Lecture Notes for Chapters 4 & 5

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1 Matrix Algebra

1. Gives us a shorthand way of writing a large system of equations.

2. Allows us to test for the existance of solutions to simultaneous systems.

3. Allows us to solve a simultaneous system.

DRAWBACK: Only works for linear systems. However, we can often covert non-linear to linear systems.

Example

$$y = ax^b$$
$$\ln y = \ln a + b \ln x$$

Matrices and Vectors

Given

$$y = 10 - x \;\; \Rightarrow \;\; x + y = 10$$
$$y = 2 + 3x \;\; \Rightarrow \;\; -3x + y = 2$$

In matrix form

$$\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$$

Matrix of Coefficients   Vector of Unknows   Vector of Constants

In general

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = d_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = d_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = d_m$$

n-unknowns $(x_{1,}x_{2,} \ldots x_n)$

Matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \ldots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}$$

Matrix shorthand

$$Ax = d$$

Where:
A= coefficient martrix or an array
x= vector of unknowns or an array
d= vector of constants or an array

Subscript notation

$$a_{ij}$$

is the coefficient found in the i-th row (i=1,...,m) and the j-th column (j=1,...,n)

1.1 Vectors as special matrices

The number of rows and the number of columns define the DIMENSION of a matrix.

A is m rows and n is columns or "mxn."

A matrix containing 1 column is called a "column VECTOR"

x is a n×1 column vector

d is a m×1 column vector

If x were arranged in a horizontal array we would have a row vector.

Row vectors are denoted by a prime

$$x' = [x_1, x_2, \ldots, x_n]$$

A 1×1 vector is known as a scalar.

$$x = [4] \text{ is a scalar}$$

Matrix Operators

If we have two matrices, A and B, then

$$A = B \quad iff \quad a_{ij} = b_{ij}$$

Addition and Subtraction of Matrices

Suppose A is an m×n matrix and B is a p×q matrix then A and B is possible only if m=p and n=q. Matrices must have the same dimensions.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11} + b_{11}) & (a_{12} + b_{12}) \\ (a_{21} + b_{21}) & (a_{22} + b_{22}) \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Subtraction is identical to addition

$$\begin{bmatrix} 9 & 4 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 7 & 2 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} (9-7) & (4-2) \\ (3-1) & (1-6) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -5 \end{bmatrix}$$

Scalar Multiplication

Suppose we want to multiply a matrix by a scalar

$$\begin{array}{ccc} k & \times & A \\ 1 \times 1 & & m \times n \end{array}$$

We multiply every element in A by the scalar k

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & & & \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

Example

Let k=[3] and A= $\begin{bmatrix} 6 & 2 \\ 4 & 5 \end{bmatrix}$

then kA=

$$kA = \begin{bmatrix} 3 \times 6 & 3 \times 2 \\ 3 \times 4 & 3 \times 5 \end{bmatrix} = \begin{bmatrix} 18 & 6 \\ 12 & 15 \end{bmatrix}$$

Multiplication of Matrices

To multiply two matrices, A and B, together it must be true that for

$$\begin{array}{ccccc} A & \times & B & = & C \\ m \times n & & n \times q & & m \times q \end{array}$$

That A must have the same number of columns (n) as B has rows (n).

The product matrix, C, will have the same number of rows as A and the same number of columns as B.

Example

$$\begin{array}{ccccc} A & \times & B & = & C \\ (1 \times 3) & & (3 \times 4) & & (1 \times 4) \\ 1 row & & 3 rows & & 1 row \\ 3 cols & & 4 cols & & 4 cols \end{array}$$

In general

$$\underset{(3\times 2)}{A} \quad \times \quad \underset{(2\times 5)}{B} \quad \times \quad \underset{(5\times 4)}{C} \quad \times \quad \underset{(4\times 1)}{D} \quad = \quad \underset{(3\times 1)}{E}$$

To multiply two matrices:

(1) Multiply each element in a given row by each element in a given column

(2) Sum up their products

Example 1

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Where:

c_{11}=$a_{11}b_{11} + a_{12}b_{21}$ (sum of row 1 times column 1)
c_{12}=$a_{11}b_{12} + a_{12}b_{22}$ (sum of row 1 times column 2)
c_{21}=$a_{21}b_{11} + a_{22}b_{21}$ (sum of row 2 times column 1)
c_{22}=$a_{21}b_{12} + a_{22}b_{22}$ (sum of row 2 times column 2)

Example 2

$$\begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (3\times 1) & +(2\times 3) & (3\times 2) & +(2\times 4) \end{bmatrix} = \begin{bmatrix} 9 & 14 \end{bmatrix}$$

Example 3

$$\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} (3\times 2) & +(2\times 1) & +(1\times 4) \end{bmatrix} = [12]$$

12 is the inner product of two vectors.

Suppose

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{then} \quad x' = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$$

therefore

$$x'x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \left[x_1^2 + x_2^2\right]$$

However

$$xx' = 2 \text{ by } 2 \text{ matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1 x_2 \\ x_2 x_1 & x_2^2 \end{bmatrix}$$

Example 4

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$Ab = \begin{bmatrix} (1 \times 5) & + & (3 \times 9) \\ (2 \times 5) & + & (8 \times 9) \\ (4 \times 5) & + & (0 \times 9) \end{bmatrix} = \begin{bmatrix} 32 \\ 82 \\ 20 \end{bmatrix}$$

Example

$$Ax = d$$

$$\underset{(3 \times 3)}{\overset{A}{\begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix}}} \quad \underset{(3 \times 1)}{\overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}} = \underset{(3 \times 1)}{\overset{d}{\begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix}}}$$

This produces

$$\begin{aligned} 6x_1 + 3x_2 + x_3 &= 22 \\ x_1 + 4x_2 - 2x_3 &= 12 \\ 4x_1 - x_2 + 5x_3 &= 10 \end{aligned}$$

1.1.1 National Income Model

$$y = c + I_0 + G_0$$

$$C = a + bY$$

Arrange as

$$\begin{aligned} &y - C = I_0 + G_0 \\ &-bY + C = a \end{aligned}$$

Matrix form

$$\begin{matrix} A & x & = d \\ \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} & \begin{bmatrix} Y \\ C \end{bmatrix} & = \begin{bmatrix} I_o + G_o \\ a \end{bmatrix} \end{matrix}$$

1.1.2 Division in Matrix Algebra

In ordinary algebra

$$\frac{a}{b} = c$$

is well defined iff b$\neq 0$.
Now $\frac{1}{b}$ can be rewritten as b^{-1}, therefore $ab^{-1} = c$, also b$^{-1}a = c$.

But in matrix algebra

$$\frac{A}{B} = C$$

is not defined. However,

$$AB^{-1} = C$$

is well defined. BUT

$$AB^{-1} \neq B^{-1}A$$

B^{-1} is called the inverse of B

$$B^{-1} \neq \frac{1}{B}$$

In some ways B^{-1} has the same properties as b^{-1} but in other ways it differs. We will explore these differences later.

1.2 Commutative, Associative, and Distributive Laws

From Highschool algebra we know commutative law of addition,

$$a + b = b + a$$

commutative law of multiplication,

$$ab = ba$$

Associative law of addition,

$$(a + b) + c = a + (b + c)$$

associative law of multiplication,

$$(ab)c = a(bc)$$

Distributive law

$$a(b + c) = ab + ac$$

In matrix algebra most, but not all, of these laws are true.

1.2.1 I) Communicative Law of Addition

$$A + B = B + A$$

Since we are adding individual elements and $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ for all i and j.

1.2.2 II) Similarly Associative Law of Addition

$$A + (B + C) = (A + B) + C$$

for the same reasons.

