Notes for Chapters 6 & 7

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1 Comparative Statistics

1.0.1 Example: "Shift in Demand"

Comparing the value of variables (P,Q) from one equilibrium point to another equilibrium point

1. Comparative statistics compares the values of P and Q at the points A and B ONLY!!!
2. Says nothing about the path they follow from A to B
3. Often, we are only interested in the direction variables move (ie. up or down, bigger or smaller)

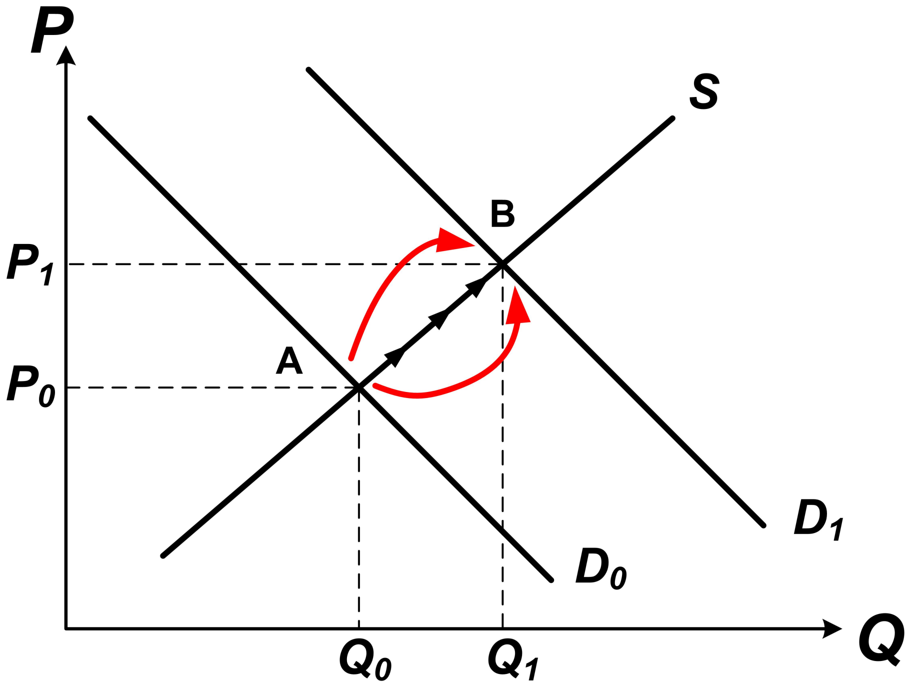
1.0.2 Find the Slope of a Non-Linear Function

Slope $= \frac{\text{rise}}{\text{run}} = \frac{\Delta Y}{\Delta X} = \frac{y_2 - y_1}{x_2 - x_1}$

Since: $y = f(x) \implies \frac{\Delta Y}{\Delta X} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

At D: Slope $= \frac{\Delta Y}{\Delta X} = \frac{f(x_1 + h_2) - f(x_1)}{(x_1 + h_2) - x_1} = \frac{f(x_1 + h_2) - f(x_1)}{h_2}$

At B: $\frac{\Delta Y}{\Delta X} = \frac{f(x_1 + h_1) - f(x_1)}{h_1}$ as h$\longrightarrow$ 0 then $(x + h) \longrightarrow x$ in the Limit



1.1 The Limit

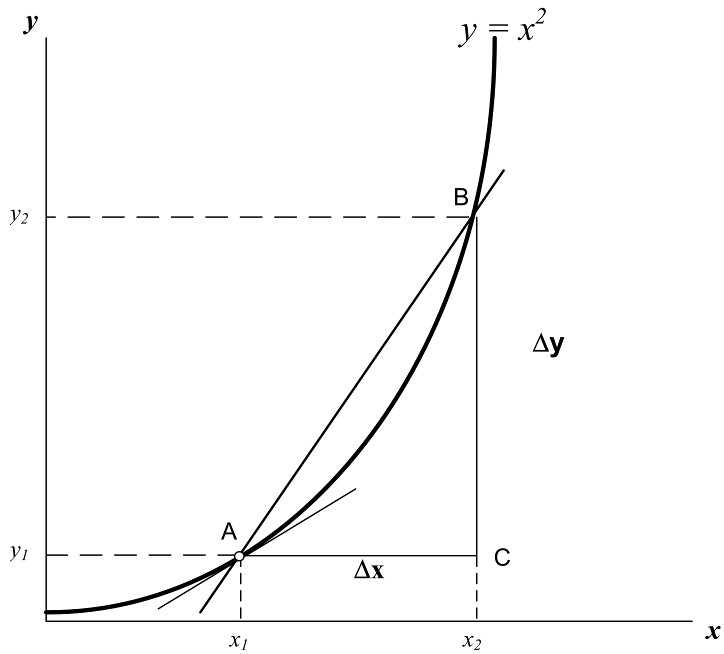
Slope=$\frac{f(x+h)-f(x)}{(x+h)-x}$

<u>for $y = f(x) = x^2$</u>

$$
\begin{aligned}
Slope &= \frac{(x+h)^2 - x^2}{(x+h) - x} \\
&= \frac{(x^2 + 2xh + h^2) - x^2}{h} \\
&= \frac{2xh + h^2}{h} \\
&= (2x + h)
\end{aligned}
$$

