PART FOUR

SIMULTANEOUS-EQUATION MODELS

A casual look at the published empirical work in business and economics will reveal that many economic relationships are of the single-equation type. That is why we devoted the first three parts of this book to the discussion of single-equation regression models. In such models, one variable (the dependent variable *Y*) is expressed as a linear function of one or more other variables (the explanatory variables, the *X*'s). In such models an implicit assumption is that the cause-and-effect relationship, if any, between *Y* and the *X*'s is unidirectional: The explanatory variables are the *cause* and the dependent variable is the *effect*.

However, there are situations where there is a two-way flow of influence among economic variables; that is, one economic variable affects another economic variable(s) and is, in turn, affected by it (them). Thus, in the regression of money *M* on the rate of interest *r*, the single-equation methodology assumes implicitly that the rate of interest is fixed (say, by the Federal Reserve System) and tries to find out the response of money demanded to the changes in the level of the interest rate. But what happens if the rate of interest depends on the demand for money? In this case, the conditional regression analysis made in this book thus far may not be appropriate because now *M* depends on *r* and *r* depends on *M*. Thus, we need to consider two equations, one relating *M* to *r* and another relating *r* to *M*. And this leads us to consider simultaneous-equation models, models in which there is more than one regression equation, one for each interdependent variable.

In **Part IV** we present a very elementary and often heuristic introduction to the complex subject of **simultaneous-equation models,** the details being left for the references.

In Chapter 18, we provide several examples of simultaneous-equation models and show why the method of ordinary least squares considered previously is generally inapplicable to estimate the parameters of each of the equations in the model.

In Chapter 19, we consider the so-called **identification problem.** If in a system of simultaneous equations containing two or more equations it is not possible to obtain numerical values of each parameter in each equation because the equations are *observationally indistinguishable*, or look too much like one another, then we have the identification problem. Thus, in the regression of quantity Q on price P, is the resulting equation a demand function or a supply function, for Q and P enter into both functions? Therefore, if we have data on Q and P only and no other information, it will be difficult if not impossible to identify the regression as the demand or supply function. It is essential to resolve the identification problem before we proceed to estimation because if we do not know what we are estimating, estimation per se is meaningless. In Chapter 19 we offer various methods of solving the identification problem.

In Chapter 20, we consider several estimation methods that are designed specifically for estimating the simultaneous-equation models and consider their merits and limitations.

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SIMULTANEOUS-EQUATION MODELS

In this and the following two chapters we discuss the simultaneous-equation models. In particular, we discuss their special features, their estimation, and some of the statistical problems associated with them.

18.1 THE NATURE OF SIMULTANEOUS-EQUATION MODELS

In **Parts I** to **III** of this text we were concerned exclusively with single-equation models, i.e., models in which there was a single dependent variable Y and one or more explanatory variables, the X's. In such models the emphasis was on estimating and/or predicting the average value of Y conditional upon the fixed values of the X variables. The cause-and-effect relationship, if any, in such models therefore ran from the X's to the Y.

But in many situations, such a one-way or unidirectional cause-and-effect relationship is not meaningful. This occurs if Y is determined by the X's, and some of the X's are, in turn, determined by Y. In short, there is a two-way, or simultaneous, relationship between Y and (some of) the X's, which makes the distinction between *dependent* and *explanatory* variables of dubious value. It is better to lump together a set of variables that can be determined simultaneously by the remaining set of variables—precisely what is done in simultaneous-equation models. In such models there is more than one equation—one for each of the *mutually,* or *jointly,* dependent or **endogenous variables.**[1] And unlike the single-equation models, in the

[1]In the context of the simultaneous-equation models, the jointly dependent variables are called **endogenous variables** and the variables that are truly nonstochastic or can be so regarded are called the **exogenous,** or **predetermined, variables.** (More on this in Chap. 19.)

simultaneous-equation models one may not estimate the parameters of a single equation without taking into account information provided by other equations in the system.

