12.2 Relativistic Mechanics

12.2.1 Proper Time and Proper Velocity

As you progress along your world line, your watch runs slow; while the clock on the wall ticks off an interval dt, your watch only advances $d\tau$:

$$d\tau = \sqrt{1 - u^2/c^2}\, dt. \tag{12.37}$$

(I'll use u for the velocity of a particular object—you, in this instance—and reserve v for the relative velocity of two inertial systems.) The time τ your watch registers (or, more generally, the time associated with the moving object) is called **proper time**. (The word suggests a mistranslation of the French *propre*, meaning "own.") In some cases τ may be a more relevant or useful quantity than t. For one thing, proper time is invariant, whereas "ordinary" time depends on the particular reference frame you have in mind.

Now, imagine you're on a flight to Los Angeles, and the pilot announces that the plane's velocity is $\frac{4}{5}c$, due South. What precisely does he mean by "velocity"? Well, of course, he means the displacement divided by the time:

$$\mathbf{u} = \frac{d\mathbf{l}}{dt}, \tag{12.38}$$

and, since he is presumably talking about the velocity relative to ground, both $d\mathbf{l}$ and dt are to be measured by the ground observer. That's the important number to know, if you're concerned about being on time for an appointment in Los Angeles, but if you're wondering whether you'll be hungry on arrival, you might be more interested in the distance covered per unit *proper* time:

$$\boldsymbol{\eta} \equiv \frac{d\mathbf{l}}{d\tau}. \tag{12.39}$$

This hybrid quantity—distance measured on the ground, over time measured in the airplane—is called **proper velocity**; for contrast, I'll call $\mathbf{u}$ the **ordinary velocity**. The two are related by Eq. 12.37:

$$\boldsymbol{\eta} = \frac{1}{\sqrt{1 - u^2/c^2}}\mathbf{u}. \tag{12.40}$$

For speeds much less than c, of course, the difference between ordinary and proper velocity is negligible.

From a theoretical standpoint, however, proper velocity has an enormous advantage over ordinary velocity: it transforms simply, when you go from one inertial system to another. In fact, $\boldsymbol{\eta}$ is the spatial part of a 4-vector,

$$\eta^\mu \equiv \frac{dx^\mu}{d\tau}, \tag{12.41}$$

whose zeroth component is

$$\eta^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1 - u^2/c^2}}. \tag{12.42}$$

For the numerator, dx^μ, is a displacement 4-vector, while the denominator, $d\tau$, is invariant. Thus, for instance, when you go from system $\mathcal{S}$ to system $\bar{\mathcal{S}}$, moving at speed v along the common $x\,\bar{x}$ axis,

$$\left.\begin{aligned}\bar{\eta}^0 &= \gamma(\eta^0 - \beta\eta^1),\\ \bar{\eta}^1 &= \gamma(\eta^1 - \beta\eta^0),\\ \bar{\eta}^2 &= \eta^2,\\ \bar{\eta}^3 &= \eta^3.\end{aligned}\right\} \tag{12.43}$$

More generally,

$$\bar{\eta}^\mu = \Lambda^\mu_\nu \eta^\nu; \tag{12.44}$$

η^μ is called the **proper velocity 4-vector**, or simply the **4-velocity**.

By contrast, the transformation rule for *ordinary* velocities is extremely cumbersome, as we found in Ex. 12.6 and Prob. 12.14:

$$\left.\begin{aligned}\bar{u}_x &= \frac{d\bar{x}}{d\bar{t}} = \frac{u_x - v}{(1 - vu_x/c^2)},\\ \bar{u}_y &= \frac{d\bar{y}}{d\bar{t}} = \frac{u_y}{\gamma(1 - vu_x/c^2)},\\ \bar{u}_z &= \frac{d\bar{z}}{d\bar{t}} = \frac{u_z}{\gamma(1 - vu_x/c^2)}.\end{aligned}\right\} \tag{12.45}$$

The *reason* for the added complexity is plain: we're obliged to transform both the numerator $d\mathbf{l}$ *and the denominator* dt, whereas for *proper* velocity the denominator $d\tau$ is invariant, so the ratio inherits the transformation rule of the numerator alone.