1.2.3 III) Matrix Multiplication

Matrix multiplication in not communtative

$$IB \neq BA$$

Example 1
Let A be 2×3 and B be 3×2

$$\underset{(2\times 3)}{A} \times \underset{(3\times 2)}{B} = \underset{(2\times 2)}{C} \quad whereas \quad \underset{(3\times 2)}{B} \times \underset{(2\times 3)}{A} = \underset{(3\times 3)}{C}$$

Example 2
Let A=$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and B=$\begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$

$$AB = \begin{bmatrix} (1 \times 10) + (2 \times 6) & (1 \times -1) + (2 \times 7) \\ (3 \times 0) + (4 \times 6) & (3 \times -1) + (4 \times 7) \end{bmatrix} = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix}$$

But

$$BA = \begin{bmatrix} (0)(1)-(1)(3) & (0)(2)-(1)(4) \\ (6)(1)+(7)(3) & (6)(2)+(7)(4) \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 27 & 40 \end{bmatrix}$$

Therefore, we realize the distinction of post multiply and pre multiply. In the case

$$AB = C$$

B is pre multiplied by A, A is post multiplied by B.

1.2.4 IV) Associative Law

Matrix multiplication is associative

$$(AB)C = A(BC) = ABC$$

as long as their dimensions conform to our earlier rules of multiplication.

$$\begin{array}{ccccc} A & \times & B & \times & C \\ (m \times n) & & (n \times p) & & (p \times q) \end{array}$$

1.2.5 V) Distributive Law

Matrix multiplication is distributive

$$\begin{array}{ll} A(B+C) = AB + AC & \text{Pre multiplication} \\ (B+C)A = BA + CA & \text{Post multiplication} \end{array}$$

1.3 Identity Matrices and Null Matrices

1.3.1 Identity matrix:

is a square matrix with ones on its principal diagonals and zeros everywhere else.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I_n = \begin{bmatrix} 1 & 0 & \dots & n \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

Identity Matrix in scalar algebra we know

$$1 \times a = a \times 1 = a$$

In matrix algebra the identity matrix plays the same role

$$IA = AI = A$$

Example 1

Let A=$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} (1 \times 1) + (0 \times 2) & (1 \times 3) + (0 \times 4) \\ (0 \times 1) + (1 \times 2) & (0 \times 3) + (1 \times 4) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Example 2

Let A=$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix}$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = A \{I_2 Case\}$$

$$AI = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = A \{I_3 Case\}$$

Furthermore,

$$\underset{(m \times n)(n \times p)}{AIB} = (AI)B = A(IB) = \underset{(m \times n)(n \times p)}{AB}$$

1.3.2 Null Matrices

A null matrix is simply a matrix where all elements equal zero.

$$0 = \underset{(2 \times 2)}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} \quad 0 = \underset{(2 \times 3)}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

The rules of scalar algebra apply to matrix algebra in this case.

Example

$$a + 0 = a \Rightarrow \{scalar\}$$

$$A + 0 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = A \qquad \{matrix\}$$

$$A \times 0 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

1.4 Idiosyncracies of matrix algebra

1) We know AB≠BA

2)ab=0 implies a or b=0

In matrix

$$AB = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

1.4.1 Transposes and Inverses

1)Transpose: is when the rows and columns are interchanged.

Transpose of A=A'or A^T

Example

If A=$\begin{bmatrix} 3 & 8 & -9 \\ 1 & 0 & 4 \end{bmatrix}$ and B=$\begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix}$

A'=$\begin{bmatrix} 3 & 1 \\ 8 & 0 \\ -9 & 4 \end{bmatrix}$ and B'=$\begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix}$

Symmetrix Matrix

If A=$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$ then A'=$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$

A is a symmetric matrix.

Properties of Transposes

1) $(A\prime)\prime = A$

2) $(A + B)\prime = A\prime + B\prime$

3) $(AB)\prime = B\prime A\prime$

Inverses and their Properties

In scalar algebra if

$$ax = b$$

then

$$x = \frac{b}{a} \text{ or } ba^{-1}$$

In matrix algebra, if

$$Ax = d$$

then

$$x = A^{-1}d$$

where A^{-1} is the inverse of A.