What happens if the parameters of each equation are estimated by applying, say, the method of OLS, disregarding other equations in the system? Recall that one of the crucial assumptions of the method of OLS is that the explanatory X variables are either nonstochastic or, if stochastic (random), are distributed independently of the stochastic disturbance term. If neither of these conditions is met, then, as shown later, the least-squares estimators are not only biased but also inconsistent; that is, as the sample size increases indefinitely, the estimators do not converge to their true (population) values. Thus, in the following hypothetical system of equations,[2]

$$Y_{1i} = \beta_{10} + \beta_{12}Y_{2i} + \gamma_{11}X_{1i} + u_{1i} \quad \textbf{(18.1.1)}$$

$$Y_{2i} = \beta_{20} + \beta_{21}Y_{1i} + \gamma_{21}X_{1i} + u_{2i} \quad \textbf{(18.1.2)}$$

where Y_1 and Y_2 are mutually dependent, or endogenous, variables and X_1 is an exogenous variable and where u_1 and u_2 are the stochastic disturbance terms, the variables Y_1 and Y_2 are both stochastic. Therefore, unless it can be shown that the stochastic explanatory variable Y_2 in (18.1.1) is distributed independently of u_1 and the stochastic explanatory variable Y_1 in (18.1.2) is distributed independently of u_2, application of the classical OLS to these equations individually will lead to inconsistent estimates.

In the remainder of this chapter we give a few examples of simultaneous-equation models and show the bias involved in the direct application of the least-squares method to such models. After discussing the so-called identification problem in Chapter 19, in Chapter 20 we discuss some of the special methods developed to handle the simultaneous-equation models.

18.2 EXAMPLES OF SIMULTANEOUS-EQUATION MODELS

EXAMPLE 18.1

DEMAND-AND-SUPPLY MODEL

As is well known, the price P of a commodity and the quantity Q sold are determined by the intersection of the demand-and-supply curves for that commodity. Thus, assuming for simplicity that the demand-and-supply curves are linear and adding the stochastic disturbance terms u_1 and u_2, we may write the empirical demand-and-supply functions as

Demand function: $$Q_t^d = \alpha_0 + \alpha_1 P_t + u_{1t} \qquad \alpha_1 < 0 \quad \textbf{(18.2.1)}$$

Supply function: $$Q_t^s = \beta_0 + \beta_1 P_t + u_{2t} \qquad \beta_1 > 0 \quad \textbf{(18.2.2)}$$

Equilibrium condition: $$Q_t^d = Q_t^s$$

(*Continued*)

[2]These economical but self-explanatory notations will be generalized to more than two equations in Chap. 19.

EXAMPLE 18.1 (*Continued*)

where Q^d = quantity demanded
Q^s = quantity supplied
t = time

and the α's and β's are the parameters. A priori, α_1 is expected to be negative (downward-sloping demand curve), and β_1 is expected to be positive (upward-sloping supply curve).

Now it is not too difficult to see that P and Q are jointly dependent variables. If, for example, u_{1t} in (18.2.1) changes because of changes in other variables affecting Q_t^d (such as income, wealth, and tastes), the demand curve will shift upward if u_{1t} is positive and downward if u_{1t} is negative. These shifts are shown in Figure 18.1.

As the figure shows, a shift in the demand curve changes both P and Q. Similarly, a change in u_{2t} (because of strikes, weather, import or export restrictions, etc.) will shift the supply curve, again affecting both P and Q. Because of this simultaneous dependence between Q and P, u_{1t} and P_t in (18.2.1) and u_{2t} and P_t in (18.2.2) cannot be independent. Therefore, a regression of Q on P as in (18.2.1) would violate an important assumption of the classical linear regression model, namely, the assumption of no correlation between the explanatory variable(s) and the disturbance term.

