3 ENUMERATION

Objectives

After studying this chapter you should

- understand the basic principles of enumeration;
- know and be able to use the $\binom{n}{r}$ notation;
- be able to apply this notation to the solution of various problems.

3.0 Introduction

Enumeration is really just a fancy name for counting, and the aim of this chapter is to teach you how to count! But that's not quite as silly as it may seem, since enumeration is concerned particularly with counting the number of ways in which something can be done - often, the number of ways in which a particular choice can be made.

There is an important linguistic convention here. When a mathematician asks (for example) how many ways there are of choosing three students from a class of 21, she doesn't want a list like

- choose the three oldest
- choose the first three on the register
- use random selection
- let them fight for it

and so on. This might be something to discuss, but is not enumeration.

The question "How many ways?" in this area of mathematics is understood to refer not to the mechanism of choosing but to the number of different results. Thus the choices could be

- Mary, John and Jasmine, or
- Mary, Lloyd and Martin, or
- Trevor, Mary and Desmond,

or any of 1327 other possibilities. You will see shortly how this figure can be calculated.

One final point before you begin to study this topic. We have chosen to use the title **Enumeration** to describe it, but this name is not universal. Some textbooks use the label **combinatorics** to describe enumeration (or often to describe enumeration and graph theory taken together), while many others use the older title **permutations** and **combinations**. If you are using other books alongside this one (as you should, from time to time), you need to be familiar with this variation.

3.1 The multiplicative principle

A restaurant offers a **Special Business Lunch menu** as shown opposite for the price of £5.95.

Special Business Lunch

Fruit Juice
or
Soup of the Day

Chicken Curry with Rice
or
Pork Chow Mein
or
Liver with Onions and Mashed Potatoes

Fruit Salad and Cream
or
Ice Cream
or
Cheese and Biscuits

£5.95

How many different meals could be served, each chosen from this menu?

You might start by trying to list all the meals:

Soup, Curry, Fruit Salad
Soup, Liver, Cheese
Juice, Curry, Ice Cream
Soup, Chow Mein, Cheese ....

That is clearly not going to be very reliable, because if the meals are listed in this almost random order it is very difficult to know whether or not you have included every possibility exactly once.

So a better list might be in some logical order:

Juice, Curry, Fruit Salad
Juice, Curry, Ice Cream
Juice, Curry, Cheese
Juice, Chow Mein, Fruit Salad
....
Juice, Liver, Cheese
Soup, Curry, Fruit Salad
....
Soup, Liver, Cheese

What meals will fill the gaps? How many meals are there altogether?

What you will find if you write out the full list - you may be able to see it already in your imagination - is that because there are three 'desserts' there will be three meals for each combination of starter and main course; and because there are three main courses there will be three such combinations for each starter; and there are two starters.

So the total number of different meals is

2 (starters) $\times$ 3 (main courses) $\times$ 3 (desserts) = 18.

This is an example of what is often called the **multiplicative principle.** Where several independent choices have to be made one after another, and there are *r* options for the first choice, *s* options for the second choice, *t* options for the third choice, and so on, the total number of possibilities is

$$r \times s \times t \times \ldots$$

Example

Sandra has six different sweaters and five different pairs of trousers. How many different combinations of these can she wear?

Solution

Assuming Sandra chooses sweater and trousers independently, she has $6 \times 5 = 30$ possibilities. In reality there are probably some particular combinations (the pink sweater with the orange trousers, for example) that she wouldn't be seen dead in! But even that can be taken into account as long as the limitations are spelled out in advance.

Example

In a tennis club there are 16 men (three of whom are County players) and 23 women (including six County players). In how many ways can the club select a mixed doubles team for a competition, if it must not include more than one County player?

Solution

First of all, ignore the restriction. The club must choose one of 16 men, then one of 23 women, so there are $16 \times 23 = 368$ possible pairs. But some of these pairs contain two County players, which is illegal. In fact there are three male and six female County players, so there are $3 \times 6 = 18$ pairs in which both players are County players.

So there are $368 - 18 = 350$ legal mixed doubles teams the club can choose.

Exercise 3A

1. A group of friends go into a cafe together. Each of them orders either tea or coffee, with or without milk, with no sugar, one sugar or two sugars. If no two of them have exactly the same, how many friends could there be?
2. How many different six-figure telephone numbers are possible, if the first figure cannot be 0, 1 or 9?
3. An advertisement for a computer printer claims that it offers more than 100 different fonts. When you read the small print, you find that each combination of style, emphasis and size has been counted separately. If each letter can have any of four emphases (normal, italic, bold and italic bold) and any of three sizes (6 cpi, 12 cpi and 24 cpi), how many styles must there be?
4. An examination paper is divided into three sections. Each section contains six questions, three of which are starred. The rubric says "Answer only one question from each section. Answer at least one starred question and at least one unstarred question." How many different choices are possible?
5. The 37 members of a club have to elect from among themselves a chairman, a secretary and a treasurer. Assuming no one can be elected to more than one of these positions, how many different results are possible?

3.2 Arrangements

Holy Trinity Church in Seaton Carew has four bells, which are rung before the main service every Sunday morning. Traditionally, English bellringers don't actually play tunes on the bells; instead, they 'play maths' by trying to ring the bells in every possible order.

How many possible orders are there for four bells, each rung once? This problem can be solved in a similar way to the last question of the exercise above. There are four choices for the bell that comes first, then three choices for the bell that comes second, then two choices for the bell that comes third, then one choice (!) for the bell that comes fourth. So by the multiplicative principle, there are $4\times3\times2\times1=24$ possible orders altogether.

Activity 1

Make a list of the 24 possible orders.