Properties of Inverses

1) Not all matrices have inverses

non-singular: if there is an inverse

singular: if there is no inverse

2) A matrix must be square in order to have an inverse. (Necessary but not sifficient)

3) In scalar algebra $\frac{a}{a} = 1$, in matrix algebra $AA^{-1} = A^{-1}A = I$

4) If an inverse exists then it must be unique.

Example

Let A=$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} \\ 0 & \frac{1}{2} \end{bmatrix}$

$A^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$ by factoring $\{\frac{1}{6}$ is a scalar$\}$

Post Multiplication

$$AA^{-1} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\frac{1}{6} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\frac{1}{6} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Pre Multiplication

$$A^{-1}A = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Further properties

If A and B are square and non-singular then:

1) $(A^{-1})^{-1} = A$
2) $(AB)^{-1} = B^{-1}A^{-1}$
3) $(A\prime)^{-1} = (A^{-1})^{1}$

Solving a linear system

Suppose

$$\underset{(3\times 3)}{A}\ \underset{(3\times 1)}{x} = \underset{(3\times 1)}{d}$$

then

$$\underset{(3\times 3)}{A^{-1}}\ \underset{(3\times 3)}{A}\ \underset{(3\times 1)}{x} = \underset{(3\times 3)}{A^{-1}}\ \underset{(3\times 1)}{d}$$

$$\underset{(3\times 3)}{I}\ \underset{(3\times 1)}{x} = \underset{(3\times 3)}{A^{-1}}\ \underset{(3\times 1)}{d}$$

$$x = A^{-1}d$$

Example

$$Ax = d$$

$$A = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad d = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix} \quad A^{-1} = \frac{1}{52}\begin{bmatrix} 18 & -16 & -10 \\ -13 & 26 & 13 \\ -17 & 18 & 21 \end{bmatrix}$$

then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{52}\begin{bmatrix} 18 & -16 & -10 \\ -13 & 26 & 13 \\ -17 & 18 & 21 \end{bmatrix}\begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$x_1^* = 2 \quad x_2^* = 3 \quad x_3^* = 1$$

1.5 Linear Dependence and Determinants

Suppose we have the following

1. $x_1 + 2x_2 = 1$
2. $2x_1 + 4x_2 = 2$

where equation two is twice equation one. Therefore, there is no solution for x_1, x_2.

In matrix form:

$$Ax = d$$

$$\overset{A}{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \overset{d}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

The determinant of the coefficient matrix is

$$|A| = (1)(4) - (2)(2) = 0$$

a determinant of zero tells us that the equations are linearly dependent. Sometimes called a "vanishing determinant."

In general, the determinant of a square matrix, A is written as $|A|$ or detA.

For two by two case

$$|A| = \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} = a_{11}a_{22} - a_{12}a_{21} = k$$

where k is unique
any $k \neq 0$ implies linear independence

Example 1

A= $\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$

$$|A| = (3 \times 5) - (1 \times 2) = 13 \qquad \{\text{Non-singular}\}$$

Example 2

B= $\begin{bmatrix} 2 & 6 \\ 8 & 24 \end{bmatrix}$

$$|B| = (2 \times 24) - (6 \times 8) = 0 \qquad \{\text{Singular}\}$$

Three by three case

Given A= $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

then

$$|A| = (a_1b_2c_3) + (a_2b_3c_1) + (b_1c_2a_3) - (a_3b_2c_1) - (a_2b_1c_3) - (b_3c_2a_1)$$

Cross-diagonals

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Use viso to display cross diagonals

Multiple along the diagonals and add up their products

⇒The product along the BLUE lines are given a positive sign

⇒The product of the RED lines are negative.

1.6 Using Laplace expansion

⇒ The cross diagonal method does not work for matrices greater than three by three

⇒ Laplace expansion evaluates the determinant of a matrix, A, by means of subdeterminants of A.

Subdeterminants or Minors

Given A= $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

By deleting the first row and first column, we get

$$|M_{11}| = \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix}$$

The determinant of this matrix is the minor element a_1.