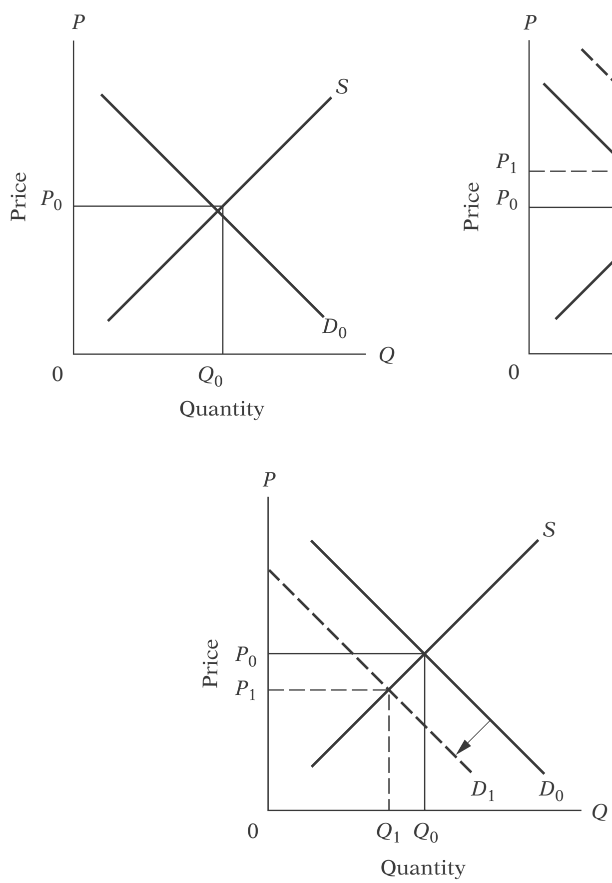

FIGURE 18.1 Interdependence of price and quantity.

EXAMPLE 18.2

KEYNESIAN MODEL OF INCOME DETERMINATION

Consider the simple Keynesian model of income determination:

$$\textit{Consumption function:} \quad C_t = \beta_0 + \beta_1 Y_t + u_t \qquad 0 < \beta_1 < 1 \tag{18.2.3}$$

$$\textit{Income identity:} \quad Y_t = C_t + I_t\,(= S_t) \tag{18.2.4}$$

where
$C =$ consumption expenditure
$Y =$ income
$I =$ investment (assumed exogenous)
$S =$ savings
$t =$ time
$u =$ stochastic disturbance term
β_0 and $\beta_1 =$ parameters

The parameter β_1 is known as the *marginal propensity to consume* (MPC) (the amount of extra consumption expenditure resulting from an extra dollar of income). From economic theory, β_1 is expected to lie between 0 and 1. Equation (18.2.3) is the (stochastic) consumption function; and (18.2.4) is the national income identity, signifying that total income is equal to total consumption expenditure plus total investment expenditure, it being understood that total investment expenditure is equal to total savings. Diagrammatically, we have Figure 18.2.

From the postulated consumption function and Figure 18.2 it is clear that C and Y are interdependent and that Y_t in (18.2.3) is not expected to be independent of the disturbance term because when u_t shifts (because of a variety of factors subsumed in the error term), then the consumption function also shifts, which, in turn, affects Y_t. Therefore, once again the classical least-squares method is inapplicable to (18.2.3). If applied, the estimators thus obtained will be inconsistent, as we shall show later.

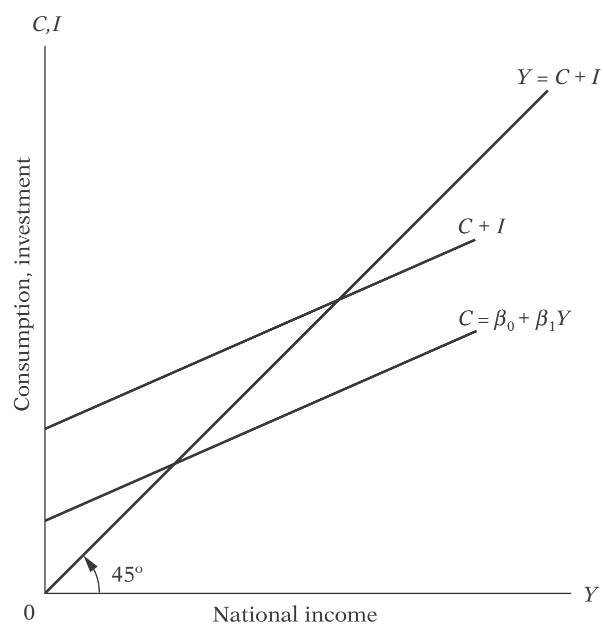

FIGURE 18.2 Keynesian model of income determination.

EXAMPLE 18.3

WAGE–PRICE MODELS

Consider the following Phillips-type model of money-wage and price determination:

$$\dot{W}_t = \alpha_0 + \alpha_1 \text{UN}_t + \alpha_2 \dot{P}_t + u_{1t} \quad \textbf{(18.2.5)}$$

$$\dot{P}_t = \beta_0 + \beta_1 \dot{W}_t + \beta_2 \dot{R}_t + \beta_3 \dot{M}_t + u_{2t} \quad \textbf{(18.2.6)}$$

where $\dot{W}$ = rate of change of money wages
UN = unemployment rate, %
$\dot{P}$ = rate of change of prices
$\dot{R}$ = rate of change of cost of capital
$\dot{M}$ = rate of change of price of imported raw material
t = time
u_1, u_2 = stochastic disturbances

Since the price variable $\dot{P}$ enters into the wage equation and the wage variable $\dot{W}$ enters into the price equation, the two variables are jointly dependent. Therefore, these stochastic explanatory variables are expected to be correlated with the relevant stochastic disturbances, once again rendering the classical OLS method inapplicable to estimate the parameters of the two equations individually.