Let h go to zero (or take the limit)

lim h$\longrightarrow$ 0 $\quad (2x + h) = 2x$



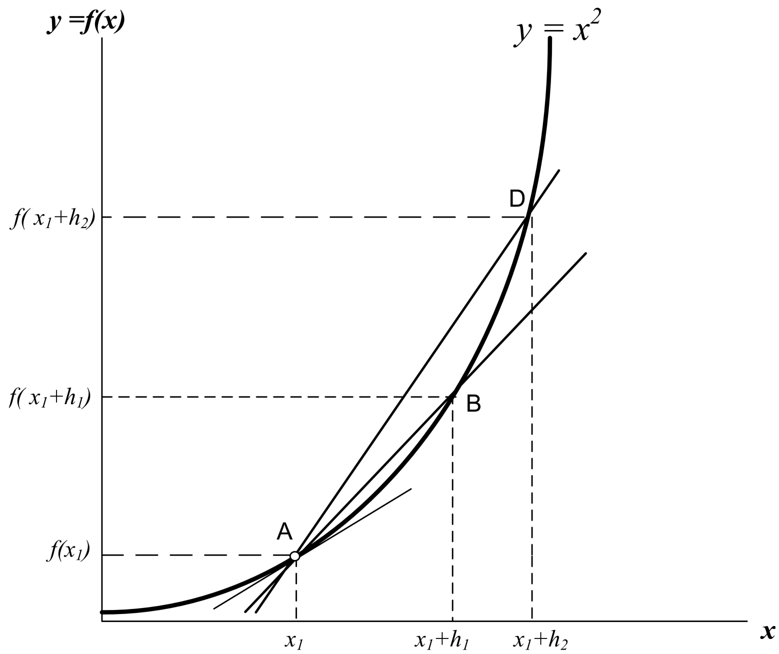
$2x$ is the lop of x^2 at x_1
Generally:

$$\lim\ \mathrm{h} \longrightarrow 0 \left(\frac{\Delta Y}{\Delta X}\right) = \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{dy}{dx} = f'(x) \text{ the derivative}$$

1.2 Left Hand, Right Hand Limit

1.2.1 Left Hand Limit

Start at $x - h$



Slope

$$
\begin{aligned}
&= \frac{f(x) - f(x-h)}{x-(x-h)} \\
&= \frac{x^2 - (x-h)^2}{h} \\
&= x^2 - x^2 + 2xh - h^2 \\
&= 2x - h \\
\lim h &\longrightarrow 0(2x-h) = 2x = f'(x)
\end{aligned}
$$

1.2.2 Right Hand Limit

Start at x+h

$$
\begin{aligned}
&= \frac{f(x+h)-f(x)}{(x+h)-x} \\
&= \frac{(x+h)^2-x^2}{h} \\
&= \frac{2xh+h^2}{h} \\
&= 2x+h \\
\lim h &\longrightarrow 0(2xh+h) = 2x = f'(x)
\end{aligned}
$$

Therefore: $RHL = LHL = 2x = f`(x)$

1.3 Continuity and Differentiability of a Function

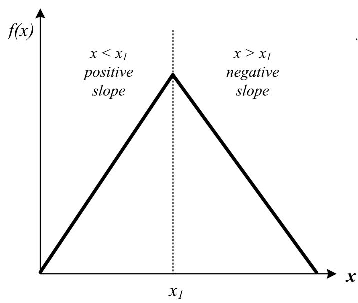
The Result: Right Hand Limit = Left Hand Limit, or

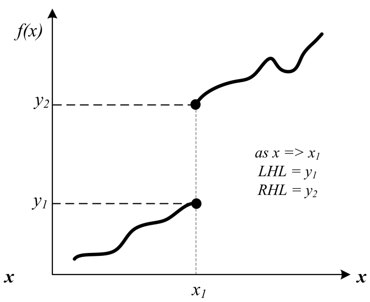
$$\lim h \longrightarrow 0\left[f\left(x_1+h\right)\right] = \lim h \longrightarrow -0\left[f(x_1-h)\right] \quad \text{for } x = x_1$$