$|M_{ij}| \equiv$ is the subdeterminant from deleting the i-th row and the j-th column.

Given A= $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

then

$$M_{21} \equiv \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} \quad M_{31} \equiv \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

1.6.1 Cofactors

A cofactor is a minor with a specific algebraic sign.

$$C_{ij} = (-1)^{i+j} |M_{ij}|$$

therefore

$$C_{11} = (-1)^2 |M_{11}| = |M_{11}|$$
$$C_{21} = (-1)^3 |M_{21}| = -|M_{21}|$$

The determinant by Laplace Expanding down the first column

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = a_{11}\,|C_{11}| + a_{21}\,|C_{21}| + a_{31}\,|C_{31}| = \sum_{i=1}^{3} a_{i1}\,|C_{i1}|$$

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Note: minus sign $(-1)^{(1+2)}$

$$|A| = a_{11}\,[a_{22}a_{33} - a_{23}a_{32}] - a_{21}\,[a_{12}a_{33} - a_{13}a_{32}] + a_{31}\,[a_{12}a_{23} - a_{13}a_{22}]$$

Laplace expansion can be used to expand along any row or any column.

Example: Third row

$$|A| = a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Example

$$A = \begin{bmatrix} 8 & 1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 3 \end{bmatrix}$$

(1)Expand the first column

$$|A| = 8 \begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} - 4 \begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} + 6 \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}$$

$$|A| = (8 \times 0) - (4 \times 3) + (6 \times 1) = -6$$

(2) Expand the second column

$$|A| = -1 \begin{vmatrix} 4 & 1 \\ 6 & 3 \end{vmatrix} + 0 \begin{vmatrix} 8 & 3 \\ 6 & 3 \end{vmatrix} - 0 \begin{vmatrix} 8 & 3 \\ 4 & 1 \end{vmatrix}$$

$$|A| = (-1 \times 6) + (0) - (0) = -6$$

Suggestion: Try to choose an easy row or column to expand. (i.e. the ones with zero's in it.)

1.7 Matrix Inversion

Given an n×n matrix, A, the inverse of A is

$$A^{-1} = \frac{1}{|A|} \bullet AdjA$$

where AdjA is the adjoint matrix of A. AdjA is the transpose of matrix A's cofactor matrix. It is also the adjoint, which is an n×n matrix

Cofactor Matrix (denoted C)

The cofactor matrix of A is a matrix who's elements are the cofactors of the elements of A

$$If\ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ then } C = \begin{bmatrix} |C_{11}| & |C_{12}| \\ |C_{21}| & |C_{22}| \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

Example

Let A=$\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \Rightarrow |A|$=-2

Step 1: Find the cofactor matrix

$$C = \begin{bmatrix} |C_{11}| & |C_{12}| \\ |C_{21}| & |C_{22}| \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix}$$

Step 2: Transpose the cofactor matrix

$$C^T = Adj A = \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix}$$

Step 3: Multiply all the elements of AdjA by $\frac{1}{|A|}$ to find A^{-1}

$$A^{-1} = \frac{1}{|A|} \bullet Adj A = \left(-\frac{1}{2}\right) \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Step 4: Check by $\text{AA}^{-1} = I$

$$\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} (3)(0) + (2)(\frac{1}{2}) & (3)(1) + (2)(-\frac{3}{2}) \\ (1)(0) + (0)(\frac{1}{2}) & (1)(1) + (0)(-\frac{3}{2}) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1.8 Cramer's Rule

