Euler Method

Computational Physics

Euler's Method

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

$$\text{Slope} = \frac{Rise}{Run}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$
$$= y_0 + f(x_0, y_0)h$$

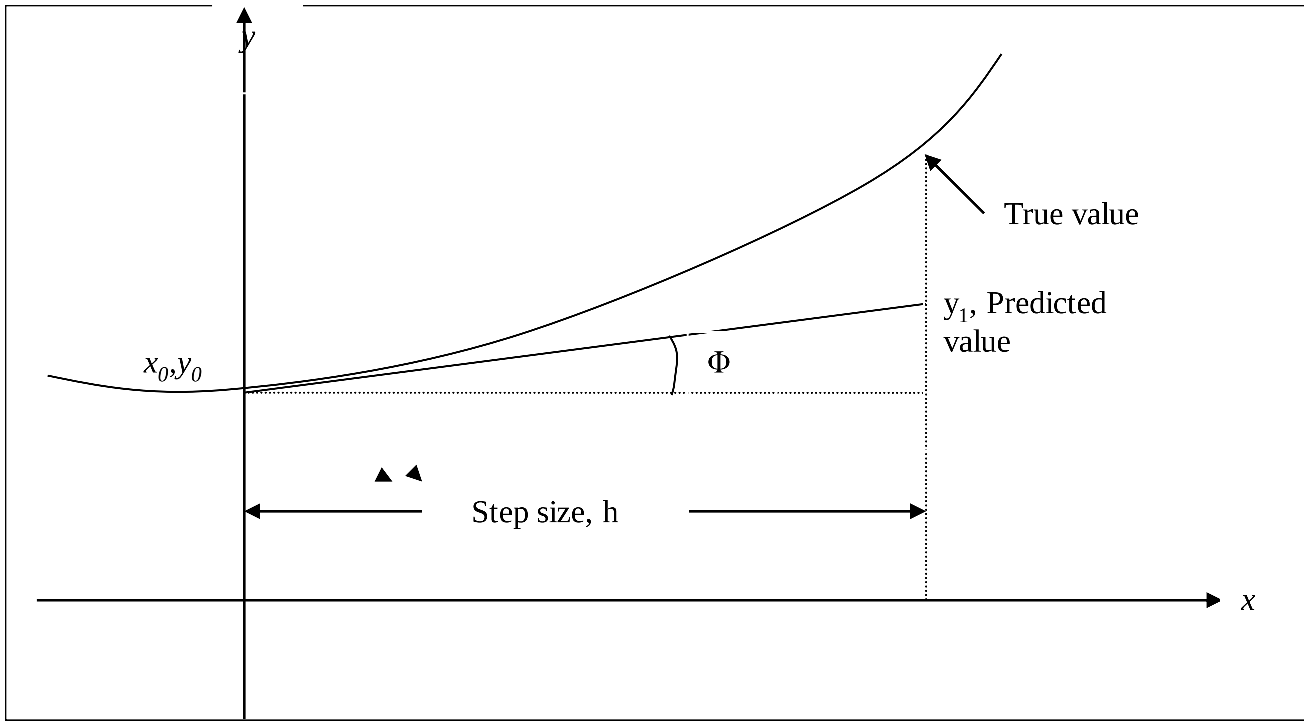

Figure 1 Graphical interpretation of the first step of Euler's method

Euler's Method

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$$h = x_{i+1} - x_i$$

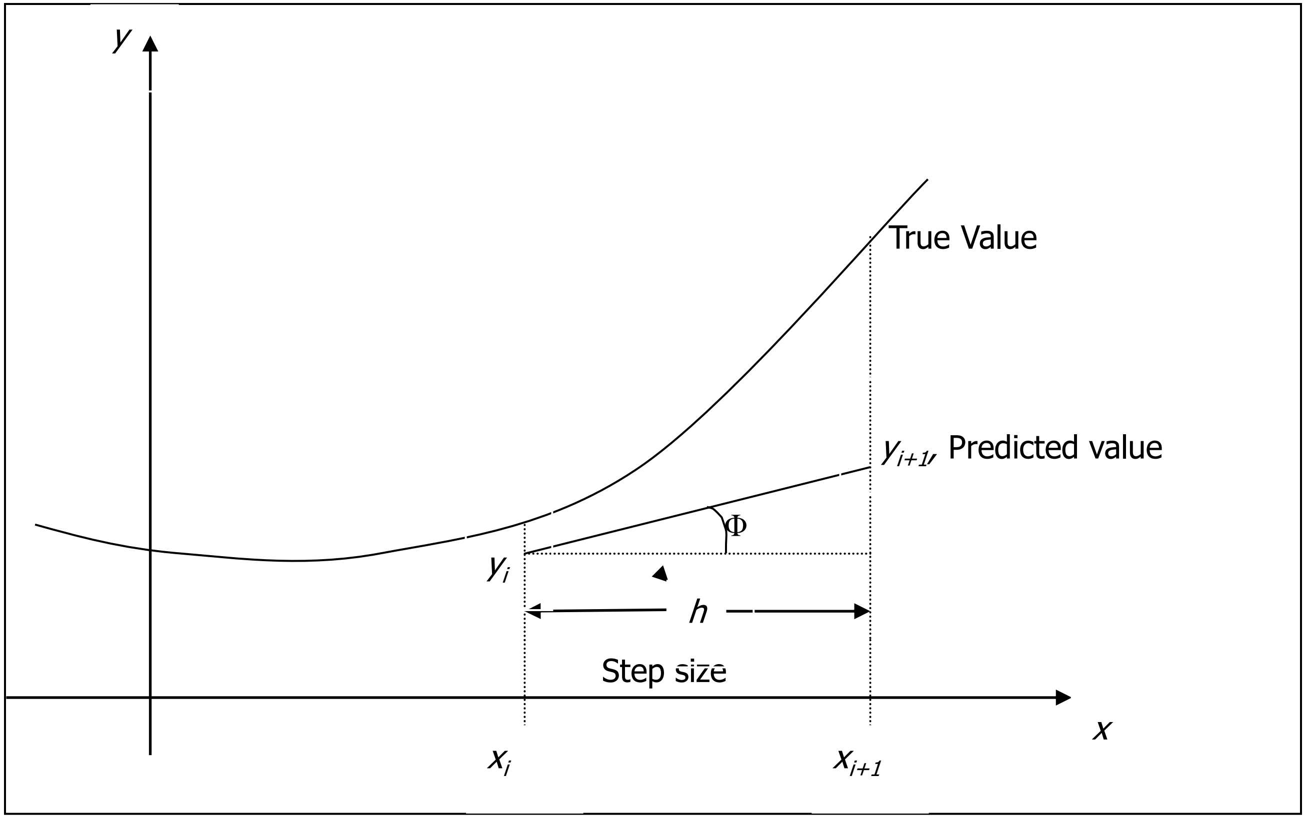

Figure 2. General graphical interpretation of Euler's method

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right), \theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds using Euler's method. Assume a step size of $h = 240$ seconds.

Solution

Step 1:

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + f(t_i, \theta_i)h$$