Then try to arrange these orders one after another so that no bell changes more than one place forward or back between one order and the next. That is,

1 3 4 2

1 4 3 2

is all right, because bells 3 and 4 each change just one place, while bells 1 and 2 do not move at all.

1 3 4 2

1 4 2 3

is not allowed, because bell 3 changes two places back.

The same basic method can be used to deal with other arrangements. If there are ten boats in a race, for example, the number of different orders in which they can finish (assuming dead heats are not allowed) is

$$10\times9\times8\times7\times6\times5\times4\times3\times2\times1=3\ 628\ 800.$$

Writing the multiplication out in full like this every time can obviously be very boring, and it is customary to adopt a mathematical shorthand:

$N!$ (spoken as 'N factorial')

$$= N\times(N-1)\times(N-2)\times(N-3)\times\ \ldots\ \times2\times1.$$

The table below shows the value of N! for values of N between 1 and 20.

1! =	1	11! =	39 916 800
2! =	2	12! =	479 001 600
3! =	6	13! =	6 227 020 800
4! =	24	14! =	87 178 291 200
5! =	120	15! =	1 307 674 368 000
6! =	720	16! =	20 922 789 888 000
7! =	5 040	17! =	355 687 428 096 000
8! =	40 320	18! =	6 402 373 705 728 000
9! =	362 880	19! =	121 645 100 408 832 000
10! =	3 628 800	20! =	2 432 902 008 176 640 000

Does 0! have a meaning? If not, why does your calculator give it a value?

Activity 2

Which grows faster as x increases: 10^x, or $x!$?

Don't jump to conclusions from just a few small-number values - see what happens when x gets quite large.

Activity 3

Calculating $x!$ for large values of x is quite time-consuming: even a pocket calculator gives up at 70! in most cases. But there is a formula called Stirling's formula which gives an approximate value of $x!$ for large x:

$$x! \approx \sqrt{(2\pi x)} \times \left(\frac{x}{e}\right)^x$$

where e ($\approx$ 2.718 ...) is the base of natural logarithms. Use the formula to calculate approximations to 20!, 50! and 100!, and compare them with the most accurate answers you can find elsewhere.

Using this factorial notation, it is easy to state a general rule for the number of possible arrangements of a finite set:

N different objects can be arranged in order in N! ways.

Example

If six people have to share the six seats in a railway compartment, in how many ways can they sit?

Solution

Assuming no one sits on anyone else's lap, this is just a problem of arranging six different objects. This can be done in

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720 \text{ ways.}$$

Example

How many nine-letter words (which needn't actually make sense) can be made from the letters of SPREADING?

Solution

Nine different objects can be arranged in 9! = 362 880 orders.

Example

How many of these 'words' have all three vowels together?

Solution

Imagine the three vowels locked together as a single unit, called V, say. Then the problem is to arrange SPRDNGV, and this can be done in 7! ways. But the three vowels can be arranged in 3! ways among themselves, so altogether there are $7! \times 3! = 30\ 240$ possible arrangements.

If the objects are not all different, of course, the problem becomes more complicated.

Suppose the last example had asked for words formed from the letters of TOTTERING - what would the answer have been then?

There are still nine letters, but they are not all different. There are three identical Ts, and there will be fewer possibilities for rearrangement.

Suppose the Ts are made different by giving them labels - call them T_1, T_2 and T_3, say.

Now there are nine different letters once again, which can be formed into 362 880 words just as in the previous case. But look at some of these:

$T_1OT_2T_3ERING$	$T_1OT_3T_2ERING$	$T_2OT_1T_3ERING$
$T_2OT_3T_1ERING$	$T_3OT_1T_2ERING$	$T_3OT_2T_1ERING$

There are six apparent words here, each of them giving the same word TOTTERING once the labels are removed, so the arrangement TOTTERING has been counted six times among the 362 880. And this will be true for any other arrangement too, because for every arrangement the three Ts can be rearranged among themselves in $3! = 6$ ways.

So if every arrangement has been counted six times, the real number of words that can be formed from TOTTERING is

$$9! \div 3! = 362\ 880 \div 6 = 60\ 480$$

This idea can be extended to cases where there is more than one repetition, leading to the general rule:

If there are N objects, of which R are the same in one way, S the same in another way, and so on, the number of different arrangements is

$$\frac{N!}{R! \times S! \times \ldots}.$$

Example

In how many ways can the letters of MISSISSIPPI be arranged?

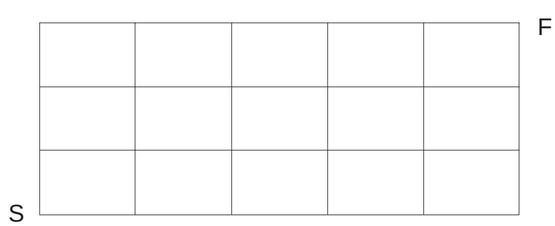
Solution

There are 11 letters here, including 4 Is, 4 Ss and 2 Ps. The total number of arrangements is therefore

$$11! \div (4! \times 4! \times 2!) = 39\ 916\ 800 \div 1152 = 34\ 650$$

Example

How many ways are there of getting from S to F in the diagram, by moving 'north' or 'east' only?



Solution

The journey from S to F involves five eastward moves and three northward moves, and so is equivalent to an arrangement of EEEEENNN. The number of such arrangements is

$8! \div (5! \times 3!) = 56$, so there are 56 possible routes.

Exercise 3B

(Answers greater than 10^6 may be given correct to three significant figures.)