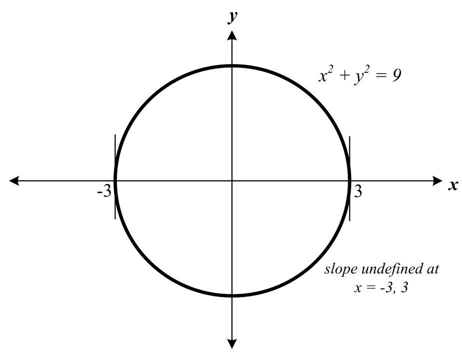
IS NOT ALWAYS TRUE

If it is true then the derivative at $x = x_1$ exists. In general, the derivative of a function exists if:

1. f(x) is a well defined function at x=x$_1$ $\{$ie. f(x)=$\frac{1}{x}$ and x$_1 = 0\}$
2. $\lim x \longrightarrow x_1 \;\; (f(x)) = f(x_1)$
3. x$_1$ is the in the domain of f(x)







1.3.1 Examples of Discontinuous Functions

1.3.2 Rules of Differentiation

1. Constant Function

$$\text{If } y = f(x) = k \quad \text{Then } \frac{dy}{dx} = f'(x) = 0$$

[1]

2. Power Function

$$\text{If } y = ax^n \ \{a, n \text{ are constants}\} \ \text{Then } \frac{dy}{dx} = anx^{n-1}$$

Example

[1] insert first graph beside #1 on Page 8

(a) y=x^2 $\frac{dy}{dx} = 2x$

(b) y=3x^4 $\frac{dy}{dx} = 12x^3$

(c) y=x^{-1} $\frac{dy}{dx} = (-1)x^{-2}$
[2]

3. Sum-Difference Rule

$$\text{If} y = f(x) \pm g(x) \text{ then } \frac{dy}{dx} = \frac{d}{dx}\left[f(x) \pm g(x)\right] \;\; = f'(x) \pm g'(x)$$

Examples

(a) Let

$$f(x) = x^3 \;\; g(x) = x^{1/2}$$

Therefore

$$\begin{aligned} y &= x^3 + x^{1/2} \\ \frac{dy}{dx} &= 3x^2 + \frac{1}{2}x^{-1/2} \end{aligned}$$

(b) If

$$y = f(x) - g(x)$$

Where

$$f(x) = 2x^3 \text{ and } g(x) = x^4$$

Then

$$\frac{dy}{dx} = 6x^2 - 4x^3$$

[2] Insert 2nd graph on page 8

4. Product Rule

   If

$$y = f(x)g(x)$$

   Then

$$\frac{dy}{dx} = f(x)g'(x) + f'(x)g(x)$$

   Example: Let

$$f(x) = \left(x^2 + x\right) \quad g(x) = x^3$$

   Then

$$\begin{aligned} y &= \left(x^2 + x\right) + \left(x^3\right) \\ \frac{dy}{dx} &= \underbrace{(2x+1)(x^3)}_{f'(x)g(x)} + \underbrace{(x^2+x)(3x^2)}_{f(x)g'(x)} \end{aligned}$$

   3 Function Case

   if

$$y = f(x)g(x)h(x)$$

   Then

$$\frac{dy}{dx} = \underbrace{f'(x)g(x)h(x)}_{f'gh} + \underbrace{f(x)g'(x)h(x)}_{fg'h} + \underbrace{f(x)g(x)h'(x)}_{fgh'}$$

5. Quotient Rule

   If

$$y = \frac{f(x)}{g(x)}$$

Then

$$\frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Example:

$$\begin{aligned} y &= \frac{\left(x^3+2\right)}{x^2} \\ \frac{dy}{dx} &= \frac{(3x^2)(x^2) - (x^3+2)(2x)}{x^4} = \frac{x^3-4}{x^3} \end{aligned}$$

Quotient Rule is a special case of PRODUCT RULE.

$$\begin{aligned} y &= \frac{\left(x^3+2\right)}{x^2} = \left(x^3+2\right)\left(x^{-2}\right) \\ \frac{dy}{dx} &= \left(3x^2+2\right)\left(x^{-2}\right) + \left(x^3+2\right)\left(-2x^{-3}\right) = 3 - 2 - 4x^{-3} \\ &= 1 - 4x^{-3} = \frac{x^3-4}{x^3} \end{aligned}$$

6. Chain Rule

Suppose	$y = f(x)$
and	$x = g(z)$
Then	$y = f(g(z))$

Therefore:

$$\frac{dy}{dx} = \left(\frac{dy}{dx}\right)\left(\frac{dx}{dz}\right) = f'(g(z))g'(z)$$

Chain effect

$$\Delta Y \longleftarrow \Delta X \longleftarrow \Delta Z$$

Example:Let

$$
\begin{aligned}
y &= f(x) = x^2 \\
x &= g(z) = (x+2)
\end{aligned}
$$

Then

$$y = f(g(z)) = (x+2)^2$$

And

$$\frac{dy}{dz} = 2(x+2)$$

1.4 Monotonic Functions and the Inverse Function Rule

If $x_1 < x_2$ and $f(x_1) < f(x_2)$ (for all x), then $f(x)$ is Monotonically increasing.