$$\begin{aligned}\theta_1 &= \theta_0 + f(t_0, \theta_0)h \\ &= 1200 + f(0,1200)240 \\ &= 1200 + \left(-2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right)\right)240 \\ &= 1200 + (-4.5579)240 \\ &= 106.09K\end{aligned}$$

θ_1 is the approximate temperature at $t = t_1 = t_0 + h = 0 + 240 = 240$

$$\theta(240) \approx \theta_1 = 106.09K$$

Solution Cont

Step 2: For $i=1, \quad t_1 = 240, \quad \theta_1 = 106.09$

$$\begin{aligned}\theta_2 &= \theta_1 + f(t_1, \theta_1)h \\ &= 106.09 + f(240,106.09)240 \\ &= 106.09 + \left(-2.2067\times 10^{-12}\left(106.09^4 - 81\times 10^8\right)\right)240 \\ &= 106.09 + (0.017595)240 \\ &= 110.32K\end{aligned}$$

θ_2 is the approximate temperature at $t = t_2 = t_1 + h = 240 + 240 = 480$

$$\theta(480) \approx \theta_2 = 110.32K$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta-300}{\theta+300}-1.8519\tan^{-1}\left(0.00333\theta\right)=-0.22067\times10^{-3}t-2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480)=647.57K$$

