its inventor, who regarded it as one of his most important contributions. Of course, we could as well take care of the minus sign by switching to covariant b:

$$a_\mu b^\mu = a^\mu b_\mu = -a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3. \tag{12.33}$$

• **Problem 12.17** Check Eq. 12.29, using Eq. 12.27. [This only proves the invariance of the scalar product for transformations along the x direction. But the scalar product is also invariant under *rotations*, since the first term is not affected at all, and the last three constitute the three-dimensional dot product $\mathbf{a}\cdot\mathbf{b}$. By a suitable rotation, the x direction can be aimed any way you please, so the four-dimensional scalar product is actually invariant under *arbitrary* Lorentz transformations.]

Problem 12.18

(a) Write out the matrix that describes a *Galilean* transformation (Eq. 12.12).

(b) Write out the matrix describing a Lorentz transformation along the y axis.

(c) Find the matrix describing a Lorentz transformation with velocity v along the x axis followed by a Lorentz transformation with velocity $\bar{v}$ along the y axis. Does it matter in what order the transformations are carried out?

Problem 12.19 The parallel between rotations and Lorentz transformations is even more striking if we introduce the **rapidity**:

$$\theta \equiv \tanh^{-1}(v/c). \tag{12.34}$$

(a) Express the Lorentz transformation matrix Λ (Eq. 12.24) in terms of θ, and compare it to the rotation matrix (Eq. 1.29).

In some respects rapidity is a more natural way to describe motion than velocity. [See E. F. Taylor and J. A. Wheeler, *Spacetime Physics* (San Francisco: W. H. Freeman, 1966).] For one thing, it ranges from $-\infty$ to $+\infty$, instead of $-c$ to $+c$. More significantly, rapidities add, whereas velocities do not.

(b) Express the Einstein velocity addition law in terms of rapidity.

(ii) The invariant interval. Suppose event A occurs at $(x_A^0, x_A^1, x_A^2, x_A^3)$, and event B at $(x_B^0, x_B^1, x_B^2, x_B^3)$. The difference,

$$\Delta x^\mu \equiv x_A^\mu - x_B^\mu, \tag{12.35}$$

is the **displacement 4-vector**. The scalar product of Δx^μ with itself is a quantity of special importance; we call it the **interval** between two events:

$$I \equiv (\Delta x)_\mu (\Delta x)^\mu = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = -c^2t^2 + d^2, \tag{12.36}$$

where t is the time difference between the two events and d is their spatial separation. When you transform to a moving system, the *time* between A and B is altered ($\bar{t} \neq t$), and so is the *spatial separation* ($\bar{d} \neq d$), but the interval I remains the same.

Depending on the two events in question, the interval can be positive, negative, or zero:

1. If $I < 0$ we call the interval **timelike**, for this is the sign we get when the two occur at the *same place* ($d = 0$), and are separated only temporally.
2. If $I > 0$ we call the interval **spacelike**, for this is the sign we get when the two occur at the *same time* ($t = 0$), and are separated only spatially.
3. If $I = 0$ we call the interval **lightlike**, for this is the relation that holds when the two events are connected by a signal traveling at the speed of light.

If the interval between the two events is timelike, there exists an inertial system (accessible by Lorentz transformation) in which they occur at the same point. For if I hop on a train going from (A) to (B) at the speed $v = d/t$, leaving event A when it occurs, I shall be just in time to pass B when *it* occurs; in the train system, A and B take place at the same point. You cannot do this for a *spacelike* interval, of course, because v would have to be greater than c, and no observer can exceed the speed of light (γ would be imaginary and the Lorentz transformations would be nonsense). On the other hand, if the interval is spacelike, then there exists a system in which the two events occur at the same time (see Prob. 12.21).

Problem 12.20

(a) Event A happens at point ($x_A = 5, y_A = 3, z_A = 0$) and at time t_A given by $ct_A = 15$; event B occurs at $(10, 8, 0)$ and $ct_B = 5$, both in system $\mathcal{S}$.

(i) What is the invariant interval between A and B?

(ii) Is there an inertial system in which they occur *simultaneously*? If so, find its velocity (magnitude and direction) relative to $\mathcal{S}$.

(iii) Is there an inertial system in which they occur at the same point? If so, find its velocity relative to $\mathcal{S}$

(b) Repeat part (a) for $A = (2, 0, 0)$, $ct = 1$; and $B = (5, 0, 0)$, $ct = 3$.

Problem 12.21 The coordinates of event A are $(x_A, 0, 0)$, t_A, and the coordinates of event B are $(x_B, 0, 0)$, t_B. Assuming the interval between them is spacelike, find the velocity of the system in which they are simultaneous.

