dangerous for it suggests problems with causality. Surely everyone must agree that a person must be born before they die? And indeed they must. It is a remarkable feature of Special Relativity that although the time ordering of events can be a matter for debate this is only the case for causally disconnected events, i.e. events which cannot influence each other. We shall return to this interesting discussion in Part IV. For now we content ourselves with a thought experiment which illustrates the breakdown of simultaneity.

Consider a train travelling along at a speed u relative to the platform. An observer is standing in the middle of the train. Suppose that a flashlight is attached to each end of the train and that the flashlights flash on for a brief instant. If the observer receives the light from each flashlight at the same time then she will conclude that the flashes occurred simultaneously, for the light from each flashlight had to travel the same distance (half the length of the train) at the same speed. Now consider a second observer standing on the platform watching proceedings. They must observe that our first observer does indeed receive the light from either end of the train at a particular instant in time. However, from their viewpoint the light from the front of the train has less distance to travel than the light from the rear of the train since the observer on the train is moving towards the point of emission at the front of the train and away from the point of emission at the rear of the train. None of what has been said so far is controversial; it holds in classical theory too. Here comes the difference. As a result of the 2nd postulate, the observer on the platform still sees each pulse of light travel at the same speed c. Now since both pulses arrive at the centre of the train at the same time, and the pulse from the front had less distance to travel, it follows that it must have been emitted later than the light from the rear of the train. Classical physics avoids this conclusion because although the light from the front has less distance to travel it is travelling more slowly (its speed is $c - u$) than the light from the rear (its speed is $c + u$) and the reduction in speed compensates the reduction in distance. You might like to check that this compensation is exact and that both observers agree that the pulses were emitted at the same time according to classical physics.

6.2 LORENTZ TRANSFORMATIONS

In Section 5.1 we derived the Galilean transformation equations which relate the co-ordinates of an event in one inertial frame to the co-ordinates in a second inertial frame. For their derivation we relied upon the idea of absolute time and, as the last section showed, this is a flawed concept in Special Relativity. We must therefore seek new equations to replace the Galilean transformations. These new equations are the so-called Lorentz transformations.

To derive the Lorentz transformations we shall follow the methods of Section 5.1. We shall define our two inertial frames S and S' exactly as before, and as illustrated in Figure 5.1, i.e. S' is moving along the positive x axis at a speed v relative to S. Since the motion is parallel to the x and x' axes it follows that

$$y' = y \tag{6.17}$$

$$z' = z \tag{6.18}$$

as before. Recall that we want to express the co-ordinates in S' in terms of those measured in S. Again in order for the 1st postulate to remain valid the transformations must be of the form

$$x' = ax + bt, \tag{6.19a}$$

$$t' = dx + et. \tag{6.19b}$$

Notice that we have not assumed that there exists a unique time variable, i.e. we allow for $t' \neq t$. Our goal is to solve for the coefficients a, b, d and e. As with the derivation of the Galilean transforms we require that the origin O' (i.e. the point $x' = 0$) move along the x-axis according to $x = vt$. Substituting this information into Eq. (6.19a) yields

$$-b/a = v. \tag{6.20}$$

Similarly we require that the origin O move along the line $x' = -vt'$. From Eqs. (6.19) the point $x = 0$ satisfies $x' = bt$ and $t' = et$ such that $x' = -vt'$ implies that

$$-b/e = v. \tag{6.21}$$

Eqs. (6.20) and (6.21) imply that $e = a$ and $b = -av$. Substituting these into Eqs. (6.19) gives

$$\begin{aligned} x' &= ax - avt, \\ t' &= dx + at. \end{aligned} \tag{6.22}$$

We have two unknowns, a and d, remaining and have two postulates to implement. Let us first implement the 2nd postulate. We shall do this by considering a pulse of light emitted at the origins O and O' when they are coincident, i.e. when $t = t' = 0$. We know that this pulse must travel outwards along the x and x' axes such that it satisfies $x = ct$ and $x' = ct'$, i.e. it travels out at the same speed c in both frames. These two equations must be simultaneous solutions to Eqs. (6.22) and so we require that

$$\begin{aligned} ct' &= act - avt, \\ t' &= dct + at. \end{aligned} \tag{6.23}$$

From which it follows directly that

$$d = -\frac{av}{c^2}. \tag{6.24}$$

It only remains to determine the value of a. Let us summarise progress so far. We have reduced Eqs. (6.19a) and (6.19b) to

$$x' = a(x - vt), \tag{6.25a}$$

$$t' = a\left(t - \frac{vx}{c^2}\right). \tag{6.25b}$$

Now it is time to make use of the 1st postulate which says that if Eqs. (6.25) are true then so necessarily are

$$x = a(x' + vt'), \tag{6.26a}$$

$$t = a\left(t' + \frac{vx'}{c^2}\right). \tag{6.26b}$$

This makes manifest the equivalence of the two frames. It can be seen by considering Figure 5.1 and swapping the primed and unprimed co-ordinate labels around whilst at the same time reversing the direction of v. We can determine the coefficient a now by substituting for x' and t' using Eqs. (6.25) into either of Eqs. (6.26), i.e.

$$x = a\left(ax - avt + avt - \frac{av^2x}{c^2}\right) = a^2x\left(1 - \frac{v^2}{c^2}\right)$$

$$\Rightarrow a = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma. \tag{6.27}$$

We have succeeded in deriving the Lorentz transformations:

$$x' = \gamma(x - vt), \tag{6.28a}$$

$$t' = \gamma(t - vx/c^2), \tag{6.28b}$$

$$y' = y, \tag{6.28c}$$

$$z' = z. \tag{6.28d}$$

Sometimes the inverse transformations will be more useful:

$$x = \gamma(x' + vt'), \tag{6.29a}$$

$$t = \gamma(t' + vx'/c^2). \tag{6.29b}$$

Eqs. (6.28) are perhaps the most important equations we have derived so far in this part of the book.