Comparison of Exact and Numerical Solutions

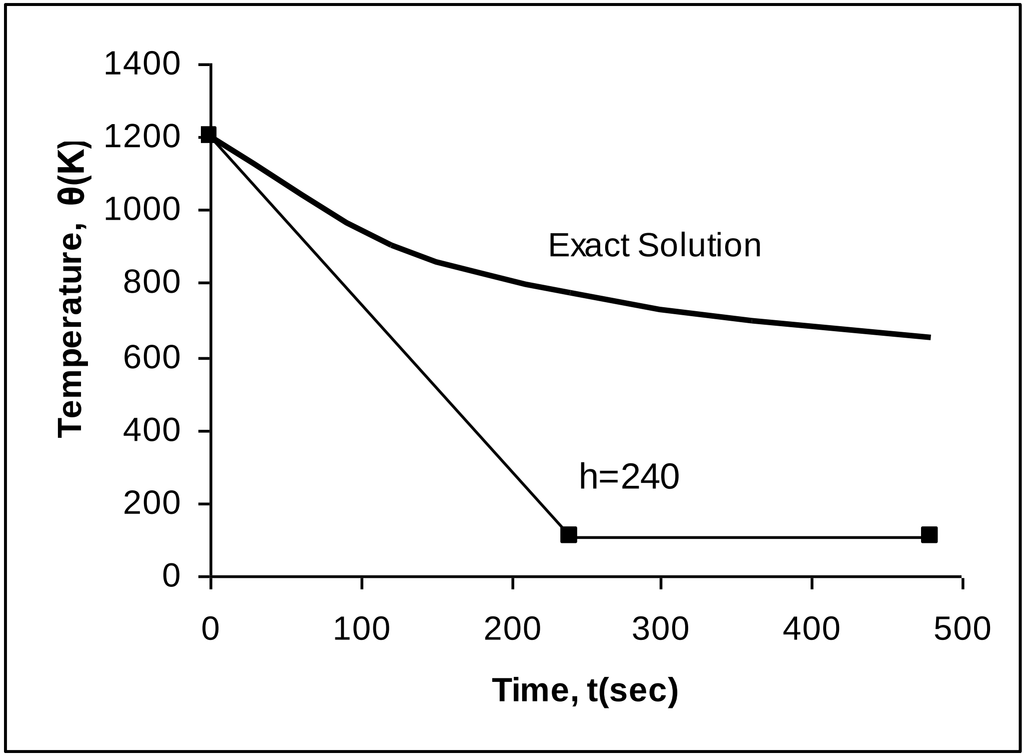

Figure 3. Comparing exact and Euler's method

Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

Step, h	$\theta(480)$	E_t	$\lvert\epsilon_t\rvert\%$
480	−987.81	1635.4	252.54
240	110.32	537.26	82.964
120	546.77	100.80	15.566
60	614.97	32.607	5.0352
30	632.77	14.806	2.2864

$\theta(480) = 647.57K$ (exact)

Comparison with exact results

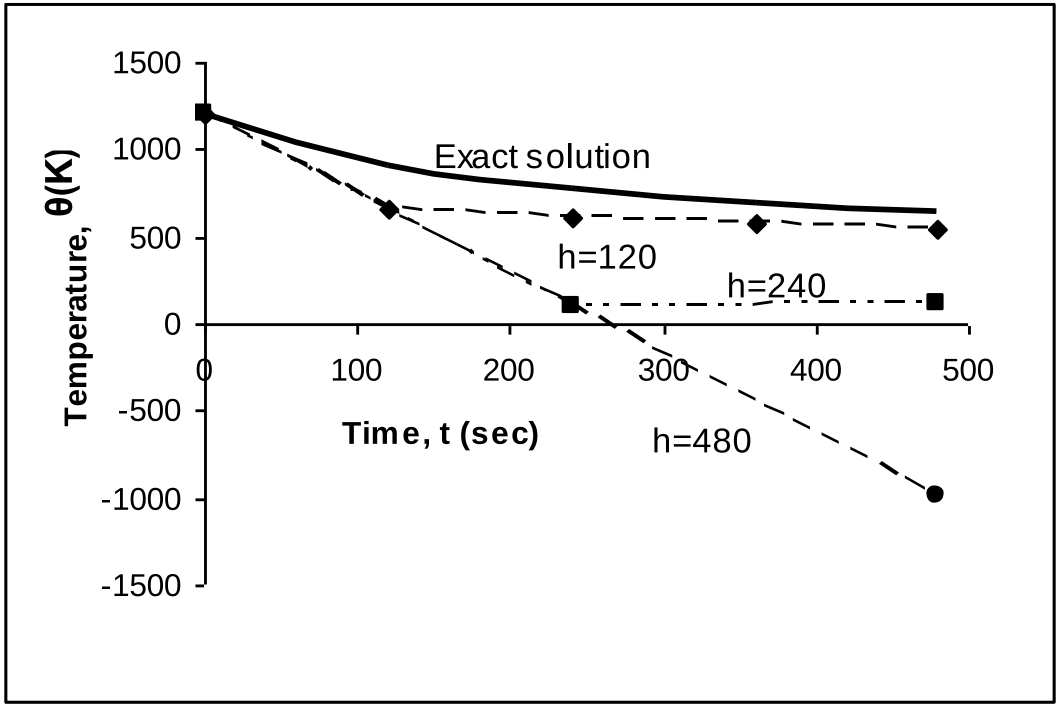

Figure 4. Comparison of Euler's method with exact solution for different step sizes