EXAMPLE 18.4

THE IS MODEL OF MACROECONOMICS

The celebrated IS, or goods market equilibrium, model of macroeconomics[3] in its nonstochastic form can be expressed as

Consumption function:	$C_t = \beta_0 + \beta_1 Y_{dt}$	$0 < \beta_1 < 1$	**(18.2.7)**
Tax function:	$T_t = \alpha_0 + \alpha_1 Y_t$	$0 < \alpha_1 < 1$	**(18.2.8)**
Investment function:	$I_t = \gamma_0 + \gamma_1 r_t$		**(18.2.9)**
Definition:	$Y_{dt} = Y_t - T_t$		**(18.2.10)**
Government expenditure:	$G_t = \bar{G}$		**(18.2.11)**
National income identity:	$Y_t = C_t + I_t + G_t$		**(18.2.12)**

where Y = national income
C = consumption spending
I = planned or desired net investment
$\bar{G}$ = given level of government expenditure
T = taxes
Y_d = disposable income
r = interest rate

(*Continued*)

[3]"The goods market equilibrium schedule, or IS schedule, shows combinations of interest rates and levels of output such that planned spending equals income." See Rudiger Dornbusch and Stanley Fischer, *Macroeconomics,* 3d ed., McGraw-Hill, New York, 1984, p. 102. Note that for simplicity we have assumed away the foreign trade sector.

EXAMPLE 18.4 (*Continued*)

If you substitute (18.2.10) and (18.2.8) into (18.2.7) and substitute the resulting equation for *C* and Eq. (18.2.9) and (18.2.11) into (18.2.12), you should obtain

$$\text{IS equation:} \quad Y_t = \pi_0 + \pi_1 r_t \tag{18.2.13}$$

where

$$\pi_0 = \frac{\beta_0 - \alpha_0\beta_1 + \gamma_0 + \bar{G}}{1 - \beta_1(1 - \alpha_1)}$$
$$\pi_1 = \frac{1}{1 - \beta_1(1 - \alpha_1)} \tag{18.2.14}$$

Equation (18.2.13) is the equation of the IS, or goods market equilibrium, that is, it gives the combinations of the interest rate and level of income such that the goods market clears or is in equilibrium. Geometrically, the IS curve is shown in Figure 18.3.

What would happen if we were to estimate, say, the consumption function (18.2.7) in isolation? Could we obtain unbiased and/or consistent estimates of β_0 and β_1? Such a result is unlikely because consumption depends on disposable income, which depends on national income *Y*, but the latter depends on *r* and $\bar{G}$ as well as the other parameters entering in π_0. Therefore, unless we take into account all these influences, a simple regression of *C* on Y_d is bound to give biased and/or inconsistent estimates of β_0 and β_1.

FIGURE 18.3 The IS curve.

EXAMPLE 18.5

THE LM MODEL

The other half of the famous IS-LM paradigm is the LM, or money market equilibrium, relation, which gives the combinations of the interest rate and level of income such that the money market is cleared, that is, the demand for money is equal to its supply. Algebraically, the model, in the nonstochastic form, may be expressed as:

$$\text{Money demand function:} \quad M_t^d = a + bY_t - cr_t \tag{18.2.15}$$

$$\text{Money supply function:} \quad M_t^s = \bar{M} \tag{18.2.16}$$

$$\text{Equilibrium condition:} \quad M_t^d = M_t^s \tag{18.2.17}$$

(*Continued*)

EXAMPLE 18.5 (*Continued*)

where $Y =$ income, $r =$ interest rate, and $\bar{M} =$ assumed level of money supply, say, determined by the Fed.

Equating the money demand and supply functions and simplifying, we obtain:

$$\textit{LM equation:} \qquad Y_t = \lambda_0 + \lambda_1 \bar{M} + \lambda_2 r_t \tag{18.2.18}$$

where

$$\begin{aligned} \lambda_0 &= -a/b \\ \lambda_1 &= 1/b \\ \lambda_2 &= c/b \end{aligned} \tag{18.2.19}$$

For a given $M = \bar{M}$, the LM curve representing the relation (18.2.18) is as shown in Figure 18.4.