Problem 12.24

(a) Equation 12.40 defines proper velocity in terms of ordinary velocity. Invert that equation to get the formula for $\mathbf{u}$ in terms of $\boldsymbol{\eta}$.

(b) What is the relation between proper velocity and *rapidity* (Eq. 12.34)? Assume the velocity is along the x direction, and find η as a function of θ.

Problem 12.25 A car is traveling along the 45° line in $\mathcal{S}$ (Fig. 12.25), at (ordinary) speed $(2/\sqrt{5})c$.

(a) Find the components u_x and u_y of the (ordinary) velocity.

(b) Find the components η_x and η_y of the proper velocity.

(c) Find the zeroth component of the 4-velocity, η^0.

System $\bar{\mathcal{S}}$ is moving in the x direction with (ordinary) speed $\sqrt{2/5}\,c$, relative to $\mathcal{S}$. By using the appropriate transformation laws:

(d) Find the (ordinary) velocity components $\bar{u}_x$ and $\bar{u}_y$ in $\bar{\mathcal{S}}$.

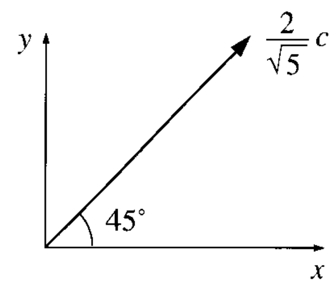

Figure 12.25

(e) Find the proper velocity components $\bar{\eta}_x$ and $\bar{\eta}_y$ in $\bar{S}$.

(f) As a consistency check, verify that

$$\bar{\boldsymbol{\eta}} = \frac{\bar{\mathbf{u}}}{\sqrt{1-\bar{u}^2/c^2}}.$$

• **Problem 12.26** Find the invariant product of the 4-velocity with itself, $\eta^\mu \eta_\mu$.

Problem 12.27 Consider a particle in hyperbolic motion,

$$x(t) = \sqrt{b^2 + (ct)^2}, \quad y = z = 0.$$

(a) Find the proper time τ as a function of t, assuming the clocks are set so that $\tau = 0$ when $t = 0$. [*Hint:* Integrate Eq. 12.37.]

(b) Find x and v (ordinary velocity) as functions of τ.

(c) Find η^μ (proper velocity) as a function of t.

12.2.2 Relativistic Energy and Momentum

In classical mechanics momentum is mass times velocity. I would like to extend this definition to the relativistic domain, but immediately a question arises: Should I use *ordinary* velocity or *proper* velocity? In classical physics $\boldsymbol{\eta}$ and $\mathbf{u}$ are identical, so there is no *a priori* reason to favor one over the other. However, in the context of relativity it is essential that we use *proper* velocity, for the law of conservation of momentum would be inconsistent with the principle of relativity if we were to define momentum as $m\mathbf{u}$ (see Prob. 12.28). Thus

$$\boxed{\mathbf{p} \equiv m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}};} \tag{12.46}$$

this is the **relativistic momentum**.

Relativistic momentum is the spatial part of a 4-vector,

$$p^{\mu} \equiv m\eta^{\mu}, \tag{12.47}$$

and it is natural to ask what the temporal component,

$$p^0 = m\eta^0 = \frac{mc}{\sqrt{1-u^2/c^2}} \tag{12.48}$$

represents. Einstein called

$$m_{\text{rel}} \equiv \frac{m}{\sqrt{1-u^2/c^2}} \tag{12.49}$$

the **relativistic mass** (so that $p^0 = m_{\text{rel}}c$ and $\mathbf{p} = m_{\text{rel}}\mathbf{u}$; m itself was then called the **rest mass**), but modern usage has abandoned this terminology in favor of **relativistic energy**:

$$\boxed{E \equiv \frac{mc^2}{\sqrt{1-u^2/c^2}}} \tag{12.50}$$

(so $p^0 = E/c$).[8] Because p^0 is (apart from the factor $1/c$) the relativistic energy, p^{μ} is called the **energy-momentum 4-vector** (or the **momentum 4-vector**, for short).