(iii) Space-time diagrams. If you want to represent the motion of a particle graphically, the normal practice is to plot the position versus time (that is, x runs vertically and t horizontally). On such a graph, the velocity can be read off as the slope of the curve. For some reason the convention is reversed in relativity: everyone plots position horizontally and time (or, better, $x^0 = ct$) vertically. Velocity is then given by the *reciprocal* of the slope. A particle at rest is represented by a vertical line; a photon, traveling at the speed of light, is described by a 45° line; and a rocket going at some intermediate speed follows a line of slope $c/v = 1/\beta$ (Fig. 12.21). We call such plots **Minkowski diagrams**.

The trajectory of a particle on a Minkowski diagram is called a **world line**. Suppose you set out from the origin at time $t = 0$. Because no material object can travel faster than light, your world line can never have a slope less than 1. Accordingly, your motion is

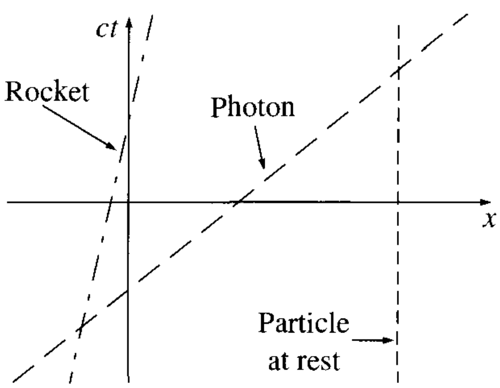

Figure 12.21

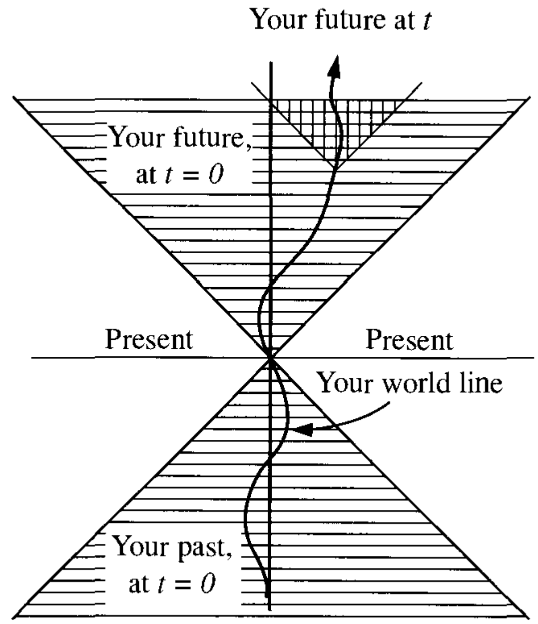

Figure 12.22

restricted to the wedge-shaped region bounded by the two 45° lines (Fig. 12.22). We call this your "future," in the sense that it is the locus of all points accessible to you. Of course, as time goes on, and you move along your chosen world line, your options progressively narrow: your "future" at any moment is the forward "wedge" constructed at whatever point you find yourself. Meanwhile, the *backward* wedge represents your "past," in the sense that it is the locus of all points from which you might have come. As for the rest (the region outside the forward and backward wedges) this is the generalized "present." You can't *get* there, and you didn't *come* from there. In fact, there's no way can can influence any event in the present (the message would have to travel faster than light); it's a vast expanse of spacetime that is absolutely inaccessible to you.

I've been ignoring the y and z directions. If we include a y axis coming out of the page, the "wedges" become cones—and, with an undrawable z axis, hypercones. Because their boundaries are the trajectories of light rays, we call them the **forward light cone** and the **backward light cone**. Your future, in other words, lies within your forward light cone, your past within your backward light cone.

Notice that the slope of the line connecting two events on a space-time diagram tells you at a glance whether the invariant interval between them is timelike (slope greater than 1), spacelike (slope less than 1), or lightlike (slope 1). For example, all points in the past and future are timelike with respect to your present location, whereas points in the present are spacelike, and points on the light cone are lightlike.

Hermann Minkowski, who was the first to recognize the full geometrical significance of special relativity, began a classic paper with the words, "Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality." It is a lovely thought, but you must be careful

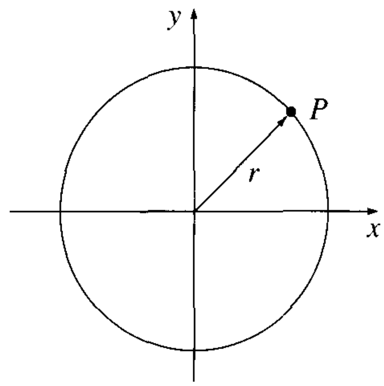

Figure 12.23

not to read too much into it. For it is not at all the case that time is "just another coordinate, on the same footing with x, y, and z" (except that for obscure reasons we measure it on clocks instead of rulers). *No:* Time is *utterly different* from the others, and the mark of its distinction is the minus sign in the invariant interval. That minus sign imparts to spacetime a hyperbolic geometry that is much richer than the circular geometry of 3-space.