1. Eight chess players take part in a league. In how many possible orders could they finish?
2. If there are eighteen cricket teams in the County Championship, in how many possible orders could they finish the season?
3. In how many different ways can the thirteen cards in a hand be arranged?
4. Three red discs, two yellow discs, one blue disc and one green disc are to be put on a spike. In how many ways can this be done?
5. On a shelf in the Maths stockroom there are six copies of 'Pure Maths', four copies of 'Discrete Maths', two copies of 'Statistics', and one each of 'Mechanics', 'Further Pure Maths', and 'Further Statistics'. How many different arrangements of these books are possible?
6. How many words of eight letters can be formed from HOPELESS?
7. How many different five-digit numbers can be made from the digits 1, 2, 2, 3, 4? How many of these numbers are even?
8. In how many ways can the three medals be awarded in a race in which eight athletes take part?
9. Trevor, June, Melissa and Barry go to the cinema and sit in a row of four seats. How many possible seating arrangements are there if June and Barry refuse to sit next to one another?

*10. In how many ways can the letters of MATHEMATICS be arranged if no two vowels are adjacent?

Activity 4

When King Arthur and his twelve knights go in to dinner, in how many different ways can they sit around the Round Table?

How many different necklaces can be made by threading ten differently-coloured beads

(i) onto a chain with a large clasp, or

(ii) onto a piece of thread tied into a circle with a very small knot?

What assumptions do you make in answering each of these questions? Discuss your assumptions and your answers with other students.

3.3 Making choices

A Sixth Form 'General Studies' programme offers thirty different courses, and each student is expected to choose three of them. How many different choices are possible?

A student has 30 possible first choices, 29 second choices, and 28 third choices, giving 24 360 possibilities altogether. But the assumption here is that the order of choosing is unimportant, because the choice 'Law, Pottery, Hockey' gives the same three courses as 'Hockey, Pottery, Law'. Each combination of three courses has thus been counted six times $(3! = 6)$, and the true number of combinations is $24\ 360 \div 6 = 4060$.

More generally, if there are n different items from which you have to choose r, where the order of choosing does not matter, the number of possible combinations is

$$\boxed{n \times (n-1) \times (n-2) \times \ \dots \ \times (n-r+1) \div r!}$$

Example

In how many ways can you choose two cakes from a plate of eight?

Solution

$8 \times 7 \div 2 = 28$ ways.

Example

In a particular company, employees can choose any four weeks (not necessarily consecutive) for their annual holiday. Last year, no two employees chose exactly the same four weeks; how many employees could there be?

Solution

If 4 weeks are chosen from 52, there are
$52 \times 51 \times 50 \times 49 \div 4! = 270\ 725$ different combinations possible; there could be this many employees.

Activity 5

Working with other students (so that each of you need only calculate a few values), complete the following table to show the number of possible combinations of r objects chosen from n.

	$n = 2$	3	4	5	6
$r = 1$	?	?	4	?	?
2	1	?	?	10	?
3	—	1	?	?	20
4	—	—	1	?	?
5	—	—	—	1	?

Where have you seen the same numbers before, perhaps in a different orientation?

The number of ways of choosing r objects from n, where all the objects are different and the order of choosing is unimportant, is denoted by $\binom{n}{r}$ (spoken as "n choose r"), or in some older books by ${}^nC_r$ or ${}_nC_r$. Then

$$\boxed{\binom{n}{r} = n \times (n-1) \times \dots \times (n-r+1) \div r!}$$

This is the same $\binom{n}{r}$ you may already have met in connection with the binomial theorem (in pure mathematics) or the binomial distribution (in statistics); you may like to discuss with your teacher the reasons for the similarity. If you calculator has a button labelled ${}^nC_r$, find out how to use it and check the answers from the examples above and below. Note also that you can write

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Example

An environment group with 26 members has to choose three members to lobby their MP. How many possible delegations are there?

Solution

$$\binom{26}{3} = 2600 \text{ delegations.}$$

Example

In how many ways can ten basketball players be split into two teams of five?

Solution

$$\binom{10}{5} \div 2 = 126$$

The number of ways of choosing a single team is divided by 2 because choosing ABCDE gives the same two teams as choosing FGHIJ.

Example

In how many ways can a committee of four be chosen from eight men and nine women, if the committee must include at least one member of each sex?

Solution

There are 17 members, so $\binom{17}{4}$ possible committees altogether.

But $\binom{8}{4}$ of these committees are all male, and $\binom{9}{4}$ are all female, and neither of these is allowed. The number of valid choices is therefore $\binom{17}{4}-\binom{8}{4}-\binom{9}{4}=2380-70-126=2184$.

Exercise 3C

1. Evaluate the following:

   (a) $\binom{10}{3}$ (b) $\binom{10}{6}$

   (c) $\binom{18}{4}$ (d) $\binom{n}{n}$

   (e) $\binom{n}{0}$

2. In how many ways can Colleen choose three library books from a shelf of 22?

3. A cafeteria serves sausages, bacon, eggs, mushrooms, tomatoes, beans, hash browns and fried bread at breakfast time, and offers any five different items for £1.99. How many different breakfasts can be made up?

4. If there are 46 universities offering the course I want, in how many ways can I choose eight of them for my UCAS form?

5. To complete his pools coupon, Harry has to predict which 10 out of 58 football matches will end as 1-1 draws. In how many ways can he choose these ten matches?

6. In how many ways can a squad of eight be chosen from six sergeants and twelve other ranks, if the squad must include at least one sergeant?

7. From a squad of 16 players, a football team of 11 players must be chosen. In how many ways can this be done if only two of the squad can keep goal?

8. In how many ways can 16 athletes be divided into two equal teams for a tug of war?

9. Every week last term, Lucas was late to school on two of the five days, but never on the same two days. What is the longest the term could have been?

10. There are 77 applications for the 72 places on a coach trip. In how many ways can the 72 lucky applicants be chosen?

3.4 Further arrangements

Activity 6

(a) Verify by several examples that for $n \ge r \ge 0$, $\binom{n}{r}=\binom{n}{n-r}$.

By considering different choosing procedures, explain why this should be true.

