(c) And for the return leg we introduce a third event corresponding to when the ball returns. In S it has co-ordinates $(x_1, t_1 + 2L/u)$. As in part (b) we use Eq. (6.28b). Notice that $x_3 - x_2 = -\Delta x$ this time whilst $t_3 - t_2 = \Delta t$ and so

$$\begin{aligned}\Delta t'_{in} &= \gamma(\Delta t + v\Delta x/c^2),\\ &= \frac{\gamma L}{u}(1 + uv/c^2).\end{aligned} \tag{6.32}$$

Notice also that the total time for the journey is just as we would expect from time dilation, i.e $\Delta t'_{tot} = \gamma(2L/u)$ as it should be since the point of departure and point of return are one and the same place. This result confirms our earlier claim that there was nothing special about a light-clock.

6.3 VELOCITY TRANSFORMATIONS

6.3.1 Addition of Velocities

We can use the Lorentz transformations to figure out how the rules for adding velocities must change in Special Relativity. Consider an object moving with a velocity $\mathbf{v}'$ in S'. Let us determine its velocity $\mathbf{v}$ in S. The situation is illustrated in Figure 6.9. Notice that to avoid confusion the relative speed between the two frames is now u and we have simplified to the case of motion in two dimensions (in the $x - y$ plane). It is straightforward to generalize to three-dimensions. Recall that according to Galilean relativity $v_x = v'_x + u$ and $v_y = v'_y$. Neither of these holds true in Special Relativity, as we shall now see.

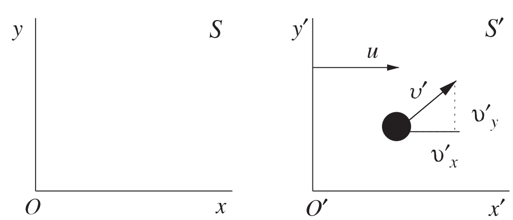

Figure 6.9 Relative velocities.

To determine the x-component of the velocity in S we make use of the Lorentz transformation formulae for x and for t:

$$\begin{aligned}v_x = \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\gamma(\mathrm{d}x' + u\mathrm{d}t')}{\gamma(\mathrm{d}t' + u\mathrm{d}x'/c^2)} = \frac{(\mathrm{d}x'/\mathrm{d}t') + u}{1 + u(\mathrm{d}x'/\mathrm{d}t')c^2}\\ &= \frac{v'_x + u}{1 + uv'_x/c^2}.\end{aligned} \tag{6.33}$$

Similarly we can determine the y-component of the velocity:

$$v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y'}{\gamma(\mathrm{d}t' + u\mathrm{d}x'/c^2)} = \frac{\mathrm{d}y'/\mathrm{d}t'}{\gamma(1 + u(\mathrm{d}x'/\mathrm{d}t')c^2)}$$

$$= \frac{v'_y}{\gamma(1 + uv'_x/c^2)}. \tag{6.34}$$

Notice that for $uv'_x \ll c^2$ and $u \ll c$ these results reduce to the expectation based on classical thinking. Eqs. (6.33) and (6.34) are known as the velocity transformation equations and their use is pretty straightforward. Perhaps the only place where there is room for error is when it comes to figuring out the signs. For example, if S' were moving in the negative x-direction then we should replace $u \to -u$ in the equations. We can quickly check to see that the velocity transformation equations satisfy the 2nd postulate, i.e. if $v'_x = c$ and $v_y = 0$ we have

$$v_x = \frac{c + u}{1 + uc/c^2} = c, \tag{6.35}$$

which is as it should be.

6.3.2 Stellar Aberration Revisited

It is at this point that we can confirm that although Einstein has abolished the ether his new theory is still capable of explaining the phenomenon of stellar aberration. To understand this, let us consider the particular situation illustrated in Figure 6.10. We imagine that the Sun, Earth and star all lie in the same plane and that the Sun is at rest in S'. Suppose that light emitted from the star arrives at an angle angle α' to the vertical in S'. We shall take the relative speed between the Earth and Sun to be u and α is the angle at which the starlight arrives on Earth.

Using the velocity addition formulae with $v'_x = -c\sin\alpha'$ and $v'_y = -c\cos\alpha'$ we have that

$$v_x = \frac{u - c\sin\alpha'}{1 - \frac{u}{c}\sin\alpha'}, \tag{6.36}$$

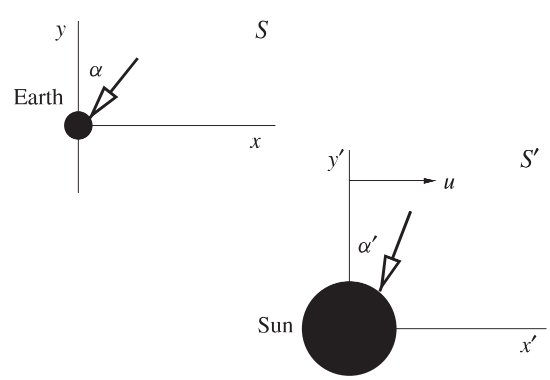

Figure 6.10 Incident starlight in the Earth and Sun rest frames.

