x)^2 \qquad \text{so that} \qquad f''(7) = 0$$
$$f'''(x) = -24(7 - x) \qquad\qquad f'''(7) = 0$$
$$f^{(4)}(x) = 24 \qquad\qquad f^{(4)}(7) = 24$$

Since 4 is an even number and since $f^{(4)}(7)$ is positive, we conclude that the point $(7, 0)$ represents a relative minimum.

As is easily verified, this function plots as a strictly convex curve. Inasmuch as the second derivative at $x = 7$ is zero (rather than positive), this example serves to illustrate our earlier statement regarding the second derivative and the curvature of a curve (Sec. 9.3) to the effect that, while a positive $f''(x)$ for all x does imply a strictly convex $f(x)$, a strictly convex $f(x)$ does *not* imply a positive $f''(x)$ for all x. More importantly, it also serves to illustrate the fact that, given a strictly convex (strictly concave) curve, the extremum found on that curve must be a minimum (maximum), because such an extremum will *either* satisfy the second-order sufficient condition, *or*, failing that, satisfy another (higher-order) sufficient condition for a minimum (maximum).

EXERCISE 9.6

1 Find the stationary values of:
(*a*) $y = x^3$ (*b*) $y = -x^4$ (*c*) $y = x^6 + 5$
Determine by the Nth-derivative test whether they represent relative maxima, relative minima, or inflection points.

2 Find the stationary values of the following functions:
(*a*) $y = (x - 1)^3 + 16$ (*b*) $y = (x - 2)^4$ (*c*) $y = (3 - x)^6 + 7$
Use the Nth-derivative test to determine the exact nature of these stationary values.