(b) Verify by several examples that for $n \ge r \ge 1$,

$\binom{n}{r}=\binom{n-1}{r}+\binom{n-1}{r-1}$. By considering different choosing procedures, explain why this result is true.

As with arrangements, the idea of choice can be extended to cases where the objects are not all different, though the extension is not particularly simple. Suppose you are asked to find the number of different selections of three items chosen from three red, two white, two blue, one black and one yellow. There are three essentially different situations:

XXX three items all the same colour. There is only one way to get this pattern, by choosing three red items.

Total: 1 selection.

XXY two items the same colour and one different. There are three ways of getting two the same (red, white or blue), and whichever of these is chosen there are then four possibilities for the third item.

Total: $3 \times 4 = 12$ selections.

XYZ all three items different. This is just a matter of choosing three of the five possible colours, and there are $\binom{5}{3}$ ways to do this.

Total: 10 selections.

So altogether there are $1 + 12 + 10 = 23$ possible selections.

The same technique can be extended without too much effort to cases in which the objects chosen must be arranged as well. If you are asked for the number of possible arrangements of three coloured items chosen from those listed above, the same three situations must be considered.

XXX can be arranged in only 1 way.

1 selection $\times$ 1 arrangement = 1

XXY can be arranged in $\frac{3!}{2!}$ = 3

12 selections $\times$ 3 arrangements = 36

XYZ can be arranged in 3! = 6

10 selections $\times$ 6 arrangements = 60

Thus there are 97 possible arrangements of three items chosen from this collection.

The technique of considering different situations individually can be applied in most such cases, and a further example should help you consolidate your understanding.

Example

In how many ways can four letters be chosen from the word REMEMBRANCE and then arranged to make a word?

Solution

Remember that in this type of problem, the 'words' need not be real English words - any arrangement of four letters will do. There are seven different letters and four situations: no letter occurs four times, so XXXX is impossible.

E occurs three times, so XXXY offers $1\times 6=6$ choices. Each of these can be arranged in $\frac{4!}{3!}=4$ ways, so there are 24 possible arrangements of this form.

E, R and M each occur at least twice, so XXYY is possible in $\binom{3}{2}=3$ ways. These letters can be arranged in $\frac{4!}{(2!\ 2!)}=6$ ways, giving 18 arrangements altogether.

The pattern XXYZ can be achieved in $\binom{3}{1}\times\binom{6}{2}=45$ ways, each of which can be arranged in $4!\div 2!=12$ ways. There are thus 540 arrangements for this pattern.

Finally, XYZW, with four different letters, presents $\binom{7}{4}=35$ choices, each of which has $4!\ =24$ arrangements, giving 840 arrangements altogether.

Putting all these results together, there are 89 possible selections but 1422 possible arrangements of four letters taken from REMEMBRANCE.

Exercise 3D

1. In a bag there are two treacle toffees, two plain toffees, two nut toffees, two liquorice toffees, and two mint toffees. If I am invited to take three toffees, how many different choices are possible?
2. How many four-letter words are there in PHOTOGRAPHY?
3. Helen is playing a game involving coloured counters. In her hand she has five red counters, three green, two blue, two yellow and one black. How many different sets of four counters can Helen make up?
4. How many different patterns can Helen make from four of these counters arranged in a square?
5. Elmer is planning his homework schedule. He has five lots of homework to do, each lasting an hour, and four three-hour evenings in which to do them. In how many different ways could he allocate subjects to evenings, if the order within any given evening is not important?

3.5 Simple probability

The methods of the previous sections can be applied to simple (and not-so-simple) problems in probability. Where a situation has many possible outcomes, you may recall, all equally likely, the probability of 'success' (however that is defined) is given by

$$\text{Probability} = \frac{\text{number of successful outcomes}}{\text{total number of outcomes}}$$

If you want to find the probability of getting a prime number with one throw of a dodecahedral (12-faced) die, for example, you note that there are five prime-number outcomes and 12 possible outcomes altogether, so that the probability is 5/12. The extension of this idea to enumeration is not difficult, as the following examples show.

Example

From a class of 21 students, four have to be selected as stewards for Open Evening. If the selection is made at random, what is the probability that the four selected include both Lee and Geoffrey?

Solution

Total number of selections $= \binom{21}{4}$.

Selections including L and G $= \binom{19}{2}$

because all that remains is to choose the other two.

Probability of including L and G

$$= \binom{19}{2} \div \binom{21}{4} = 171 \div 5985 = \frac{1}{35}.$$

Example

If ten girls are arranged at random in a line, what is the probability that Susan is directly between Chantal and Yolanda?

Solution

Total possible arrangements = 10! If CSY is treated as one block, there are 8! arrangements with these three girls together, and the block itself can be either CSY or YSC. So the probability is

$$(2 \times 8!) \div 10! = \frac{1}{45}$$

Example

In the game of bridge, a Yarborough is a hand of thirteen cards in which there is no card higher than 9. What is the probability of being dealt such a hand? (Ace counts high.)

Solution

Total number of hands $=\binom{52}{13}$, because you receive 13 of the 52 cards and the order of receiving them is not important.

Number of Yarborough hands $=\binom{32}{13}$, because there are 32 cards ranked 9 and under (four each of 2, 3, 4, 5, 6, 7, 8 and 9).

Probability of Yarborough $=\binom{32}{13}\div\binom{52}{13}\approx 0.000\ 547$.

Can you solve this last problem by an alternative method, not involving enumeration?