$$v_y = \frac{-c\cos\alpha'}{\gamma(u)(1-\frac{u}{c}\sin\alpha')}. \tag{6.37}$$

These two equations imply that

$$\tan\alpha = \frac{v_x}{v_y} = \frac{\sin\alpha' - u/c}{\cos\alpha'}\frac{1}{\sqrt{1-u^2/c^2}}. \tag{6.38}$$

Stellar aberration is greatest when $\alpha' = 0$, in which case this result simplifies to

$$\tan\alpha = -\frac{u}{c}\frac{1}{\sqrt{1-u^2/c^2}},$$

$$\text{i.e.}\ \sin\alpha = -\frac{u}{c}.$$

Now if $u \ll c$ then this is gives rise to a variation in the star's angular position of $\approx 2u/c$ over the course of one year, which is in accord with observations.

Example 6.3.1 *Consider three galaxies, A, B and C. An observer in A measures the velocities of B and C and finds they are moving in opposite directions each with a speed of* $0.7c$. *(a) At what rate does the distance between B and C increase according to A? (b) What is the speed of A observed in B? (c) What is the speed of C observed in B?*

Solution 6.3.1 *Again it really helps to draw a picture: we refer to Figure 6.11. (a) The relative speed between B and C according to A is just* $2u = 1.4c$. *We do not of course worry that this speed is in excess of c because it is not the speed of any material object. (b) According to B, A moves 'to the right' with speed u. (c) Now to determine the speed of C according to an observer in B we do need to use the addition of velocities formula since we only know the speed of C in A and the speed of A relative to B. In classical theory, the result would be* $1.4c$, *but this will clearly be modified to a value smaller than c in Special Relativity. The correct value is found using Eq. (6.33):*

$$\frac{u+u}{1+u^2/c^2} = \frac{1.4c}{1.49} = 0.94c.$$

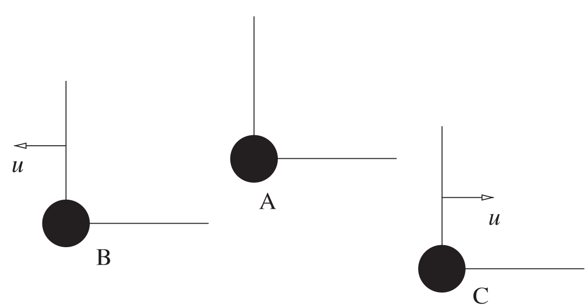

Figure 6.11 Relative motion of three galaxies viewed from an observer in A.

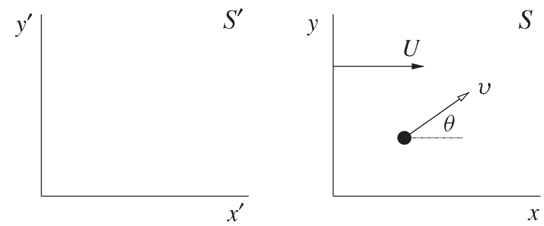

Figure 6.12 A particle moves in S at an angle θ to the x-axis at speed v.

Example 6.3.2 *A particle moves with speed v at an angle θ to the positive x-axis in the frame S. What is the direction of the particle in the frame S' given that S and S' move with relative speed U along their common x direction?*

Solution 6.3.2 *Figure 6.12 illustrates what is going on. Using the velocity addition formulae we can write down the components of the velocity of the particle in S':*

$$v'_x = \frac{v_x + U}{1 + Uv_x/c^2},$$
$$v'_y = \frac{v_y}{\gamma(U) \cdot (1 + Uv_x/c^2)}.$$

Using $\tan\theta' = v'_y/v'_x$, $v_x = v\cos\theta$ *and* $v_y = v\sin\theta$ *gives*

$$\tan\theta' = \frac{(1 - U^2/c^2)^{1/2} v \sin\theta}{U + v\cos\theta}.$$

This is an interesting result: if $U \to c$ then $\tan\theta' \to 0$ *regardless of θ. This effect would only happen in classical theory as $U \to \infty$.*

PROBLEMS 6

6.1 A spaceship moves relative to the Earth at a speed of $0.93c$. If a person on Earth spends 30 minutes reading the newspaper, how long have they been reading according to someone on the spaceship?

6.2 Pions are elementary particles, which decay with a half-life of 1.8×10^{-8} s as measured in a frame in which the pions are at rest. In a laboratory experiment, a beam of pions has a speed of $0.95c$. According to an observer in the lab, how long does it take for half of the pions to decay? Through what distance will they travel in that time?

6.3 An alien spacecraft is flying overhead at a great distance. You see its searchlight blink on for 0.190 s. Meanwhile, on board the spacecraft, the pilot observes that the searchlight was on for 12.0 ms. What is the speed of the spacecraft relative to the Earth?

6.4 In the previous question, why was it necessary to state that the spacecraft was at a great distance overhead? Suppose that the same spacecraft is flying at $0.998c$ but this time at ground level and directly away from you. If the pilot once again turns the searchlight on for 12.0 ms, how long now does the searchlight appear to stay on according to your watch?