Suppose you use protons ($mc^2 = 1$ GeV) with $E = 30$ GeV. What $\bar{E}$ do you get? What multiple of E does this amount to? (1 GeV=10^9 electron volts.) [Because of this relativistic enhancement, most modern elementary particle experiments involve **colliding beams**, instead of fixed targets.]

Problem 12.35 In a **pair annihilation** experiment, an electron (mass m) with momentum p_e hits a positron (same mass, but opposite charge) at rest. They annihilate, producing two photons. (Why couldn't they produce just *one* photon?) If one of the photons emerges at 60° to the incident electron direction, what is its energy?

12.2.4 Relativistic Dynamics

Newton's *first* law is built into the principle of relativity. His second law, in the form

$$\boxed{\mathbf{F} = \frac{d\mathbf{p}}{dt},} \tag{12.60}$$

retains its validity in relativistic mechanics, *provided we use the relativistic momentum.*

Example 12.10

Motion under a constant force. A particle of mass m is subject to a constant force F. If it starts from rest at the origin, at time $t = 0$, find its position (x), as a function of time.

Solution:

$$\frac{dp}{dt} = F \quad \Rightarrow \quad p = Ft + \text{constant},$$

but since $p = 0$ at $t = 0$, the constant must be zero, and hence

$$p = \frac{mu}{\sqrt{1 - u^2/c^2}} = Ft.$$

Solving for u, we obtain

$$u = \frac{(F/m)t}{\sqrt{1 + (Ft/mc)^2}}. \tag{12.61}$$

The numerator, of course, is the classical answer—it's approximately right, if $(F/m)t \ll c$. But the relativistic denominator ensures that u never exceeds c; in fact, as $t \to \infty$, $u \to c$.

To complete the problem we must integrate again:

$$\begin{aligned} x(t) &= \frac{F}{m}\int_0^t \frac{t'}{\sqrt{1 + (Ft'/mc)^2}}\,dt' \\ &= \frac{mc^2}{F}\sqrt{1 + (Ft'/mc)^2}\bigg|_0^t = \frac{mc^2}{F}\left[\sqrt{1 + (Ft/mc)^2} - 1\right]. \end{aligned} \tag{12.62}$$

In place of the classical parabola, $x(t) = (F/2m)t^2$, the graph is a *hyperbola* (Fig. 12.30); for this reason, motion under a constant force is often called **hyperbolic motion.** It occurs, for example, when a charged particle is placed in a uniform electric field.

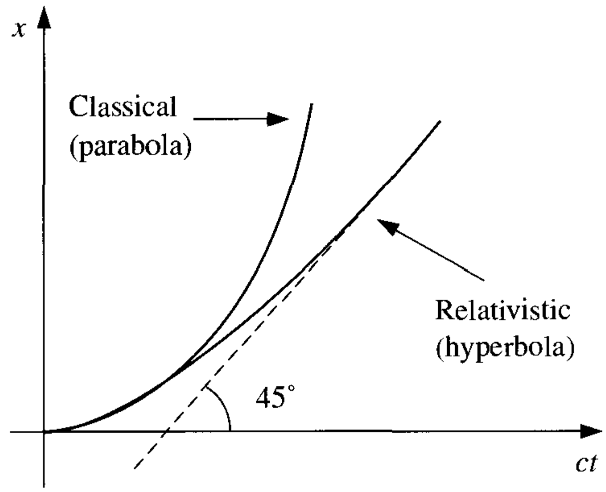

Figure 12.30

Work, as always, is the line integral of the force:

$$W \equiv \int \mathbf{F} \cdot d\mathbf{l}. \tag{12.63}$$

The **work-energy theorem** ("the net work done on a particle equals the increase in its kinetic energy") holds relativistically:

$$W = \int \frac{d\mathbf{p}}{dt} \cdot d\mathbf{l} = \int \frac{d\mathbf{p}}{dt} \cdot \frac{d\mathbf{l}}{dt}\, dt = \int \frac{d\mathbf{p}}{dt} \cdot \mathbf{u}\, dt,$$

while

$$\begin{aligned} \frac{d\mathbf{p}}{dt} \cdot \mathbf{u} &= \frac{d}{dt}\left(\frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}\right) \cdot \mathbf{u} \\ &= \frac{m\mathbf{u}}{(1-u^2/c^2)^{3/2}} \cdot \frac{d\mathbf{u}}{dt} = \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-u^2/c^2}}\right) = \frac{dE}{dt}, \end{aligned} \tag{12.64}$$

so

$$W = \int \frac{dE}{dt}\, dt = E_{\text{final}} - E_{\text{initial}}. \tag{12.65}$$