Exercise 3E

1. If four students are chosen at random from a class of ten males and eight females, what is the probability that they are all male?
2. If eleven different cars are parked in a random order in adjacent parking bays, what is the probability that the Metro is next to the Fiesta?
3. Coming home from a ten-day Scout Camp, Pete announces that it rained on only three days. On this information alone, what is the probability that the first and last days were both dry?
4. In bridge, what is the probability that your 13-card hand includes all four aces?
5. Still in bridge, what is the probability that each of the four players receives a hand consisting of cards all of one suit? (Such an occurrence is reported in the press every couple of years.)

3.6 Subsets

"Don't take all those magazines," says Mother, "but you can have some of them if you want." Assuming there are nine magazines, all different, how many different selections of "some of them" are possible?

One way of solving this problem is to work step by step through the different numbers of magazines you might take.

With one magazine, there are $\binom{9}{1}=9$ selections.

With two magazines, there are $\binom{9}{2}=36$ selections.

With three magazines, there are $\binom{9}{3} = 84$ selections.

With eight magazines, there are $\binom{9}{8} = 9$ selections.

What are the missing values?

If you add all these answers together, you will find they add up to 510, so there are 510 ways of taking some (but not all) of the magazines.

But there is another simpler way to solve the problem. For each of the nine magazines in turn, you have a simple choices: take it or leave it - 2 possibilities. And the choice for each magazine is independent, so there are $2 \times 2 \times 2 \times \ \ldots\ \times 2 = 2^9$ possibilities altogether. But two of these apparent possibilities are not actually possible at all: you cannot take all nine magazines (because Mother told you not to), nor can you leave all nine (because then you would not even have 'some'). So in the end there are $2^9 - 2 = 510$ different selections of some but not all of nine magazines.

Another problem using a similar technique is to decide how many different amounts of money you could make from (say) five £1 coins, four 20p coins, three 5p coins and four 2p coins. Notice that the numbers and values of the coins have been carefully chosen so that the same total cannot be made in more than one way.

You can take 0, 1, 2, 3, 4 or 5 £1 coins	$\Rightarrow$	6 choices.
You can take 0, 1, 2, 3 or 4 20p coins	$\Rightarrow$	5 choices.
You can take 0, 1, 2 or 3 5p coins	$\Rightarrow$	4 choices.
You can take 0, 1, 2, 3 or 4 2p coins	$\Rightarrow$	5 choices.

So there are $6 \times 5 \times 4 \times 5 = 600$ different choices, giving 599 different amounts of money if £0.00 is excluded.

A third problem involving a twist on this method is to find the number of ways in which eight children can be split into two teams (not necessarily equal) for a game. As in Section 3.3, it is enough to choose one of the two teams, and there are $2^8 - 2$ ways to do this because the first team must include some but not all of the children. But this figure counts each split twice: once when team ABCDE is chosen, for example, and again when the choice is FGH. So the true number of allocations is

$$\left(2^8 - 2\right) \div 2 = 127.$$

Exercise 3F

1. How many non-empty subsets has the set {red, orange, yellow, green, blue, violet}?
2. How many different committees could be formed from a club with 25 members if no member may sit on more than one committee?
3. In how many ways can nine books be split into two piles?
4. How many factors has 720? (Hint: Write 720 as a product of prime factors, and use the method of the second example.)
5. How many words (of any length) can you make from the letters of CHEMIST?

3.7 The pigeonhole principle

The pigeonhole principle (also known as Dirichlet's box principle) was mentioned in Chapter 1, and can now be considered in slightly greater depth. It states, you will recall, that if n objects have to be placed in m pigeonholes, where $n > m$, there must be at least one pigeonhole with more than one object in it. An extended version similarly asserts that if $n > km$, there must be at least one pigeonhole with more than k objects in it.

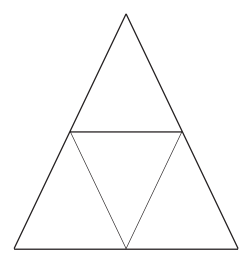
When the principle is applied to the solution of problems, the 'pigeonholes' are often mathematical ideas rather than real objects. The principle is most often used to prove assertions, as in the examples following.

Example

Prove that if five points are marked within an equilateral triangle of side 2 cm, two of them are within 1 cm of one another.

Solution

Imagine the triangle divided into four equal smaller triangles, as shown in the diagram. There are then five objects (points) to be allocated to four pigeonholes (small triangles), and by the pigeonhole principle there must be one triangle with more than one point in it. But the small triangles have side 1 cm, so any two points within such a triangle are within 1 cm of one another.



Example

Prove that among any group of six people, there are either three who all know one another or three who are all strangers to one another.

Solution

Suppose the people are A, B, C, D, E and F. Now A either knows or does not know each of the other five; if those remaining have to be put into two pigeonholes ('A knows' and 'A doesn't know') then one or other pigeonhole must contain at least three of B, C, D, E and F.

Suppose it is the 'A knows' pigeonhole, and that A knows B, C and D. Now either two of these know one another, in which case they make with A a set of three who all know one another (√ assertion proved) or none of them know one another, in which case B, C and D are a set of three mutual strangers (√).

Alternatively, if the 'A doesn't know' pigeonhole is the one with at least three members, suppose A doesn't know D, E or F. Then either all three of these know one another, in which case they are the three required (√) or at least two of them are strangers, in which case they with A are a set of three mutual strangers (√).

Either way, the assertion is proved.

* Exercise 3G

The questions in this exercise are more demanding than those normally set in A Level examinations, but provide an excellent training in mathematical thinking.

1. Show that if 51 points are marked within a square of side 7 cm, it is possible to draw a circle with radius 1 cm that contains at least three of the points.
2. Prove that in any set of ten whole numbers between 1 and 20 (inclusive), there are two whose highest common factor is greater than 1.
3. A point (x, y) in Cartesian geometry is called a lattice point if x and y are whole numbers. Prove that if five lattice points are chosen, they include two whose midpoint is also a lattice point.
4. Show that any set of n whole numbers includes a non-empty subset (which may be the whole set) whose sum is divisible by n.
5. If 17 points are joined to one another by straight lines (to form the graph K_{17}), and every line is either red, white or blue, prove that there is at least one triangle which has three sides all of the same colour.

