Simpson's 1/3rd Rule of Integration

Computational Physics

What is Integration?

Integration

The process of measuring the area under a curve.

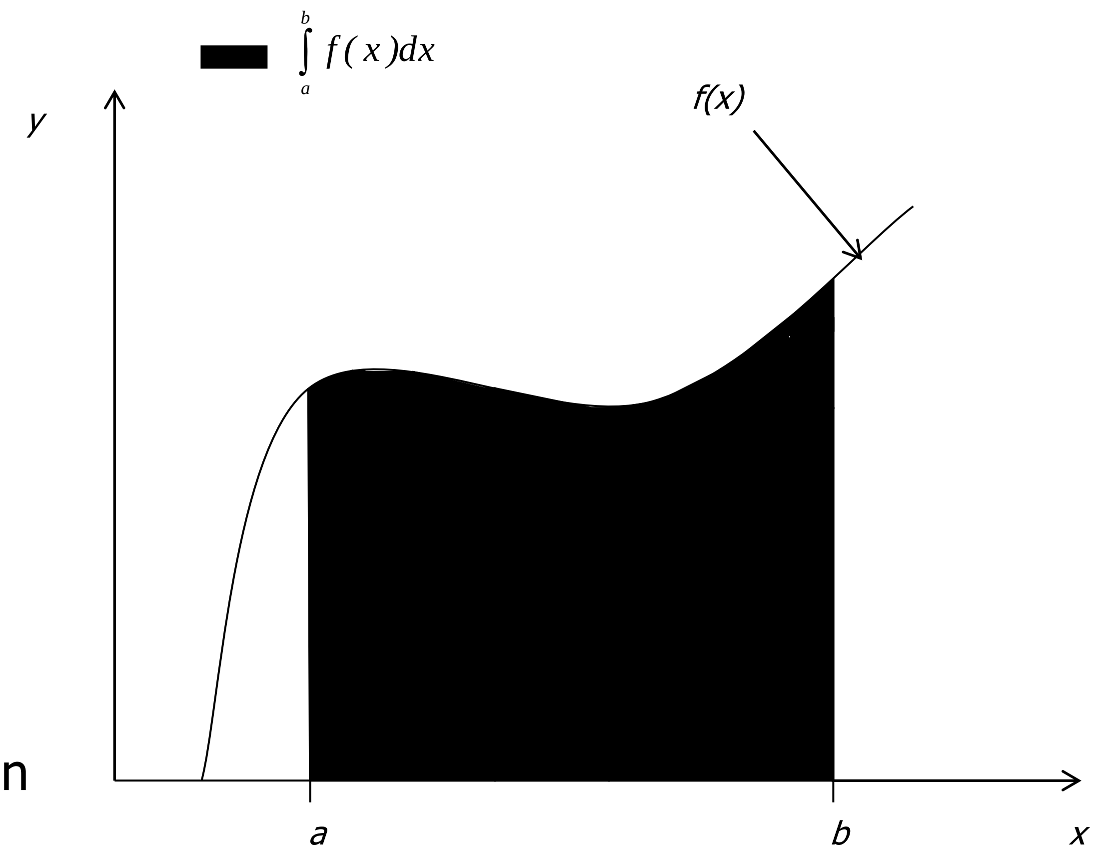
$$I = \int_a^b f(x)dx$$

Where:

f(x) is the integrand

a= lower limit of integration

b= upper limit of integration



Simpson's 1/3rd Rule

Basis of Simpson's 1/3rd Rule

Trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial in the interval of integration. Simpson's 1/3rd rule is an extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.

Hence

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

Where $f_2(x)$ is a second order polynomial.

$$f_2(x) = a_0 + a_1 x + a_2 x^2$$

Basis of Simpson's 1/3rd Rule

Choose

$$(a, f(a)), \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right), \text{ and } (b, f(b))$$

as the three points of the function to evaluate a_0, a_1 and a_2.

$$f(a) = f_2(a) = a_0 + a_1 a + a_2 a^2$$

$$f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2$$

$$f(b) = f_2(b) = a_0 + a_1 b + a_2 b^2$$

Basis of Simpson's 1/3rd Rule

Solving the previous equations for a_0, a_1 and a_2 give

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Basis of Simpson's 1/3rd Rule