Example 6.2.1 *Use the Lorentz transformations to derive the formula for time dilation.*

Solution 6.2.1 *Let us consider the situation illustrated in Figure 6.6. A clock is at rest in S', let's suppose it is at position x'_0. Now consider one tick of the clock. In S', we suppose that the tick starts at time t'_1 and ends at time t'_2 such that $\Delta t' = t'_2 - t'_1$ is the duration in the clock's rest frame. The question is: 'what is the duration of the same tick as determined by an observer in S?'*

There are two events to consider. Event 1 (start of the tick) has co-ordinates (x'_0, t'_1) in S' and event 2 (end of tick) which has co-ordinates (x'_0, t'_2) in S'. We want to know the time of each event in S. Given that we know both the location and time of the events in S' we should use Eq. (6.29b) to give us the corresponding times in S:

$$t_1 = \gamma(t'_1 + vx'_0/c^2),$$

$$t_2 = \gamma(t'_2 + vx'_0/c^2).$$

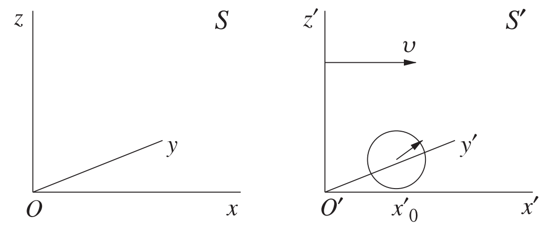

Figure 6.6 A moving clock.

Subtracting these two equations gives

$$\Delta t = t_2 - t_1 = \gamma \Delta t',$$

which is the required result. Notice that to derive this result it was crucial to be clear that the clock is at rest in S'.

Example 6.2.2 *Use the Lorentz transformations to derive the formula for length contraction.*

Solution 6.2.2 *We now consider the situation illustrated in Figure 6.7 where we have placed a ruler in S' such that it lies along the x'-axis with one end located at x'_1 and the other at x'_2. The length of the ruler in its rest frame is therefore $\Delta x' = x'_2 - x'_1$. The question now is: 'what is the length of the ruler as determined by an observer in S?'*

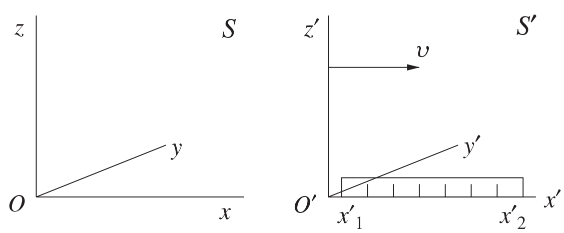

Figure 6.7 A moving ruler.

Again there are two events to consider. Event 1 (measurement of one end of the ruler) and event 2 (measurement of the other end of the ruler). The crucial point now is that both events occur at the same time in S because that is what is meant by a measurement of length. Let's call this time t_0. Given that we know the location of the two events in S' and the time of the events in S we should use Eq. (6.28a) to give us the location of the events in S:

$$x'_1 = \gamma(x_1 - vt_0),$$

$$x'_2 = \gamma(x_2 - vt_0).$$

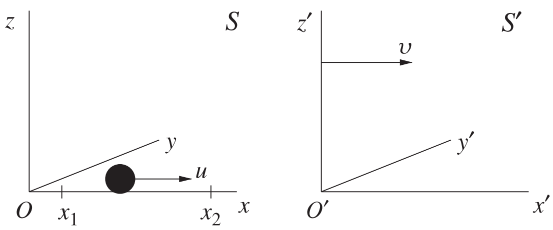

Figure 6.8 A ball bouncing back and forth between two points.

Subtracting these two equations gives

$$\Delta x' = x_2' - x_1' = \gamma \Delta x$$
$$\Rightarrow \Delta x = \frac{1}{\gamma} \Delta x',$$

which is the required result.

Example 6.2.3 *A ball is rolled at speed u from the point x_1 on the x-axis to the point $x_2 = x_1 + L$ at which point it is reflected back again elastically, as illustrated in Figure 6.8. In a frame moving with speed v along the positive x-axis compute:*

(a) The spatial separation between the point where the ball starts its journey and the point where it is reflected;
(b) The time taken for the outward part of the ball's journey;
(c) The time taken for the return part of the ball's journey.

Solution 6.2.3 *(a) Event 1 is when the ball starts on its journey and has co-ordinates (x_1, t_1) in S. Event 2 is when the ball arrives at the point of reflection. It has co-ordinates $(x_2, t_1 + L/u)$. We are asked to find $\Delta x' = x_2' - x_1'$. Note that it is not going to be given by the length contraction formula since the two events are not simultaneous in either S or S′. We know both $\Delta x = x_2 - x_1 = L$ and $\Delta t = L/u$ and need $\Delta x'$. We therefore need to use Eq. (6.28a) which informs us that*

$$\Delta x' = \gamma(\Delta x - v\Delta t),$$
$$= \gamma L(1 - v/u) \tag{6.30}$$

and γ is of course evaluated using the relative speed of the two frames v.

(b) To get the time taken for the outward part of the journey we should use Eq. (6.28b) (we hope that by now the reader is getting the hang of selecting the correct equation to use), i.e.

$$\Delta t'_{\text{out}} = \gamma(\Delta t - v\Delta x/c^2),$$
$$= \frac{\gamma L}{u}(1 - uv/c^2). \tag{6.31}$$