3.8 Inclusion and exclusion

You have already seen a number of enumeration problems solved by a 'back-door' approach, in which the solution involves calculating an answer more than the real answer, and then taking some away. The tennis example in Section 3.1 is such a problem, as is Question 6 in Exercise 3C. In this section, the 'Inclusion-exclusion principle' is developed a little further.

The principle itself can be expressed in various ways. Its clearest expression is probably in set notation, where $n(\mathrm{A})$ represents the number of elements in the set A and $\sum$ represents a summation.

The principle states that

$$n(\mathrm{A}_1 \cup \mathrm{A}_2 \cup \ldots \cup \mathrm{A}_r) = \sum n(\mathrm{A}_i) - \sum n(\mathrm{A}_i \cap \mathrm{A}_j) + \sum n(\mathrm{A}_i \cap \mathrm{A}_j \cap \mathrm{A}_k)$$
$$- \ldots \pm n(\mathrm{A}_1 \cap \mathrm{A}_2 \cap \ldots \cap \mathrm{A}_r)$$

where the signs continue alternately + and $-$ to the end.

Example

In a large group of children, there are 23 who hate cabbage and 14 who hate semolina. If 6 children hate both cabbage and semolina, how many hate at least one of them?

Solution

Using the obvious notation,

$$n(\mathrm{C} \cup \mathrm{S}) = [n(\mathrm{C}) + n(\mathrm{S})] - n(\mathrm{C} \cap \mathrm{S})$$

$$= [23 + 14] - 6$$

$$= 31$$

This agrees with the common-sense answer, because the six children who hate both have been included in the 23 and again in the 14, and so are counted twice unless 6 is subtracted from the total.

Example

A batch of cars went for their M.O.T. test. 17 cars had faulty lights, 21 had faulty tyres, and 16 had faulty steering. 9 cars failed on lights and tyres, 12 failed on tyres and steering, and 8 failed on steering and lights. 4 cars failed on all three points.

How many cars failed the test altogether?

Solution

$$\begin{aligned} n(\mathrm{L} \cup \mathrm{T} \cup \mathrm{S}) &= [n(\mathrm{L}) + n(\mathrm{T}) + n(\mathrm{S})] - \\ &\quad [n(\mathrm{L} \cap \mathrm{T}) + n(\mathrm{T} \cap \mathrm{S}) + n(\mathrm{S} \cap \mathrm{L})] + n(\mathrm{L} \cap \mathrm{T} \cap \mathrm{S}) \\ &= (17 + 21 + 16) - (9 + 12 + 8) + 4 \\ &= 29. \end{aligned}$$

The principle can be applied to other problems too, where the set connection is less obvious.

Example

In a class there are 23 children, including four pairs of twins. In how many ways can five children be chosen if they must not include both of any pair?

Solution

There are $\binom{23}{5} = 33\ 649$ choices altogether.

Among these, there are $\binom{21}{3} = 1330$ choices which include the A twins and three other children; similarly, there are 1330 choices including the B twins, 1330 with the C twins, and 1330 with the D twins.

There are $\binom{19}{1} = 19$ choices including both the A twins and the B twins with one other child, and similarly for each of the six ways of including two pairs of twins.

So using the inclusion-exclusion principle, the number of permissible choices is

$$33\ 649 - 4 \times 1330 + 6 \times 19 = 28\ 443.$$

Exercise 3H

1. How many hands of seven cards which contain neither an ace nor a spade can be dealt from an ordinary pack?
2. In a Sixth Form tutor group of 18 students, 6 are taking French, 5 are taking German, and 4 are taking Spanish. 3 students are taking at least two of these languages, and 2 students are taking all three. How many are taking no languages at all?
3. On an exam paper there are five questions in Section A and five in Section B. Candidates must answer four questions, including at least one question from each section. In how many ways can such a choice be made?
4. In a political debating society there are 12 Conservatives, 14 Labour supporters, and 9 Liberal Democrats. In how many ways can a committee of 5 members be chosen if it must include at least one supporter of each party?
5. A tennis club has 16 male and 21 female members, including six married couples. In how many ways can the club select a team of three men and three women, not including both halves of any married couple?

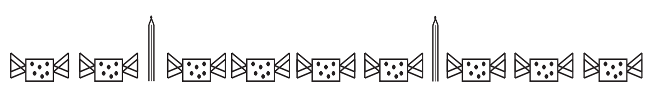
3.9 Unequal division

If Elsie, Lacie and Tillie have nine treacle toffees to share between them, in how many ways can they do it? If the shares have to be equal, there is obviously just one way - each girl must have three toffees, and since the toffees are all the same it doesn't matter which three.

But what if the shares need not all be the same size?

How many divisions are possible then?

The answer depends on whether or not you insist that every girl should have at least one toffee. If you do, then imagine the toffees set out in a line with two pencils somewhere in the line to separate the toffees into three groups.



The first group - the one on the left - will be Elsie's share, the middle group will be Lacie's, and the last group will be for Tillie. Clearly, each placing of the pencils will give a different division of the toffees.

So how many placings of the two pencils are there? The nine toffees create eight gaps between them, and two of these gaps must be chosen for the pencils. This can be done in $\binom{8}{2} = 28$ ways, so there are 28 ways to share out the toffees with each girl getting at least one.

Activity 7

Check this result by listing all the possible divisions.