If $x_1 < x_2$ and $f(x_1) > f(x_2)$ then $f(x)$ is Monotonically decreasing.

If a function is Monotonic the an inverse function exists. Ie. If $y = f(x)$, then $x = f^{-1}(y)$.

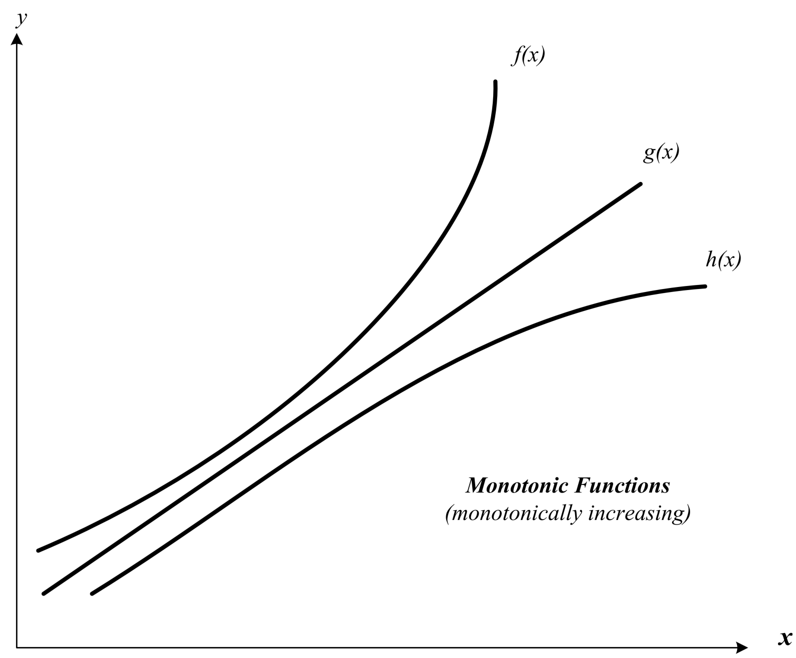
Example $y = x^2 \;\; (x \geqslant 0)$, $x = \sqrt{y}$

1.4.1 Derivative of Inverse Functions

If $y = f(x)$ and $x = f^{-1}(y)$, then $\frac{dy}{dx} = f'(x)$ and $\frac{dx}{dy} = \frac{1}{f'(x)}$

Example 1:

$$
\begin{aligned}
y &= 3x + 2 \Rightarrow \frac{dy}{dx} = 3 \\
x &= \frac{1}{3}y - \frac{2}{3} \Rightarrow \frac{dx}{dy} = \frac{1}{3} = \frac{1}{\frac{dy}{dx}}
\end{aligned}
$$



Example 2:

$$\text{If:} \quad y = x^2 \text{ and } \frac{dy}{dx} = 2x$$

$$\text{then:} \quad x = y^{1/2} \text{ and } \frac{dx}{dy} = \frac{1}{2}y^{-1/2} = \frac{1}{2y^{1/2}}$$

$$\text{so:} \quad \frac{dx}{dy} = \frac{1}{2x} = \frac{1}{\frac{dy}{dx}}$$

Application: Revenue Functions

$$\text{Demand Function} \;:\; Q = 10 - P$$
$$\text{Inverse Demand Function} \;:\; P = 10 - Q$$

Average Revenue

$$AR = P = 10 - Q \text{ Inverse demand function}$$

Total Revenue

$$\begin{aligned} TR &= P \cdot Q = (10 - Q)Q = 10Q - Q^2 \\ TR &= 10Q - Q^2 \text{ is a quadratic function} \end{aligned}$$

Marginal Revenue

$$MR = \frac{d(TR)}{dQ} = 10 - 2Q$$

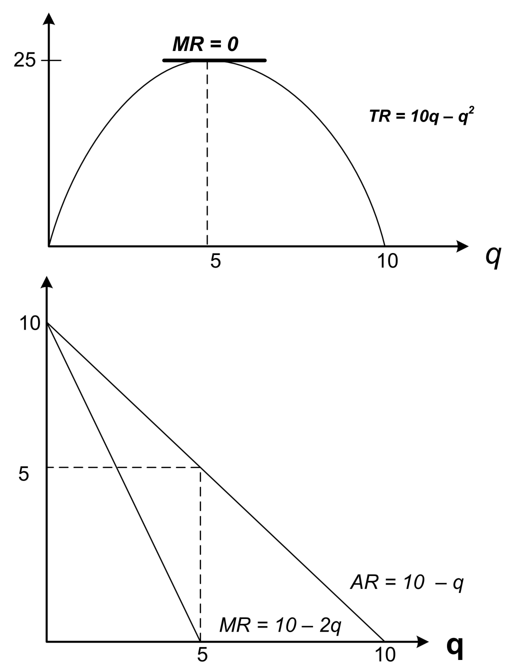
Given $AR = 10 - Q$ and $MR = 10 - 2Q$ MR falls twice as fast as AR.

