

### 12.1.4 The Structure of Spacetime

(i) **Four-vectors.** The Lorentz transformations take on a simpler appearance when expressed in terms of the quantities

$$x^0 \equiv ct, \quad \beta \equiv \frac{v}{c}. \quad (12.21)$$

Using  $x^0$  (instead of  $t$ ) and  $\beta$  (instead of  $v$ ) amounts to changing the unit of time from the *second* to the *meter*—1 meter of  $x^0$  corresponds to the time it takes light to travel 1 meter (in vacuum). If, at the same time, we number the  $x, y, z$  coordinates, so that

$$x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad (12.22)$$

then the Lorentz transformations read

$$\left. \begin{aligned} \bar{x}^0 &= \gamma(x^0 - \beta x^1), \\ \bar{x}^1 &= \gamma(x^1 - \beta x^0), \\ \bar{x}^2 &= x^2, \\ \bar{x}^3 &= x^3. \end{aligned} \right\} \quad (12.23)$$

Or, in matrix form:

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}. \quad (12.24)$$

Letting Greek indices run from 0 to 3, this can be distilled into a single equation:

$$\bar{x}^\mu = \sum_{\nu=0}^3 (\Lambda^\mu_\nu) x^\nu, \quad (12.25)$$

where  $\Lambda$  is the **Lorentz transformation matrix** in Eq. 12.24 (the superscript  $\mu$  labels the row, the subscript  $\nu$  labels the column). One virtue of writing things in this abstract manner is that we can handle in the same format a more general transformation, in which the relative motion is *not* along a common  $x \bar{x}$  axis; the matrix  $\Lambda$  would be more complicated, but the structure of Eq. 12.25 is unchanged.

If this reminds you of the *rotations* we studied in Sect. 1.1.5, it's no accident. There we were concerned with the change in components when you switch to a *rotated* coordinate system; here we are interested in the change of components when you go to a *moving*

system. In Chapter 1 we defined a (3-) vector as any set of three components that transform under rotations the same way ( $x, y, z$ ) do; by extension, we now define a **4-vector** as any set of *four* components that transform in the same manner as  $(x^0, x^1, x^2, x^3)$  under Lorentz transformations:

$$\bar{a}^\mu = \sum_{\nu=0}^3 \Lambda_\nu^\mu a^\nu. \quad (12.26)$$

For the particular case of a transformation along the  $x$  axis:

$$\left. \begin{aligned} \bar{a}^0 &= \gamma(a^0 - \beta a^1), \\ \bar{a}^1 &= \gamma(a^1 - \beta a^0), \\ \bar{a}^2 &= a^2, \\ \bar{a}^3 &= a^3. \end{aligned} \right\} \quad (12.27)$$

There is a 4-vector analog to the dot product ( $\mathbf{A} \cdot \mathbf{B} \equiv A_x B_x + A_y B_y + A_z B_z$ ), but it's not just the sum of the products of like components; rather, the zeroth components have a minus sign:

$$-a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3. \quad (12.28)$$

This is the **four-dimensional scalar product**; you should check for yourself (Prob. 12.17) that it has the same value in all inertial systems:

$$-\bar{a}^0 \bar{b}^0 + \bar{a}^1 \bar{b}^1 + \bar{a}^2 \bar{b}^2 + \bar{a}^3 \bar{b}^3 = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3. \quad (12.29)$$

Just as the ordinary dot product is **invariant** (unchanged) under rotations, this combination is invariant under Lorentz transformations.

To keep track of the minus sign it is convenient to introduce the **covariant** vector  $a_\mu$ , which differs from the **contravariant**  $a^\mu$  only in the sign of the zeroth component:

$$a_\mu = (a_0, a_1, a_2, a_3) \equiv (-a^0, a^1, a^2, a^3). \quad (12.30)$$

You must be scrupulously careful about the placement of indices in this business: *upper* indices designate *contravariant* vectors; *lower* indices are for *covariant* vectors. Raising or lowering the temporal index costs a minus sign ( $a_0 = -a^0$ ); raising or lowering a spatial index changes nothing ( $a_1 = a^1, a_2 = a^2, a_3 = a^3$ ). The scalar product can now be written with the summation symbol,

$$\sum_{\mu=0}^3 a_\mu b^\mu, \quad (12.31)$$

or, more compactly still,

$$a_\mu b^\mu. \quad (12.32)$$

Summation is *implied* whenever a Greek index is repeated in a product—once as a covariant index and once as contravariant. This is called the **Einstein summation convention**, after

its inventor, who regarded it as one of his most important contributions. Of course, we could as well take care of the minus sign by switching to covariant  $b$ :

$$a_\mu b^\mu = a^\mu b_\mu = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3. \quad (12.33)$$

- **Problem 12.17** Check Eq. 12.29, using Eq. 12.27. [This only proves the invariance of the scalar product for transformations along the  $x$  direction. But the scalar product is also invariant under *rotations*, since the first term is not affected at all, and the last three constitute the three-dimensional dot product  $\mathbf{a} \cdot \mathbf{b}$ . By a suitable rotation, the  $x$  direction can be aimed any way you please, so the four-dimensional scalar product is actually invariant under *arbitrary* Lorentz transformations.]

**Problem 12.18**

- Write out the matrix that describes a *Galilean* transformation (Eq. 12.12).
- Write out the matrix describing a Lorentz transformation along the  $y$  axis.
- Find the matrix describing a Lorentz transformation with velocity  $v$  along the  $x$  axis followed by a Lorentz transformation with velocity  $\bar{v}$  along the  $y$  axis. Does it matter in what order the transformations are carried out?

**Problem 12.19** The parallel between rotations and Lorentz transformations is even more striking if we introduce the **rapidity**:

$$\theta \equiv \tanh^{-1}(v/c). \quad (12.34)$$

- Express the Lorentz transformation matrix  $\Lambda$  (Eq. 12.24) in terms of  $\theta$ , and compare it to the rotation matrix (Eq. 1.29).

In some respects rapidity is a more natural way to describe motion than velocity. [See E. F. Taylor and J. A. Wheeler, *Spacetime Physics* (San Francisco: W. H. Freeman, 1966).] For one thing, it ranges from  $-\infty$  to  $+\infty$ , instead of  $-c$  to  $+c$ . More significantly, rapidities add, whereas velocities do not.

- Express the Einstein velocity addition law in terms of rapidity.

**(ii) The invariant interval.** Suppose event  $A$  occurs at  $(x_A^0, x_A^1, x_A^2, x_A^3)$ , and event  $B$  at  $(x_B^0, x_B^1, x_B^2, x_B^3)$ . The difference,

$$\Delta x^\mu \equiv x_A^\mu - x_B^\mu, \quad (12.35)$$

is the **displacement 4-vector**. The scalar product of  $\Delta x^\mu$  with itself is a quantity of special importance; we call it the **interval** between two events:

$$I \equiv (\Delta x)_\mu (\Delta x)^\mu = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = -c^2 t^2 + d^2, \quad (12.36)$$

where  $t$  is the time difference between the two events and  $d$  is their spatial separation. When you transform to a moving system, the *time* between  $A$  and  $B$  is altered ( $\bar{t} \neq t$ ), and so is the *spatial separation* ( $\bar{d} \neq d$ ), but the interval  $I$  remains the same.