Suppose:

$$\text{Equation 1} \quad a_1 x_1 + a_2 x_2 = d_1$$
$$\text{Equation 2} \quad b_1 x_1 + b_2 x_2 = d_2$$

or

$$\begin{matrix} A & x & = & d \\ \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = & \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \end{matrix}$$

where

$$A = a_1 b_2 - a_2 b_1 \neq 0$$

Solve for x_1 by substitution

From equation 1

$$x_2 = \frac{d_1 - a_1 x_1}{a_2}$$

and equation 2

$$x_2 = \frac{d_2 - b_1 x_1}{b_2}$$

therefore:

$$\frac{d_1 - a_1 x_1}{a_2} = \frac{d_2 - b_1 x_1}{b_2}$$

Cross multiply

$$d_1 b_2 - a_1 b_2 x_1 = d_2 a_2 - b_1 a_2 x_1$$

Collect terms

$$d_1 b_2 - d_2 a_2 = (a_1 b_2 - b_1 a_2) x_1$$

$$x_1 = \frac{d_1 b_2 - d_2 a_2}{a_1 b_2 - b_1 a_2}$$

The denominator is the determinant of $|A|$ and the numerator is the same as the denominator except $d_1 d_2$ replaces $a_1 b_1$.

Cramer's Rule

$$x_1 = \frac{\begin{vmatrix} d_1 & a_2 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = \frac{d_1 b_2 - d_2 a_2}{a_1 b_2 - b_1 a_2}$$

Where the d vector replaces column 1 in the A matrix

To find x_2 replace column 2 with the d vector

$$x_2 = \frac{\begin{vmatrix} a_1 & d_1 \\ b_1 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = \frac{a_1 d_2 - d_1 b_1}{a_1 b_2 - b_1 a_2}$$

Generally: to find x_i,replace column i with vector d; find the determinant.

x_i= the ratio of two determinants

$x_i = \frac{|A_i|}{|A|}$

1.8.1 Example: The Market Model

$$\text{Equation 1} \quad Q^d = 10 - P \quad \text{Or} \quad Q + P = 10$$

$$\text{Equation 2} \quad Q^s = P - 2 \quad \text{Or} \quad -Q + P = 2$$

Matrix form

$$\begin{matrix} A & x & = & d \\ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} & \begin{bmatrix} Q \\ P \end{bmatrix} & = & \begin{bmatrix} 10 \\ 2 \end{bmatrix} \end{matrix}$$

$$|A| = (1)(1) - (-1)(1) = 2$$

Find Q^e

$$Q^e = \frac{\begin{vmatrix} 10 & 1 \\ 2 & 1 \end{vmatrix}}{2} = \frac{10 - 2}{2} = 4$$

Find P^e

$$P^e = \frac{\begin{vmatrix} 1 & 10 \\ -1 & 2 \end{vmatrix}}{2} = \frac{2 - (-10)}{2} = 6$$

Substitute P and Q into either equation 1 or equation 2 to verify

$$Q^d = 10 - P$$
$$10 - 6 = 4$$

1.8.2 Example: National Income Model

$$Y = C + I_0 + G_0 \quad \text{Or} \quad Y - C = I_0 + G_0$$
$$C = a + bY \quad \text{Or} \quad -bY + c = a$$

In matrix form

$$\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$$

Solve for Y^e

$$Y^e = \frac{\begin{vmatrix} I_0 + G_0 & -1 \\ a & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{I_0 + G_0 + a}{1 - b}$$

Solve for C^e

$$C^e = \frac{\begin{vmatrix} 1 & I_0 + G_0 \\ -b & a \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{a + b(I_0 + G_0)}{1 - b}$$

Numeric example:

Let $C = 100 + 0.75Y$, $I = 150$ and $G = 250$. Then the model is

$$\begin{aligned} Y - C &= I + G \\ Y - C &= 400 \end{aligned}$$

and

$$\begin{aligned} C &= 100 + 0.75Y \\ 0.75Y - C &= 100 \end{aligned}$$

In Matrix form

$$\begin{bmatrix} 1 & -1 \\ -0.75 & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} 400 \\ 100 \end{bmatrix}$$

Solve for Y^e

$$Y^e = \frac{\begin{vmatrix} 400 & -1 \\ 100 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -0.75 & 1 \end{vmatrix}} = \frac{500}{0.25} = 2000$$

Solve for C^e

$$C^e = \frac{\begin{vmatrix} 1 & 400 \\ -0.75 & 100 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -0.75 & 1 \end{vmatrix}} = \frac{100 + 0.75(400)}{0.25} = 1600$$