Generally:

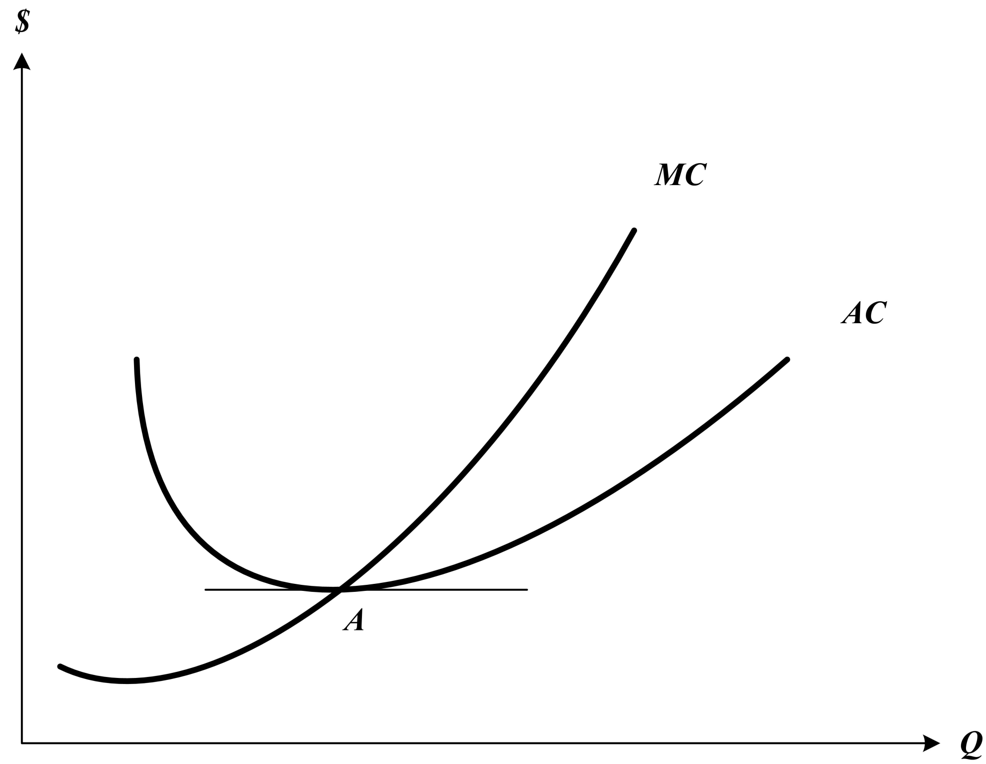
$$\begin{aligned} TR &= aQ - bQ^2 \text{ (general form quadratic)} \\ AR &= \frac{TR}{Q} = a - bQ \text{ (inverse demand function)} \\ MR &= \frac{d(TR)}{dQ} = a - 2bQ \text{ (1st derivative)} \end{aligned}$$

Graphically

1. TR is at a MAX when $MR = 0$
2. $MR = 10 - 2Q = 0$
   $Q = 5$
3. $TR = 10Q - Q^2 = 25$
4. $AR = 10 - Q = 5$



1.4.2 Average cost and Marginal Cost



1. Total Cost = $C(Q)$
2. Marginal Cost = $\frac{dC(Q)}{dQ}$
3. Average Cost = $\frac{C(Q)}{Q}$
4. Average costs are minimized when the slop of AC=0 (point A)

$$\begin{aligned}
\text{Slope of AC} &= \frac{dAC}{dQ} = \frac{C'(Q)Q - C(Q)}{Q^2} \quad \text{Quotient Rule} \\
&= \frac{1}{Q}\left[C'(Q) - \frac{C(Q)}{Q}\right] \text{ Factor out Q} \\
&= \frac{1}{Q}\left[MC - AC\right]
\end{aligned}$$

Slope of AC is:

1. (a) i. negative if MC $<$ AC
   ii. positive if MC $>$ AC
   iii. zero if MC $=$ AC

2 Partial Derivatives

Single variable calculus is really just a "special case" of multivariable calculus. For the function $y = f(x)$, we assumed that y was the endogenous variable, x was the exogenous variable and everything else was a parameter. For example, given the equations

$$y = a + bx$$

or

$$y = ax^n$$

we automatically treated a, b, and n as constants and took the derivative of y with respect to x (dy/dx). However, what if we decided to treat x as a constant and take the derivative with respect to one of the other variables? Nothing precludes us from doing this. Consider the equation

$$y = ax$$

where

$$\frac{dy}{dx} = a$$

Now suppose we find the derivative of y with respect to a, *but TREAT x as the constant.* Then

$$\frac{dy}{da} = x$$

Here we just "reversed" the roles played by a and x in our equation.