Effects of step size on Euler's Method

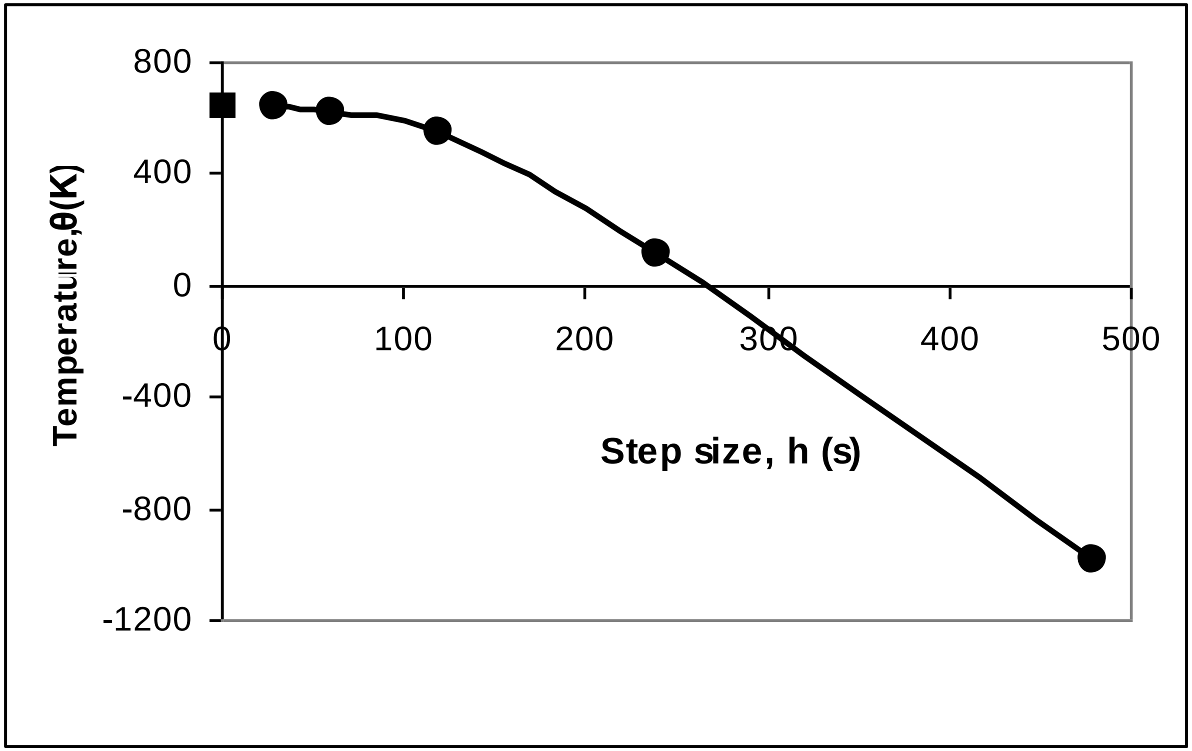

Figure 5. Effect of step size in Euler's method.

Errors in Euler's Method

It can be seen that Euler's method has large errors. This can be illustrated using Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i,y_i}(x_{i+1}-x_i) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i,y_i}(x_{i+1}-x_i)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i,y_i}(x_{i+1}-x_i)^3 + \ldots$$

$$y_{i+1} = y_i + f(x_i,y_i)(x_{i+1}-x_i) + \frac{1}{2!}f'(x_i,y_i)(x_{i+1}-x_i)^2 + \frac{1}{3!}f''(x_i,y_i)(x_{i+1}-x_i)^3 + \ldots$$

As you can see the first two terms of the Taylor series

$y_{i+1} = y_i + f(x_i, y_i)h$ are the Euler's method.

The true error in the approximation is given by

$$E_t = \frac{f'(x_i,y_i)}{2!}h^2 + \frac{f''(x_i,y_i)}{3!}h^3 + \ldots \qquad E_t \propto h^2$$

THE END

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