Under rotations about the z axis, a point P in the xy plane describes a *circle:* the locus of all points a fixed distance $r = \sqrt{x^2 + y^2}$ from the origin (Fig. 12.23). Under Lorentz transformations, however, it is the interval $I = (x^2 - c^2t^2)$ that is preserved, and the locus of all points with a given value of I is a *hyperbola*—or, if we include the y axis, a *hyperboloid of revolution.* When the interval is *timelike*, it's a "hyperboloid of two sheets" (Fig. 12.24a); when the interval is *spacelike*, it's a "hyperboloid of one sheet" (Fig. 12.24b). When you perform a Lorentz transformation (that is, when you go into a moving inertial system), the coordinates (x, t) of a given event will change to $(\bar{x}, \bar{t})$, but these new coordinates *will lie on the same hyperbola* as (x, t). By appropriate combinations of Lorentz transformations

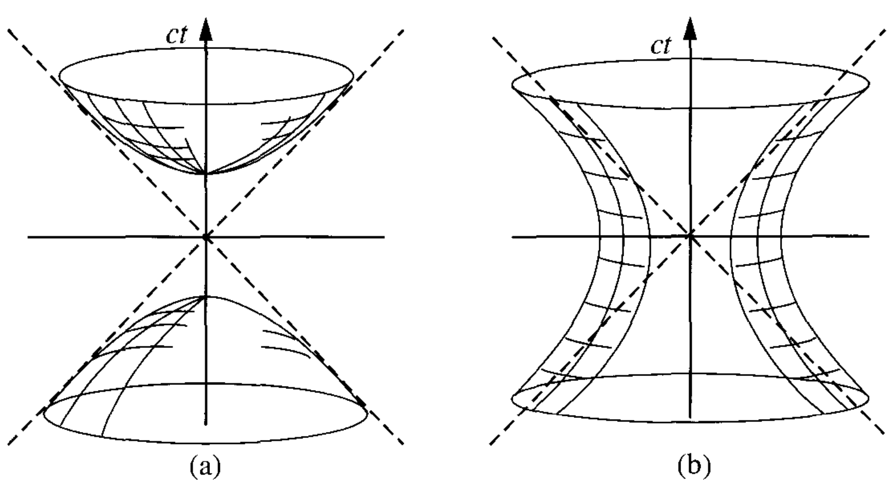

Figure 12.24

and rotations, a spot can be moved around at will over the surface of a given hyperboloid, but no amount of transformation will carry it, say, from the upper sheet of the timelike hyperboloid to the lower sheet, or to a spacelike hyperboloid.

When we were discussing simultaneity I pointed out that the time ordering of two events can, at least in certain cases, be reversed, simply by going into a moving system. But we now see that this is not *always* possible: *If the invariant interval between two events is timelike, their ordering is absolute; if the interval is spacelike, their ordering depends on the inertial system from which they are observed.* In terms of the space-time diagram, an event on the upper sheet of a timelike hyperboloid *definitely* occurred *after* (0, 0), and one on the lower sheet certainly occurred *before;* but an event on a spacelike hyperboloid occurred at positive t, or negative t, depending on your reference frame. This is not an idle curiosity, for it rescues the notion of **causality**, on which all physics is based. If it were *always* possible to reverse the order of two events, then we could never say "A caused B," since a rival observer would retort that *B preceded A.* This embarrassment is avoided, provided the two events are timelike-separated. And causally related events *are* timelike-separated—otherwise no influence could travel from one to the other. *Conclusion:* The invariant interval between causally related events is always timelike, and their temporal ordering is the same for all inertial observers.

Problem 12.22

(a) Draw a space-time diagram representing a game of catch (or a conversation) between two people at rest, 10 ft apart. How is it possible for them to communicate, given that their separation is spacelike?

(b) There's an old limerick that runs as follows:

> There once was a girl named Ms. Bright,
> Who could travel much faster than light.
> She departed one day,
> The Einsteinian way,
> And returned on the previous night.

What do you think? Even if she *could* travel faster than the speed of light, could she return before she set out? Could she arrive at some intermediate destination before she set out? Draw a space-time diagram representing this trip.

Problem 12.23 Inertial system $\bar{S}$ moves in the x direction at speed $\frac{3}{5}c$ relative to system S. (The $\bar{x}$ axis slides long the x axis, and the origins coincide at $t = \bar{t} = 0$, as usual.)

(a) On graph paper set up a Cartesian coordinate system with axes ct and x. Carefully draw in lines representing $\bar{x} = -3, -2, -1, 0, 1, 2$, and 3. Also draw in the lines corresponding to $c\bar{t} = -3, -2, -1, 0, 1, 2$, and 3. Label your lines clearly.

(b) In $\bar{S}$, a free particle is observed to travel from the point $\bar{x} = -2$ at time $c\bar{t} = -2$ to the point $\bar{x} = 2$ at $c\bar{t} = +3$. Indicate this displacement on your graph. From the slope of this line. determine the particle's speed in S.

(c) Use the velocity addition rule to determine the velocity in S algebraically, and check that your answer is consistent with the graphical solution in (b).