The IS and LM curves show, respectively, that a whole array of interest rates is consistent with goods market equilibrium and a whole array of interest rates is compatible with equilibrium in the money market. Of course, only one interest rate and one level of income will be consistent simultaneously with the two equilibria. To obtain these, all that needs to be done is to equate (18.2.13) and (18.2.18). In exercise 18.4 you are asked to show the level of the interest rate and income that is simultaneously compatible with the goods and money market equilibrium.

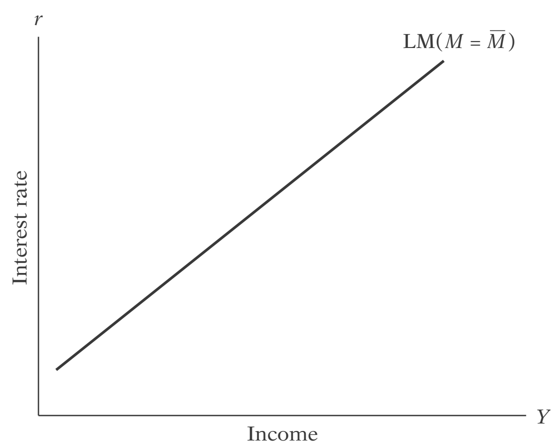

FIGURE 18.4 The LM curve.

EXAMPLE 18.6

ECONOMETRIC MODELS

An extensive use of simultaneous-equation models has been made in the econometric models built by several econometricians. An early pioneer in this field was Professor Lawrence Klein of the Wharton School of the University of Pennsylvania. His initial model, known as **Klein's model I,** is as follows:

$$\textit{Consumption function:} \qquad C_t = \beta_0 + \beta_1 P_t + \beta_2 (W + W')_t + \beta_3 P_{t-1} + u_{1t}$$

(*Continued*)

EXAMPLE 18.6 (*Continued*)

$$\text{Investment function:} \quad I_t = \beta_4 + \beta_5 P_t + \beta_6 P_{t-1} + \beta_7 K_{t-1} + u_{2t}$$

$$\begin{aligned}\text{Demand for labor:} \quad W_t &= \beta_8 + \beta_9 (Y + T - W')_t \\ &+ \beta_{10}(Y + T - W')_{t-1} + \beta_{11} t + u_{3t}\end{aligned} \tag{18.2.20}$$

$$\text{Identity:} \quad Y_t + T_t = C_t + I_t + G_t$$

$$\text{Identity:} \quad Y_t = W'_t + W_t + P_t$$

$$\text{Identity:} \quad K_t = K_{t-1} + I_t$$

where
C = consumption expenditure
I = investment expenditure
G = government expenditure
P = profits
W = private wage bill
W' = government wage bill
K = capital stock
T = taxes
Y = income after tax
t = time
u_1, u_2, and u_3 = stochastic disturbances[4]

In the preceding model the variables *C, I, W, Y, P,* and *K* are treated as jointly dependent, or endogenous, variables and the variables P_{t-1}, K_{t-1}, and Y_{t-1} are treated as predetermined.[5] In all, there are six equations (including the three identities) to study the interdependence of six endogenous variables.

In Chapter 20 we shall see how such econometric models are estimated. For the time being, note that because of the interdependence among the endogenous variables, in general they are not independent of the stochastic disturbance terms, which therefore makes it inappropriate to apply the method of OLS to an individual equation in the system. As shown in Section 18.3, the estimators thus obtained are inconsistent; they do not converge to their true population values even when the sample size is very large.

18.3 THE SIMULTANEOUS-EQUATION BIAS: INCONSISTENCY OF OLS ESTIMATORS

As stated previously, the method of least squares may not be applied to estimate a single equation embedded in a system of simultaneous equations if one or more of the explanatory variables are correlated with the disturbance term in that equation because the estimators thus obtained are inconsistent. To show this, let us revert to the simple Keynesian model of income determination given in Example 18.2. Suppose that we want to

[4]L. R. Klein, *Economic Fluctuations in the United States, 1921–1941,* John Wiley & Sons, New York, 1950.