2.1 Partial Derivatives

Suppose $y = f(x_1, x_2, ...x_n)$

ie. $y = 2x_1^2 + 3x_2 + 2x_1x_2$

What is the change in y when we change x_i $(i = 1, n)$ hold all other x's constant?

or: Find $\frac{\Delta y}{\Delta x_1} = \frac{\partial y}{\partial x_1} = f_1$ (holding $x_2, ...x_n$ fixed)

Rule: Treat all other variables as constants and use ordinary rules of differentation.

Example:

$$\begin{aligned} y &= 2x_1^2 + 3x_2 + 2x_1x_2 \\ \frac{dy}{dx_1} &= 4x_1 + 2x_2 (= f_1) \\ \frac{dy}{dx_2} &= 3 + 2x_1 (= f_2) \end{aligned}$$

2.2 Two Variable Case:

let $z = f(x, y)$, which means **"z is a function of x and y"**. In this case z is the endogenous (dependent) variable and both x and y are the exogenous (independent) variables.

To measure the the effect of a change in a single independent variable (x or y) on the dependent variable (z) we use what is known as the *PARTIAL DERIVATIVE*.

The partial derivative of z with respect to x measures the instantaneous change in the function as x changes while *HOLDING y constant*. Similarly, we would hold x constant if we wanted to evaluate the effect of a change in y on z. Formally:

- $\frac{\partial z}{\partial x}$ is the **"partial derivative"** of z with respect to x, treating y as a constant. Sometimes written as f_x.
- $\frac{\partial z}{\partial y}$ is the **"partial derivative"** of z with respect to y, treating x as a constant. Sometimes written as f_y.

The "∂" symbol ("bent over" lower case D) is called the "partial" symbol. It is interpreted in exactly the same way as $\frac{dy}{dx}$ from single variable calculus. The ∂ symbol simply serves to remind us that there are other variables in the equation, but for the purposes of the current exercise, these other variables are held constant.

EXAMPLES:

$$z = x + y \quad \partial z/\partial x = 1 \quad \partial z/\partial y = 1$$
$$z = xy \quad \partial z/\partial x = y \quad \partial z/\partial y = x$$
$$z = x^2y^2 \quad \partial z/\partial x = 2(y^2)x \quad \partial z/\partial y = 2(x^2)y$$
$$z = x^2y^3 + 2x + 4y \quad \partial z/\partial x = 2xy^3 + 2 \quad \partial z/\partial y = 3x^2y^2 + 4$$

- **REMEMBER:** When you are taking a partial derivative you treat the other variables in the equation as constants!

2.3 Rules of Partial Differentiation

Product Rule: given $z = g(x, y) \cdot h(x, y)$

$$\frac{\partial z}{\partial x} = g(x, y) \cdot \frac{\partial h}{\partial x} + h(x, y) \cdot \frac{\partial g}{\partial x}$$
$$\frac{\partial z}{\partial y} = g(x, y) \cdot \frac{\partial h}{\partial y} + h(x, y) \cdot \frac{\partial g}{\partial y}$$

Quotient Rule: given $z = \frac{g(x,y)}{h(x,y)}$ and $h(x, y) \neq 0$

$$\frac{\partial z}{\partial x} = \frac{h(x,y) \cdot \frac{\partial g}{\partial x} - g(x,y) \cdot \frac{\partial h}{\partial x}}{[h(x,y)]^2}$$
$$\frac{\partial z}{\partial y} = \frac{h(x,y) \cdot \frac{\partial g}{\partial y} - g(x,y) \cdot \frac{\partial h}{\partial y}}{[h(x,y)]^2}$$

Chain Rule: given $z = [g(x, y)]^n$

$$\frac{\partial z}{\partial x} = n\,[g(x, y)]^{n-1} \cdot \frac{\partial g}{\partial x}$$
$$\frac{\partial z}{\partial y} = n\,[g(x, y)]^{n-1} \cdot \frac{\partial g}{\partial y}$$

2.4 Further Examples:

For the function $U = U(x, y)$ find the the partial derivates with respect to x and y

for each of the following examples

Example 1

$$U = -5x^3 - 12xy - 6y^5$$

Answer:

$$\frac{\partial U}{\partial x} = U_x = 15x^2 - 12y$$
$$\frac{\partial U}{\partial y} = U_y = -12x - 30y^4$$

Example 2

$$U = 7x^2y^3$$

Answer:

$$\begin{aligned} \frac{\partial U}{\partial x} &= U_x = 14xy^3 \\ \frac{\partial U}{\partial y} &= U_y = 21x^2y^2 \end{aligned}$$