Notice that the relativistic energy is nonzero *even when the object is stationary*; we call this **rest energy**:

$$E_{\text{rest}} \equiv mc^2. \tag{12.51}$$

The remainder, which is attributable to the *motion*, we call **kinetic energy**

$$E_{\text{kin}} \equiv E - mc^2 = mc^2\left(\frac{1}{\sqrt{1-u^2/c^2}} - 1\right). \tag{12.52}$$

In the nonrelativistic régime ($u \ll c$) the square root can be expanded in powers of u^2/c^2, giving

$$E_{\text{kin}} = \frac{1}{2}mu^2 + \frac{3}{8}\frac{mu^4}{c^2} + \cdots; \tag{12.53}$$

the leading term reproduces the classical formula.

So far, this is all just *notation*. The *physics* resides in the experimental fact that E and $\mathbf{p}$, as defined by Eqs. 12.46 and 12.50, are *conserved:*

In every closed[9] system, the total relativistic energy and momentum are conserved.

[8] Since E and m_{rel} differ only by a constant factor (c^2), there's nothing to be gained by keeping both terms in circulation, and m_{rel} has gone the way of the two dollar bill.

[9] If there are *external* forces at work, then (just as in the classical case) the energy and momentum of the system itself will *not*, in general, be conserved.

"Relativistic mass" (if you care to use that term) is *also* conserved—but this is equivalent to conservation of energy. *Rest* mass is *not* conserved—a fact that has been painfully familiar to everyone since 1945 (though the so-called "conversion of mass into energy" is really a conversion of *rest* energy into *kinetic* energy). Note the distinction between an **invariant** quantity (same value in all inertial systems) and a **conserved** quantity (same value before and after some process). Mass is invariant, but not conserved; energy is conserved but not invariant; electric charge (as we shall see) is both conserved *and* invariant; velocity is neither conserved *nor* invariant.

The scalar product of p^μ with itself is

$$p^\mu p_\mu = -(p^0)^2 + (\mathbf{p} \cdot \mathbf{p}) = -m^2c^2, \tag{12.54}$$

as you can quickly check using the result of Prob. 12.26. In terms of the relativistic energy,

$$\boxed{E^2 - p^2c^2 = m^2c^4.} \tag{12.55}$$

This result is extremely useful, for it enables you to calculate E (if you know p), or p (knowing E), without ever having to determine the velocity.

Problem 12.28

(a) Repeat Prob. 12.2 using the (incorrect) definition $\mathbf{p} = m\mathbf{u}$, but with the (correct) Einstein velocity addition rule. Notice that if momentum (so defined) is conserved in $\mathcal{S}$, it is *not* conserved in $\bar{\mathcal{S}}$. Assume all motion is along the x axis.

(b) Now do the same using the correct definition, $\mathbf{p} = m\boldsymbol{\eta}$. Notice that if momentum (so defined) is conserved in $\mathcal{S}$ it is automatically also conserved in $\bar{\mathcal{S}}$. [*Hint:* Use Eq. 12.43 to transform the proper velocity.] What must you assume about relativistic energy?

Problem 12.29 If a particle's kinetic energy is n times its rest energy, what is its speed?

Problem 12.30 Suppose you have a collection of particles, all moving in the x direction, with energies E_1, E_2, E_3, ...and momenta p_1, p_2, p_3, ... Find the velocity of the **center of momentum** frame, in which the total momentum is zero.

12.2.3 Relativistic Kinematics

In this section we'll explore some applications of the conservation laws to particle decays and collisions.

Example 12.7

Two lumps of clay, each of (rest) mass m, collide head-on at $\frac{3}{5}c$ (Fig. 12.26). They stick together. *Question*: what is the mass (M) of the composite lump?