[5]The model builder will have to specify which of the variables in a model are endogenous and which are predetermined. K_{t-1} and Y_{t-1} are predetermined because at time t their values are known. (More on this in Chap. 19.)

estimate the parameters of the consumption function (18.2.3). Assuming that $E(u_t) = 0$, $E(u_t^2) = \sigma^2$, $E(u_t u_{t+j}) = 0$ (for $j \neq 0$), and $\text{cov}(I_t, u_t) = 0$, which are the assumptions of the classical linear regression model, we first show that Y_t and u_t in (18.2.3) are correlated and then prove that $\hat{\beta}_1$ is an inconsistent estimator of β_1.

To prove that Y_t and u_t are correlated, we proceed as follows. Substitute (18.2.3) into (18.2.4) to obtain

$$Y_t = \beta_0 + \beta_1 Y_t + u_t + I_t$$

that is,

$$Y_t = \frac{\beta_0}{1-\beta_1} + \frac{1}{1-\beta_1} I_t + \frac{1}{1-\beta_1} u_t \tag{18.3.1}$$

Now

$$E(Y_t) = \frac{\beta_0}{1-\beta_1} + \frac{1}{1-\beta_1} I_t \tag{18.3.2}$$

where use is made of the fact that $E(u_t) = 0$ and that I_t being exogenous, or predetermined (because it is fixed in advance), has as its expected value I_t.

Therefore, subtracting (18.3.2) from (18.3.1) results in

$$Y_t - E(Y_t) = \frac{u_t}{1-\beta_1} \tag{18.3.3}$$

Moreover,

$$u_t - E(u_t) = u_t \qquad \text{(Why?)} \tag{18.3.4}$$

whence

$$\begin{aligned} \text{cov}(Y_t, u_t) &= E[Y_t - E(Y_t)][u_t - E(u_t)] \\ &= \frac{E(u_t^2)}{1-\beta_1} \qquad \text{from (18.3.3) and (18.3.4)} \\ &= \frac{\sigma^2}{1-\beta_1} \end{aligned} \tag{18.3.5}$$

Since σ^2 is positive by assumption (why?), the covariance between Y and u given in (18.3.5) is bound to be different from zero.[6] As a result, Y_t and u_t in (18.2.3) are expected to be correlated, which violates the assumption of the classical linear regression model that the disturbances are independent or at least uncorrelated with the explanatory variables. As noted previously, the OLS estimators in this situation are inconsistent.

[6]It will be greater than zero as long as β_1, the MPC, lies between 0 and 1, and it will be negative if β_1 is greater than unity. Of course, a value of MPC greater than unity would not make much economic sense. In reality therefore the covariance between Y_t and u_t is expected to be positive.

To show that the OLS estimator $\hat{\beta}_1$ is an inconsistent estimator of β_1 because of correlation between Y_t and u_t, we proceed as follows:

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum(C_t - \bar{C})(Y_t - \bar{Y})}{\sum(Y_t - \bar{Y})^2} \\ &= \frac{\sum c_t y_t}{\sum y_t^2} \\ &= \frac{\sum C_t y_t}{\sum y_t^2}\end{aligned} \tag{18.3.6}$$

where the lowercase letters, as usual, indicate deviations from the (sample) mean values. Substituting for C_t from (18.2.3), we obtain

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum(\beta_0 + \beta_1 Y_t + u_t)y_t}{\sum y_t^2} \\ &= \beta_1 + \frac{\sum y_t u_t}{\sum y_t^2}\end{aligned} \tag{18.3.7}$$

where in the last step use is made of the fact that $\sum y_t = 0$ and $(\sum Y_t y_t / \sum y_t^2) = 1$ (why?).

If we take the expectation of (18.3.7) on both sides, we obtain

$$E(\hat{\beta}_1) = \beta_1 + E\left[\frac{\sum y_t u_t}{\sum y_t^2}\right] \tag{18.3.8}$$

Unfortunately, we cannot evaluate $E(\sum y_t u_t / \sum y_t^2)$ since the expectations operator is a linear operator. [*Note:* $E(A/B) \neq E(A)/E(B)$.] But intuitively it should be clear that unless the term $(\sum y_t u_t / \sum y_t^2)$ is zero, $\hat{\beta}_1$ is a biased estimator of β_1. But have we not shown in (18.3.5) that the covariance between Y and u is nonzero and therefore would $\hat{\beta}_1$ not be biased? The answer is, not quite, since $\text{cov}(Y_t, u_t)$, a population concept, is not quite $\sum y_t u_t$, which is a sample measure, although as the sample size increases indefinitely the latter will tend toward the former. But if the sample size increases indefinitely, then we can resort to the concept of consistent estimator and find out what happens to $\hat{\beta}_1$ as n, the sample size, increases indefinitely. In short, when we cannot explicitly evaluate the expected value of an estimator, as in (18.3.8), we can turn our attention to its behavior in the large sample.