Example 3

$$U = 3x^2(8x - 7y)$$

Answer:

$$\begin{aligned} \frac{\partial U}{\partial x} &= U_x = 3x^2(8) + (8x - 7y)(6x) = 72x^2 - 42xy \\ \frac{\partial U}{\partial y} &= U_y = 3x^2(-7) + (8x - 7y)(0) = -21x^2 \end{aligned}$$

Example 4

$$U = (5x^2 + 7y)(2x - 4y^3)$$

Answer:

$$\begin{aligned} \frac{\partial U}{\partial x} &= U_x = (5x^2 + 7y)(2) + (2x - 4y^3)(10x) \\ \frac{\partial U}{\partial y} &= U_y = (5x^2 + 7y)(-12y^2) + (2x - 4y^3)(7) \end{aligned}$$

Example 5

$$U = \frac{9y^3}{x - y}$$

Answer:

$$\begin{aligned} \frac{\partial U}{\partial x} &= U_x = \frac{(x - y)(0) - 9y^3(1)}{(x - y)^2} = \frac{-9y^3}{(x - y)^2} \\ \frac{\partial U}{\partial y} &= U_y = \frac{(x - y)(27y^2) - 9y^3(-1)}{(x - y)^2} = \frac{27xy^2 - 18y^3}{(x - y)^2} \end{aligned}$$

Example 6

$$U = (x - 3y)^3$$

Answer:

$$\begin{aligned}\frac{\partial U}{\partial x} &= U_x = 3(x-3y)^2(1) = 3(x-3y)^2 \\ \frac{\partial U}{\partial y} &= U_y = 3(x-3y)^2(-3) = -9(x-3y)^2\end{aligned}$$

2.5 A Special Function: Cobb-Douglas

The Cobb-douglas function is a mathematical function that is very popular in economic models. The general form is

$$z = x^a y^b$$

and its partial derivatives are

$\partial z/\partial x = a x^{a-1} y^b$ and $\partial z/\partial y = b x^a y^{b-1}$

Furthermore, the absolute value of the slope of the level curve of a Cobb-douglas is given by

$$\frac{\partial z/\partial x}{\partial z/\partial y} = MRS = \frac{a}{b}\frac{y}{x}$$

Example: Production Function Let $Q = f(K, L)$

$f_L = \frac{dQ}{dK}$ = Marginal product of labour (Δ in Q from a Δ in L)

$f_K = \frac{dQ}{dK}$ = Marginal product of capital (Δ in Q from a Δ in K)

Let $Q = K^a L^b$ (Cobb-Douglas Technology)