(Since the *rest* energy is constant, it doesn't matter whether we use the total energy, here, or the kinetic energy.)

Unlike to the first two, Newton's *third* law does *not*, in general, extend to the relativistic domain. Indeed, if the two objects in question are separated in space, the third law is incompatible with the relativity of simultaneity. For suppose the force of A on B at some instant t is $\mathbf{F}(t)$, and the force of B on A at the same instant is $-\mathbf{F}(t)$; then the third law applies, *in this reference frame*. But a moving observer will report that these equal and opposite forces occurred at *different times;* in his system, therefore, the third law is *violated.* Only in the case of contact interactions, where the two forces are applied at the *same physical point* (and in the trivial case where the forces are *constant*), can the third law be retained.

Because **F** is the derivative of momentum with respect to *ordinary* time, it shares the ugly behavior of (ordinary) velocity, when you go from one inertial system to another: both the numerator *and the denominator* must be transformed. Thus,[12]

$$\bar{F}_y = \frac{d\bar{p}_y}{d\bar{t}} = \frac{dp_y}{\gamma\, dt - \dfrac{\gamma\beta}{c}dx} = \frac{dp_y/dt}{\gamma\left(1 - \dfrac{\beta}{c}\dfrac{dx}{dt}\right)} = \frac{F_y}{\gamma(1-\beta u_x/c)}, \tag{12.66}$$

and similarly for the z component:

$$\bar{F}_z = \frac{F_z}{\gamma(1-\beta u_x/c)}.$$

The x component is even worse:

$$\bar{F}_x = \frac{d\bar{p}_x}{d\bar{t}} = \frac{\gamma\, dp_x - \gamma\beta\, dp^0}{\gamma\, dt - \dfrac{\gamma\beta}{c}dx} = \frac{\dfrac{dp_x}{dt} - \beta\dfrac{dp^0}{dt}}{1 - \dfrac{\beta}{c}\dfrac{dx}{dt}} = \frac{F_x - \dfrac{\beta}{c}\left(\dfrac{dE}{dt}\right)}{1-\beta u_x/c}.$$

We calculated dE/dt in Eq. 12.64; putting that in,

$$\bar{F}_x = \frac{F_x - \beta(\mathbf{u}\cdot\mathbf{F})/c}{1-\beta u_x/c}. \tag{12.67}$$

Only in one special case are these equations reasonably tractable: *If the particle is (instantaneously) at rest in* $\mathcal{S}$, so that $\mathbf{u} = 0$, then

$$\bar{\mathbf{F}}_\perp = \frac{1}{\gamma}\mathbf{F}_\perp, \quad \bar{F}_\parallel = F_\parallel. \tag{12.68}$$

That is, the component of **F** *parallel* to the motion of $\bar{\mathcal{S}}$ is unchanged, whereas components perpendicular are divided by γ.

It has perhaps occurred to you that we could avoid the bad transformation behavior of **F** by introducing a "proper" force, analogous to proper velocity, which would be the derivative of momentum with respect to *proper* time:

$$K^\mu \equiv \frac{dp^\mu}{d\tau}. \tag{12.69}$$

This is called the **Minkowski force**; it is plainly a 4-vector, since p^μ is a 4-vector and proper time is invariant. The spatial components of K^μ are related to the "ordinary" force by

$$\mathbf{K} = \left(\frac{dt}{d\tau}\right)\frac{d\mathbf{p}}{dt} = \frac{1}{\sqrt{1-u^2/c^2}}\mathbf{F}, \tag{12.70}$$

[12]Remember: γ and β pertain to the motion of $\bar{\mathcal{S}}$ with respect $\mathcal{S}$—they are *constants;* **u** is the velocity of the *particle* with respect to $\mathcal{S}$.

while the zeroth component,

$$K^0 = \frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{d\tau}, \tag{12.71}$$

is, apart from the $1/c$, the (proper) rate at which the energy of the particle increases—in other words, the (proper) *power* delivered to the particle.