Now an estimator is said to be consistent if its **probability limit,**[7] or **plim** for short, is equal to its true (population) value. Therefore, to show that $\hat{\beta}_1$ of (18.3.7) is inconsistent, we must show that its plim is not equal to the true β_1.

[7]See **App. A** for the definition of probability limit.

Applying the rules of probability limit to (18.3.7), we obtain[8]

$$\begin{aligned}\operatorname{plim}(\hat{\beta}_1) &= \operatorname{plim}(\beta_1) + \operatorname{plim}\left(\frac{\sum y_t u_t}{\sum y_t^2}\right)\\ &= \operatorname{plim}(\beta_1) + \operatorname{plim}\left(\frac{\sum y_t u_t/n}{\sum y_t^2/n}\right)\\ &= \beta_1 + \frac{\operatorname{plim}\left(\sum y_t u_t/n\right)}{\operatorname{plim}\left(\sum y_t^2/n\right)}\end{aligned} \tag{18.3.9}$$

where in the second step we have divided $\sum y_t u_t$ and $\sum y_t^2$ by the total number of observations in the sample n so that the quantities in the parentheses are now the sample covariance between Y and u and the sample variance of Y, respectively.

In words, (18.3.9) states that the probability limit of $\hat{\beta}_1$ is equal to true β_1 plus the ratio of the plim of the sample covariance between Y and u to the plim of the sample variance of Y. Now as the sample size n increases indefinitely, one would expect the sample covariance between Y and u to approximate the true population covariance $E[Y_t - E(Y_t)][u_t - E(u_t)]$, which from (18.3.5) is equal to $[\sigma^2/(1-\beta_1)]$. Similarly, as n tends to infinity, the sample variance of Y will approximate its population variance, say σ_Y^2. Therefore, Eq. (18.3.9) may be written as

$$\begin{aligned}\operatorname{plim}(\hat{\beta}_1) &= \beta_1 + \frac{\sigma^2/(1-\beta_1)}{\sigma_Y^2}\\ &= \beta_1 + \frac{1}{1-\beta_1}\left(\frac{\sigma^2}{\sigma_Y^2}\right)\end{aligned} \tag{18.3.10}$$

Given that $0 < \beta_1 < 1$ and that σ^2 and σ_Y^2 are both positive, it is obvious from Eq. (18.3.10) that plim $(\hat{\beta}_1)$ will always be greater than β_1; that is, $\hat{\beta}_1$ will overestimate the true β_1.[9] In other words, $\hat{\beta}_1$ is a biased estimator, and the bias will not disappear no matter how large the sample size.

18.4 THE SIMULTANEOUS-EQUATION BIAS: A NUMERICAL EXAMPLE

To demonstrate some of the points made in the preceding section, let us return to the simple Keynesian model of income determination given in Example 18.2 and carry out the following **Monte Carlo** study.[10] Assume that

[8]As stated in **App. A**, the plim of a constant (for example, β_1) is the same constant and the plim of $(A/B) = \operatorname{plim}(A)/\operatorname{plim}(B)$. Note, however, that $E(A/B) \neq E(A)/E(B)$.

[9]In general, however, the direction of the bias will depend on the structure of the particular model and the true values of the regression coefficients.

[10]This is borrowed from Kenneth J. White, Nancy G. Horsman, and Justin B. Wyatt, *SHAZAM: Computer Handbook for Econometrics for Use with Basic Econometrics,* McGraw-Hill, New York, 1985, pp. 131–134.

TABLE 18.1

Y_t (1)	C_t (2)	I_t (3)	u_t (4)
18.15697	16.15697	2.0	−0.3686055
19.59980	17.59980	2.0	−0.8004084E-01
21.93468	19.73468	2.2	0.1869357
21.55145	19.35145	2.2	0.1102906
21.88427	19.48427	2.4	−0.2314535E-01
22.42648	20.02648	2.4	0.8529544E-01
25.40940	22.80940	2.6	0.4818807
22.69523	20.09523	2.6	−0.6095481E-01
24.36465	21.56465	2.8	0.7292983E-01
24.39334	21.59334	2.8	0.7866819E-01
24.09215	21.09215	3.0	−0.1815703
24.87450	21.87450	3.0	−0.2509900E-01
25.31580	22.11580	3.2	−0.1368398
26.30465	23.10465	3.2	0.6092946E-01
25.78235	22.38235	3.4	−0.2435298
26.08018	22.68018	3.4	−0.1839638
27.24440	23.64440	3.6	−0.1511200
28.00963	24.40963	3.6	0.1926739E-02
30.89301	27.09301	3.8	0.3786015
28.98706	25.18706	3.8	−0.2588852E-02