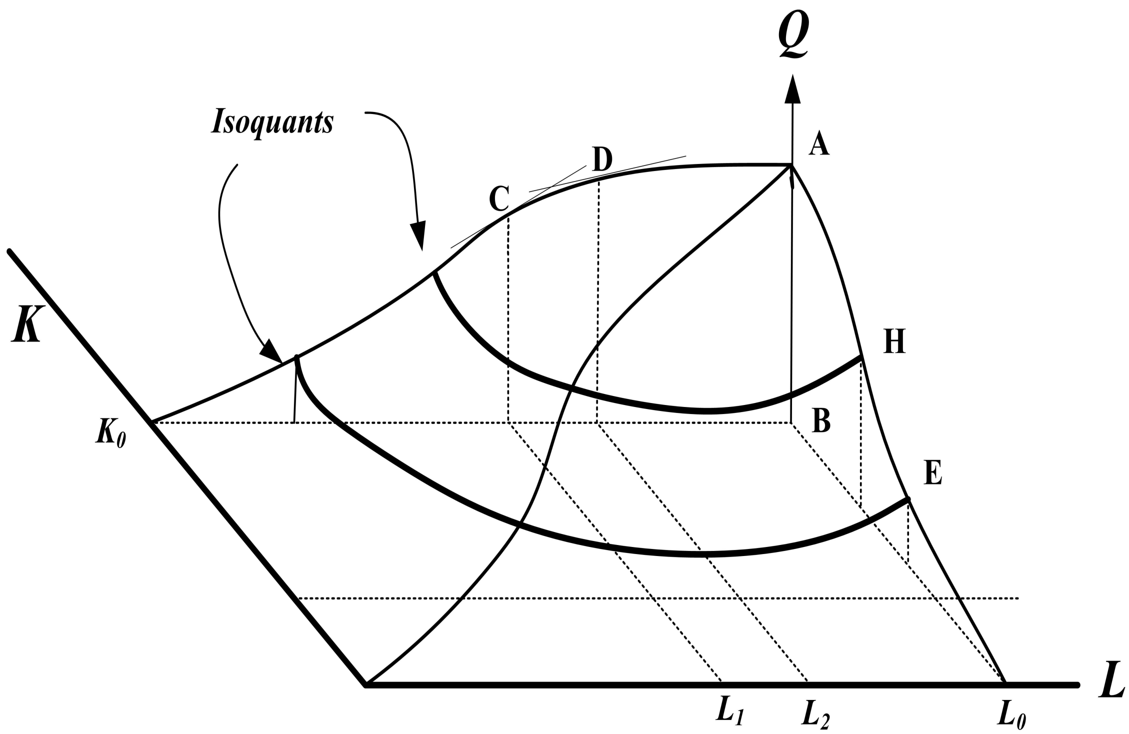
Then

$$\begin{aligned}MP_L &= bK^a L^{b-1} \qquad (\text{for } K = \bar{K}) \\ MP_K &= aK^{a-1} L^b \qquad (\text{for } L = \bar{L})\end{aligned}$$

Isoquant: Δ's in K and L that keep $Q = \bar{Q}$
Then

$$\begin{aligned}
\Delta L \cdot MP_L &= -MP_K \cdot \Delta K \\
&\text{or} \\
\Delta L \left(\frac{\partial Q}{\partial L} \right) &= \left(\frac{-\partial Q}{\partial K} \right) \Delta K
\end{aligned}$$

$$\begin{aligned}
\frac{\Delta K}{\Delta L} &= MRTS = \frac{MP_L}{MP_K} \\
&= \frac{bK^a L^{b-1}}{aK^{a-1} L^b} \\
&= \frac{b}{a} K^{(a-a+1)} L^{b-1-b} \\
&= \frac{b}{a} K^1 L^{-1} = \frac{b}{a} \frac{K}{L}
\end{aligned}$$



Point C: $\frac{\partial Q}{\partial L} = MP_L$ at $L = L_1$ and $K = K_0$
Point D: $\frac{\partial Q}{\partial L}$ at $L = L_2$ and $K = K_0$
Point E: $\frac{\partial Q}{\partial K} = MP_K$ at $L = L_0$
MP_L = marginal product of labour
MP_K = marginal product of capital

3 National Income Model

Consider the linear model of a simple economy

$$\begin{aligned} Y &= C + I_0 + G_0 \\ C &= a + bY \end{aligned}$$

where Y, C are the endogenous variables and a, b, I_0 and G_0 are the exogenous variables and parameters.

In Equilibrium:

$$\begin{aligned} Y^e &= \frac{a + I_0 + G_0}{1-b} = \frac{a}{1-b} + \frac{I_0}{1-b} + \frac{G_0}{1-b} \\ C^e &= \frac{a + bI_0 + bG_0}{1-b} = \frac{a}{1-b} + \frac{bI_0}{1-b} + \frac{bG_0}{1-b} \end{aligned}$$

$\frac{\partial Y^e}{\partial G_0} = \frac{1}{1-b} \qquad \frac{\partial C^e}{\partial G_0} = \frac{b}{1-b}$ The Multipliers

$\frac{\partial Y^e}{\partial b}$?

$$\begin{aligned} Y^e &= (a + I_0 + G_0)(1-b)^{-1} \\ \frac{\partial Y^e}{\partial b} &= (a + I_0 + G_0)(1-b)^{-2}(-1)(-1) \qquad \text{Chain Rule} \\ \frac{\partial Y^e}{\partial b} &= + \left[\frac{a + I_0 + G_0}{(1-b)^2} \right] \\ &= \text{The income multiplier with respect to a change in the MPC} \end{aligned}$$

$$\begin{aligned} Y &= C + I_0 + G_0 \\ C &= a + bY \end{aligned}$$

$$
\begin{aligned}
Y - C &= I_0 + G_0 \\
-bY + C &= a
\end{aligned}
$$

$$
\begin{pmatrix} 1 & -1 \\ -b & 1 \end{pmatrix} \begin{pmatrix} Y \\ C \end{pmatrix} = \begin{pmatrix} I_0 + G_0 \\ a \end{pmatrix} \qquad |A| = 1 - b
$$

$$
\begin{aligned}
Y^e &= \frac{a + I_0 + G_0}{1 - b} \\
C^e &= \frac{a + bI_0 + bG_0}{1 - b}
\end{aligned}
$$

$$
\tfrac{\partial Y}{\partial G_0} = \tfrac{1}{1-b} \qquad \tfrac{\partial C}{\partial G_0} = \tfrac{b}{1-b}
$$

$$
\begin{aligned}
\frac{\partial Y}{\partial b} &= \frac{\partial}{\partial b}\left[(a + I_0 + G_0)(1 - b)^{-1}\right] \\
&= (a + I_0 + G_0)(1 - b)^{-2}(-1)(-1) \\
&= \frac{a + I_0 + G_0}{(1 - b)^2}
\end{aligned}
$$