Relativistic dynamics can be formulated in terms of the ordinary force *or* in terms of the Minkowski force. The latter is generally much *neater,* but since in the long run we are interested in the particle's trajectory as a function of *ordinary* time, the former is often more useful. When we wish to generalize some classical force law, such as Lorentz's, to the relativistic domain, the question arises: Does the classical formula correspond to the *ordinary* force or to the Minkowski force? In other words, should we write

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}),$$

or should it rather be

$$\mathbf{K} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})?$$

Since proper time and ordinary time are identical in classical physics, there is no way at this stage to decide the issue. The Lorentz force law, as it turns out, is an *ordinary* force—later on I'll explain why this is so, and show you how to construct the electromagnetic Minkowski force.

Example 12.11

The typical trajectory of a charged particle in a uniform *magnetic* field is **cyclotron motion** (Fig. 12.31). The magnetic force pointing toward the center,

$$F = QuB,$$

provides the centripetal acceleration necessary to sustain circular motion. Beware, however—in special relativity the centripetal force is *not* mu^2/R, as in classical mechanics. Rather, as you can see from Fig. 12.32, $dp = p\,d\theta$, so

$$F = \frac{dp}{dt} = p\frac{d\theta}{dt} = p\frac{u}{R}.$$

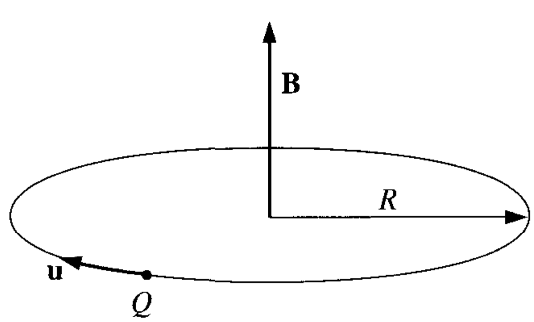

Figure 12.31

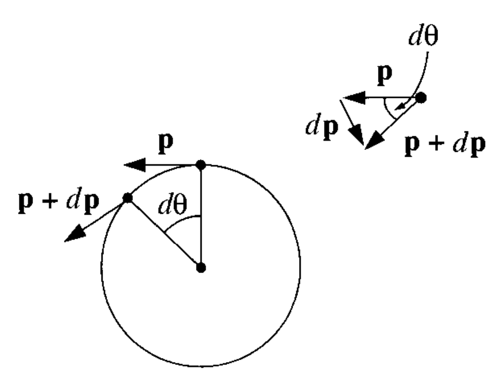

Figure 12.32

(Classically, of course, $p = mu$, so $F = mu^2/R$.) Thus,

$$QuB = p\frac{u}{R},$$

or

$$p = QBR. \tag{12.72}$$

In this form the relativistic cyclotron formula is identical to the nonrelativistic one, Eq. 5.3—the only difference is that p is now the relativistic momentum.

Example 12.12

Hidden momentum. As a model for a magnetic dipole **m**, consider a rectangular loop of wire carrying a steady current. Picture the current as a stream of noninteracting positive charges that move freely within the wire. When a uniform electric field **E** is applied (Fig. 12.33), the charges accelerate in the left segment and decelerate in the right one.[13] Find the total momentum of all the charges in the loop.