If there is no restriction on the sharing - if zero shares are allowed - there are clearly more than 28 possibilities. The same model can be used, however, and the problem now is simply to arrange nine toffees and two pencils in a line. The two pencils might be next to each other (in which case Lacie gets no toffees), or at one end (in which case Elsie or Tillie goes without). The number of ways of arranging 11 objects where 9 are the same (toffees) and 2 are the same (pencils) is just $11! \div (9! \times 2!) = 55$, so this is the number of possible divisions in this case.

These principles can be applied to other cases of unequal division.

If n identical objects have to be shared (not necessarily equally) between k different groups, the number of possible divisions is

$\binom{n-1}{k-1}$ if every group must have at least one object, or

$\binom{n+k-1}{k-1}$ if there is no such restriction.

Example

The baker sells white rolls, brown rolls, sesame rolls and fruity rolls. If I go to buy a dozen rolls, how many different selections could I bring home?

Solution

A 'baker's dozen' is traditionally 13, so the problem is to allocate 13 rolls to 4 groups, each group representing one kind of roll. There is no minimum number for any group - I don't have to take any fruity rolls at all if I don't want them - so the second formula is the one that applies. There are $\binom{16}{3} = 560$ possibilities.

Example

How many solutions in positive integers has the equation $x+y+z=10$? (An equation whose solutions must be positive integers is sometimes called a Diophantine equation after the Greek mathematician *Diophantus*, who studied such equations in Egypt about 1750 years ago.)

Solution

This problem needs ten 'units' to be split into three groups, where each group must contain at least one (to give a positive integer). Using the first formula, there are $\binom{9}{2}=36$ ways to do this, so the equation has 36 different solutions.

Activity 8

Consider how to extend the methods above to answer the following.

(a) In how many ways can three children share twelve chocolate eclairs if each child must get at least two?

(b) An examination paper contains four questions, each worth a maximum of 10 marks. How many ways are there of scoring a total of 20 marks for the paper?

Exercise 3I

1. A test paper contains four questions, each worth a maximum of 10 marks. How many ways are there of scoring a total of 10 marks for the whole paper?
2. A forester goes out to buy ten saplings for a copse. If she wants a mixture of ash, birch and holly saplings, how many different combinations might she consider?
3. A sweet manufacturer sells 'lucky bags' each containing a mixture of chews, liquorice whirls, and fruit drops. If there are twenty sweets altogether, and at least one of each kind, how many different bags can be produced?
4. Rovers scored eight goals in their first home game last season. In how many ways could these goals have been shared among the eleven members of the team?
5. After a party, twelve students travel back to their college in three cars. If the identities of the individual students are ignored, in how many different ways could they be distributed between the cars?

* 3.10 Partitions

How many addition sums are there, using only positive integers, whose answer is 10?

This is not quite the same as the previous examples, because the problem does not specify the number of terms. What is more, it is not clear from the wording of the problem whether $6+4$ is to be considered the same as, or different from, $4+6$.

Suppose first of all that the order of terms is considered important, so that $6+4$ is different from $4+6$. Then

- with two terms, there are 9 possible sums from $1+9$ to $9+1$,
- with three terms, there are 36 sums, as in the third example above,
- with four terms, there are $\binom{9}{3}=84$ sums,

and so on.

If all these results are added together, the total number of sums turns out to be 511.

Check this result by working out the other values for yourself.

There is an easier way, though. Any sum totalling ten can be rewritten in crude 'Roman numerals' with ten |s and one or more + signs. For example, $5+1+4$ could be written

$$| \ | \ | \ | \ | + | + | \ | \ | \ |$$

If the |s are written first they leave nine gaps in which a + might be put, and at each gap there are two choices (+ or no +). So there are $2\times 2\times \ \dots \ \times 2 = 2^9$ choices for the row as a whole, giving 2^9-1 possible sums since the choice of no + signs at all is not allowed. The answer is thus 511 by this method too.

You might think that disregarding the order of terms, treating $6+4$ and $4+6$ as the same, would make the problem easier. In fact the opposite is the case: although the results obtained are smaller numbers, it is much more difficult to calculate them. The young Indian mathematician *Srinivasa Ramanujan* made considerable progress towards a solution of this problem in the early twentieth century, but there is still no general formula for unordered partitions comprehensible to most people.

Activity 9

There are six unordered partitions of 5, not including 5 itself:

$4+1 \quad 3+2 \quad 3+1+1 \quad 2+2+1 \quad 2+1+1+1 \quad 1+1+1+1+1$

By careful listing, find the number of unordered partitions of each of the whole numbers from 2 to 10.

If you can find a general formula linking the results you get, and equally valid for numbers beyond 10, you will almost certainly get your name in the mathematical history books!

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* 3.11 Derangements

Four men go to a pub, and each of them hangs his coat on a peg near the door. After half a dozen drinks or so, the men get their coats ready to leave, but each of them takes another's coat. In how many ways is this possible?

This is a typical 'derangement' problem, involving a rearrangement in which none of the objects keeps its original place. For small numbers, derangement problems can be solved in various ways.

Method 1

List all the possible derangements. There can't possibly be more than 24 of them, because that is the total number of rearrangements of four coats, including some in which at least one man has the right coat.

In this case, if the original order is ABCD, the possible derangements are BADC, BCDA, BDAC, CADB, CDAB, CDBA, DABC, DCAB and DCBA - nine of them in all.

Method 2

Use the inclusion-exclusion principle.

$$\begin{aligned}
\text{Derangements} &= 24 \text{ (total rearrangements)} \\
&\quad -4\times 6 \text{ (one coat correct)} \\
&\quad +6\times 2 \text{ (two coats correct)} \\
&\quad -4\times 1 \text{ (three coats correct)} \\
&\quad +1\times 1 \text{ (four coats correct)} \\
&= 9 \text{ as before.}
\end{aligned}$$

Activity 10

Use either of these two methods to complete the following table.