Then

$$I \approx \int_a^b f_2(x)\,dx$$

$$= \int_a^b \left(a_0 + a_1 x + a_2 x^2\right) dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]_a^b$$

$$= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3}$$

Basis of Simpson's 1/3rd Rule

Substituting values of a_0, a_1, a_2 give

$$\int_a^b f_2(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since for Simpson's 1/3rd Rule, the interval [a, b] is broken into 2 segments, the segment width

$$h = \frac{b-a}{2}$$

Basis of Simpson's 1/3rd Rule

Hence

$$\int_a^b f_2(x)dx = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Because the above form has 1/3 in its formula, it is called Simpson's 1/3rd Rule.

Example 1

The distance covered by a rocket from t=8 to t=30 is given by

$$x = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use Simpson's 1/3rd Rule to find the approximate value of x

b) Find the true error, E_t

c) Find the absolute relative true error, $|\in_t|$

Solution

a)

$$x = \int_{8}^{30} f(t)dt$$

$$x = \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

$$= \left(\frac{30-8}{6}\right)[f(8) + 4f(19) + f(30)]$$

$$= \left(\frac{22}{6}\right)[177.2667 + 4(484.7455) + 901.6740]$$

$$= 11065.72\ m$$

Solution (cont)

b) The exact value of the above integral is

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

$$= 11061.34\ m$$

True Error

$$E_t = 11061.34 - 11065.72$$

$$= -4.38\ m$$

Solution (cont)

a)c) Absolute relative true error,

$$\left|\in_t\right| = \left|\frac{11061.34 - 11065.72}{11061.34}\right| \times 100\%$$

$$= 0.0396\%$$

Multiple Segment Simpson's 1/3rd Rule

Multiple Segment Simpson's 1/3rd Rule

Just like in multiple segment Trapezoidal Rule, one can subdivide the interval [a, b] into n segments and apply Simpson's 1/3rd Rule repeatedly over every two segments. Note that n needs to be even. Divide interval [a, b] into equal segments, hence the segment width

$$h = \frac{b-a}{n} \qquad \int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx$$

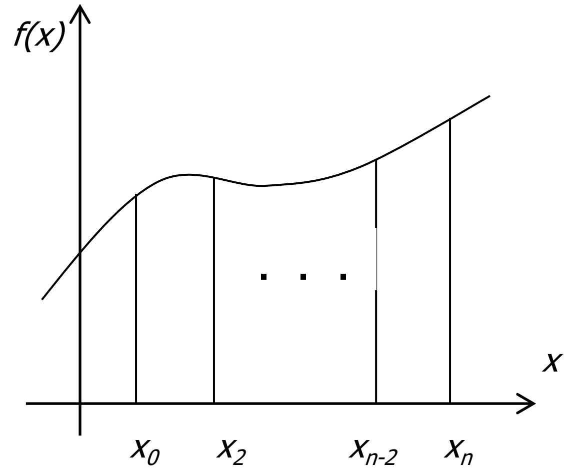
where

$$x_0 = a \qquad x_n = b$$

Multiple Segment Simpson's 1/3rd Rule

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots\ldots$$

$$\ldots + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$



Apply Simpson's 1/3rd Rule over each interval,

$$\int_a^b f(x)dx = (x_2 - x_0)\left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right] + \ldots$$

$$+ (x_4 - x_2)\left[\frac{f(x_2) + 4f(x_3) + f(x_4)}{6}\right] + \ldots$$

Multiple Segment Simpson's 1/3rd Rule

$$\ldots + (x_{n-2} - x_{n-4})\left[\frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6}\right] + \ldots$$

$$+ (x_n - x_{n-2})\left[\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}\right]$$

Since

$$x_i - x_{i-2} = 2h \qquad i = 2, 4, \ldots, n$$

Multiple Segment Simpson's 1/3rd Rule

Then

$$\int_a^b f(x)dx = 2h\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right]+\ldots$$

$$+2h\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right]+\ldots$$

$$+2h\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right]+\ldots$$

$$+2h\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

Multiple Segment Simpson's 1/3rd Rule

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0)+4\{f(x_1)+f(x_3)+\ldots+f(x_{n-1})\}+\ldots\right]$$

$$\ldots+2\{f(x_2)+f(x_4)+\ldots+f(x_{n-2})\}+f(x_n)\}]$$

$$=\frac{h}{3}\left[f(x_0)+4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(x_i)+2\sum_{\substack{i=2\\ i=even}}^{n-2} f(x_i)+f(x_n)\right]$$

$$=\frac{b-a}{3n}\left[f(x_0)+4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(x_i)+2\sum_{\substack{i=2\\ i=even}}^{n-2} f(x_i)+f(x_n)\right]$$