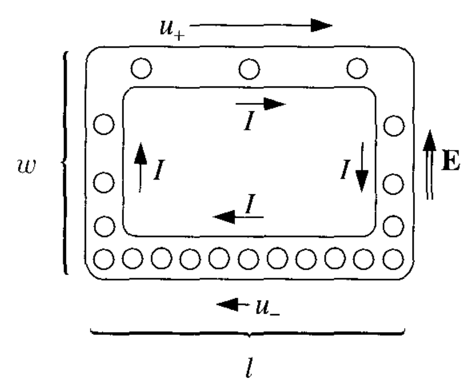

Figure 12.33

Solution: The momenta of the left and right segments cancel, so we need only consider the top and the bottom. Say there are N_+ charges in the top segment, going at speed u_+ to the right, and N_- charges in the lower segment, going at (slower) speed u_- to the left. The current ($I = \lambda u$) is the same in all four segments (or else charge would be piling up somewhere); in particular,

$$I = \frac{QN_+}{l}u_+ = \frac{QN_-}{l}u_-, \quad \text{so} \quad N_\pm u_\pm = \frac{Il}{Q},$$

where Q is the charge of each particle, and l is the length of the rectangle. *Classically,* the momentum of a single particle is $\mathbf{p} = M\mathbf{u}$ (where M is its mass), and the total momentum (to the right) is

$$p_{\text{classical}} = MN_+u_+ - MN_-u_- = M\frac{Il}{Q} - M\frac{Il}{Q} = 0,$$

[13]This is not a very realistic model for a current-carrying wire, obviously, but other models lead to exactly the same result. See V. Hnizdo, *Am. J. Phys.* **65**, 92 (1997).

as one would certainly expect (after all, the loop as a whole is not moving). But relativistically $\mathbf{p} = \gamma M\mathbf{u}$, and we get

$$p = \gamma_+ M N_+ u_+ - \gamma_- M N_- u_- = \frac{MIl}{Q}(\gamma_+ - \gamma_-),$$

which is *not* zero, because the particles in the upper segment are moving faster.

In fact, the gain in energy (γMc^2), as a particle goes up the left segment, is equal to the work done by the electric force, QEw, where w is the height of the rectangle, so

$$\gamma_+ - \gamma_- = \frac{QEw}{Mc^2},$$

and hence

$$p = \frac{IlEw}{c^2}.$$

But Ilw is the magnetic dipole moment of the loop; as vectors, **m** points into the page and **p** is to the right, so

$$\mathbf{p} = \frac{1}{c^2}(\mathbf{m} \times \mathbf{E}).$$

Thus a magnetic dipole in an electric field carries linear momentum, *even though it is not moving!* This so-called **hidden momentum** is strictly relativistic, and purely mechanical; it precisely cancels the electromagnetic momentum stored in the fields (see Ex. 8.3; note that both results can be expressed in the form $p = IlV/c^2$).

Problem 12.36 In classical mechanics Newton's law can be written in the more familiar form $\mathbf{F} = m\mathbf{a}$. The relativistic equation, $\mathbf{F} = d\mathbf{p}/dt$, *cannot* be so simply expressed. Show, rather, that

$$\mathbf{F} = \frac{m}{\sqrt{1 - u^2/c^2}}\left[\mathbf{a} + \frac{\mathbf{u}(\mathbf{u} \cdot \mathbf{a})}{c^2 - u^2}\right], \tag{12.73}$$

where $\mathbf{a} \equiv d\mathbf{u}/dt$ is the **ordinary acceleration.**

Problem 12.37 Show that it is possible to outrun a light ray, if you're given a sufficient head start, and your feet generate a constant force.

Problem 12.38 Define **proper acceleration** in the obvious way:

$$\alpha^\mu \equiv \frac{d\eta^\mu}{d\tau} = \frac{d^2 x^\mu}{d\tau^2}. \tag{12.74}$$

(a) Find α^0 and $\boldsymbol{\alpha}$ in terms of $\mathbf{u}$ and $\mathbf{a}$ (the ordinary acceleration).

(b) Express $\alpha_\mu \alpha^\mu$ in terms of $\mathbf{u}$ and $\mathbf{a}$.

(c) Show that $\eta^\mu \alpha_\mu = 0$.

(d) Write the Minkowski version of Newton's second law, Eq. 12.70, in terms of α^μ. Evaluate the invariant product $K^\mu \eta_\mu$.