Number of objects	1	2	3	4	5
Derangements	0	1		9	

Your answer for five objects probably suggests that the numbers involved are getting quite large for these fairly crude methods, and that something more sophisticated is needed. The best way to continue the table, in fact, turns out to be a recurrence relation.

Let d_n represent the number of derangements of *n* objects.

Consider one of these objects in particular - object A, say. In any derangement, A must take the place previously occupied by another object - call it B - and the object B can be chosen in $(n-1)$ ways.

Now there are two possibilities:

(i) B takes the place previously occupied by A, and the other $(n-2)$ objects are deranged among themselves: this can happen in d_{n-2} ways; or

(ii) a different object - object C, say - takes the place previously occupied by A. Every such derangement corresponds exactly to a derangement of the $(n-1)$ objects without A, where C takes the place previously occupied by B, and there are d_{n-1} derangements of this kind.

There are therefore $(n-1)$ ways of choosing B, and for any given B there are $(d_{n-1}+d_{n-2})$ possible derangements, so by multiplication we have the recurrence relation

$$d_n = (n-1)(d_{n-1}+d_{n-2})$$

(Recurrence relations, also called difference equations, are explained more fully in Chapters 14 and 15.)

Now you know already that $d_1 = 0$, $d_2 = 1$, $d_3 = 2$, $d_4 = 9$ and $d_5 = 44$, and you can check by substitution that these values do indeed satisfy the recurrence relation. For example, the relation gives $d_5 = 4\times(d_4+d_3) = 4\times(9+2) = 44$ as expected.

The recurrence relation can be used to calculate d_n for any given value of *n*, though when *n* is large it is a nuisance having to proceed step by step rather than directly to a result.

Unfortunately, the recurrence relation cannot be solved by the usual methods explained in Chapter 15, so that no explicit formula can be given here.

Activity 11

Use the recurrence relation to find the value of d_n for values of n up to 10.

One extension, however, may be of interest. The problem as commonly posed concerns an incompetent secretary who types letters to n different people, each with its own envelope, but then puts the letters into the envelopes entirely at random. What is the probability that no letter is in its correct envelope?

As *n* increases, will the probability tend to 1, or to 0, or to some other value? Discuss this with other students before reading on.

Since there are d_n possible derangements of n letters, and $n!$ possible arrangements altogether, the probability of a derangement is $d_n \div n!$.

Activity 12

Use your previous answers to help you complete the following table.

n	1	2	3	4	5	6	7	8	9	10
d_n										
$n!$										
Probability										

What do you notice? Can you predict (at least approximately) the value of d_{100}?

3.12 Miscellaneous Exercises

1. There are eight runners in the 100 metres final. Assuming there are no dead heats, in how many different orders might they finish?
2. From my collection of twelve teddy bears, I must choose just three to take on holiday with me. How many different choices might I make?
3. How many solutions in positive integers has the equation $a+b+c+d+e=10$?
4. In how many ways can the letters of the word MATHEMATICS be arranged?
5. A tennis club has 12 male and 15 female members. In how many ways can two 'mixed doubles' pairs be selected to represent the club in a tournament?
6. In how many ways can Anne, Bert, Cora, Dawn, Ella and Fred sit on a row of chairs if Bert and Fred must be kept separate?
7. A club with 24 members must elect a chairman, a secretary and a treasurer. In how many ways can it do this?
8. In how many ways can eight members of a chess club be paired off to play one against another?
9. How many factors has the number 7200?
10. What is the probability that a five-card poker hand, dealt from a single pack, contains at least one card of each suit?
11. In how many ways can eight rooks be placed on a chessboard so that no two of them attack one another? (Two rooks attack one another if they are on the same row or the same column.)
12. How many four-letter words can be made using letters from MISSISSIPPI?
13. How many numbers between 1 and 2000 inclusive will not divide by any prime number less than 10?
14. How many squares (of all sizes) are there on an ordinary chessboard? And how many rectangles?
15. Prove that in any set of eleven whole numbers there are two whose difference divides exactly by 10.
16. If four married couples sit down at a round table, men and women alternately, in how many different ways can they sit if no husband is next to his own wife?
17. If five identical cubical dice are thrown at the same time, how many different results are possible? (A 'result' for this purposes is a set of scores such as 2-2-3-5-6; the order of the dice does not matter.)

*18. Using only 'silver' coins (i.e. 50p, 20p, 10p and 5p), in how many ways can you give change for £1?

*19. On an examination paper, Question 1 carries 20 marks and Questions 2, 3 and 4 carry 10 marks each. If all four questions are to be answered, show that there are 1111 ways in which a candidate can score exactly 30 marks.

*20. If S(n, k) (called a Stirling number of the second kind) denotes the number of ways in which a set with n elements can be partitioned into k non-empty subsets, show that $\mathrm{S}(n,k)=k\,\mathrm{S}(n-1,k)+\mathrm{S}(n-1,k-1)$. Hence or otherwise, find S(6, 3).

21. There are $n\ (\geq 4)$ identical items in a row and they are split into four groups each consisting of one or more item. The first few in the row will form the first group, the next few will from the second group and so on. For example 0 0 | 0 0 0 | 0 | 0 0 0 0.

    By considering where the breaks between groups must be drawn, explain why the row can be split into four such groups in $\binom{n-1}{3}$ ways.

    Use this result to answer the following questions.

    (a) In how many different ways can 30 identical sweets be shared out amongst four children so that each child gets at least one sweet?

    (b) By first giving each of the children 5 sweets, or otherwise, calculate the number of ways in which 50 identical sweets can be shared out amongst four children so that each child gets at least 6 sweets.

    (c) By first taking a sweet from each child, or otherwise, calculate the number of ways in which 30 identical sweets can be shared out amongst four children, where this time some children may get no sweets.

(AEB)