Source: Kenneth J. White, Nancy G. Horsman, and Justin B. Wyatt, *SHAZAM Computer Handbook for Econometrics for Use with Damodar Gujarati: Basic Econometrics,* September 1985, p. 132.

the values of investment I are as shown in column 3 of Table 18.1. Further assume that

$$E(u_t) = 0$$
$$E(u_t u_{t+j}) = 0 \qquad (j \neq 0)$$
$$\text{var}(u_t) = \sigma^2 = 0.04$$
$$\text{cov}(u_t, I_t) = 0$$

The u_t thus generated are shown in column (4).

For the consumption function (18.2.3) assume that the values of the true parameters are known and are $\beta_0 = 2$ and $\beta_1 = 0.8$.

From the assumed values of β_0 and β_1 and the generated values of u_t we can generate the values of income Y_t from (18.3.1), which are shown in column 1 of Table 18.1. Once Y_t are known, and knowing β_0, β_1, and u_t, one can easily generate the values of consumption C_t from (18.2.3). The C's thus generated are given in column 2.

Since the true β_0 and β_1 are known, and since our sample errors are exactly the same as the "true" errors (because of the way we designed the Monte Carlo study), if we use the data of Table 18.1 to regress C_t on Y_t we should obtain $\beta_0 = 2$ and $\beta_1 = 0.8$, if OLS were unbiased. But from (18.3.7) we know that this will not be the case if the regressor Y_t and the disturbance u_t are correlated. Now it is not too difficult to verify from our data that the (sample) covariance between Y_t and u_t is $\sum y_t u_t = 3.8$ and that $\sum y_t^2 = 184$.

Then, as (18.3.7) shows, we should have

$$\begin{aligned}\hat{\beta}_1 &= \beta_1 + \frac{\sum y_t u_t}{\sum y_t^2} \\ &= 0.8 + \frac{3.8}{184} \\ &= 0.82065\end{aligned} \tag{18.4.1}$$

That is, $\hat{\beta}_1$ is upward-biased by 0.02065.

Now let us regress C_t on Y_t, using the data given in Table 18.1. The regression results are

$$\begin{aligned}\hat{C}_t &= \ 1.4940 \quad + \quad 0.82065Y_t \\ \text{se} &= (0.35413) \qquad (0.01434) \\ t &= (4.2188) \qquad (57.209) \qquad R^2 = 0.9945\end{aligned} \tag{18.4.2}$$

As expected, the estimated β_1 is precisely the one predicted by (18.4.1). In passing, note that the estimated β_0 too is biased.

In general the amount of the bias in $\hat{\beta}_1$ depends on β_1, σ^2 and var (Y) and, in particular, on the degree of covariance between Y and u.[11] As Kenneth White et al. note, "This is what simultaneous equation bias is all about. In contrast to single equation models, we can no longer assume that variables on the right hand side of the equation are uncorrelated with the error term."[12] Bear in mind that this bias remains even in large samples.

In view of the potentially serious consequences of applying OLS in simultaneous-equation models, is there a test of simultaneity that can tell us whether in a given instance we have the simultaneity problem? One version of the **Hausman specification test** can be used for this purpose, which we discuss in Chapter 19.

18.5 SUMMARY AND CONCLUSIONS

1. In contrast to single-equation models, in simultaneous-equation models more than one dependent, or **endogenous,** variable is involved, necessitating as many equations as the number of endogenous variables.

2. A unique feature of simultaneous-equation models is that the endogenous variable (i.e., regressand) in one equation may appear as an explanatory variable (i.e., regressor) in another equation of the system.

3. As a consequence, such an **endogenous explanatory variable** becomes stochastic and is usually correlated with the disturbance term of the equation in which it appears as an explanatory variable.

4. In this situation the classical OLS method may not be applied because the estimators thus obtained are not consistent, that is, they do not converge to their true population values no matter how large the sample size.

[11]See Eq. (18.3.5).
[12]Op. cit., pp. 133–134.