Example 2

Use 4-segment Simpson's 1/3rd Rule to approximate the distance

covered by a rocket from t= 8 to t=30 as given by

$$x = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use four segment Simpson's 1/3rd Rule to find the approximate value of *x*.
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error, $|\in_a|$ for part (a).

Solution

a) Using n segment Simpson's 1/3rd Rule,

$$h = \frac{30-8}{4} = 5.5$$

So

$$f(t_0) = f(8)$$

$$f(t_1) = f(8+5.5) = f(13.5)$$

$$f(t_2) = f(13.5+5.5) = f(19)$$

$$f(t_3) = f(19+5.5) = f(24.5)$$

$$f(t_4) = f(30)$$

Solution (cont.)

$$x = \frac{b-a}{3n}\left[f(t_0) + 4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(t_i) + 2\sum_{\substack{i=2\\ i=even}}^{n-2} f(t_i) + f(t_n) \right]$$

$$= \frac{30-8}{3(4)}\left[f(8) + 4\sum_{\substack{i=1\\ i=odd}}^{3} f(t_i) + 2\sum_{\substack{i=2\\ i=even}}^{2} f(t_i) + f(30) \right]$$

$$= \frac{22}{12}[f(8) + 4f(t_1) + 4f(t_3) + 2f(t_2) + f(30)]$$

Solution (cont.)

cont.

$$= \frac{11}{6}\left[f(8) + 4f(13.5) + 4f(24.5) + 2f(19) + f(30)\right]$$

$$= \frac{11}{6}\left[177.2667 + 4(320.2469) + 4(676.0501) + 2(484.7455) + 901.6740\right]$$

$$= 11061.64\ m$$

Solution (cont.)

b) In this case, the true error is

$$E_t = 11061.34 - 11061.64 = -0.30\ m$$

c) The absolute relative true error

$$|\in_t| = \left|\frac{11061.34 - 11061.64}{11061.34}\right| \times 100\%$$

$$= 0.0027\%$$

Solution (cont.)

Table 1: Values of Simpson's 1/3rd Rule for Example 2 with multiple segments

n	Approximate Value	E_t	$\lvert \in_t \rvert$
2	11065.72	4.38	0.0396%
4	11061.64	0.30	0.0027%
6	11061.40	0.06	0.0005%
8	11061.35	0.01	0.0001%
10	11061.34	0.00	0.0000%

Error in the Multiple Segment Simpson's 1/3rd Rule

The true error in a single application of Simpson's 1/3rd Rule is given as

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b$$

In Multiple Segment Simpson's 1/3rd Rule, the error is the sum of the errors in each application of Simpson's 1/3rd Rule. The error in n segment Simpson's 1/3rd Rule is given by

$$E_1 = -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1) = -\frac{h^5}{90} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2$$

$$E_2 = -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2) = -\frac{h^5}{90} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4$$

Error in the Multiple Segment Simpson's 1/3rd Rule

$$E_i = -\frac{(x_{2i} - x_{2(i-1)})^5}{2880} f^{(4)}(\zeta_i) = -\frac{h^5}{90} f^{(4)}(\zeta_i), \quad x_{2(i-1)} < \zeta_i < x_{2i}$$

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$$E_{\frac{n}{2}-1} = -\frac{(x_{n-2} - x_{n-4})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right) = -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right), \quad x_{n-4} < \zeta_{\frac{n}{2}-1} < x_{n-2}$$

$$E_{\frac{n}{2}} = -\frac{(x_n - x_{n-2})^5}{2880} f^{4}\left(\zeta_{\frac{n}{2}}\right) = -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}}\right), \quad x_{n-2} < \zeta_{\frac{n}{2}} < x_n$$

Error in the Multiple Segment Simpson's 1/3rd Rule

Hence, the total error in Multiple Segment Simpson's 1/3rd Rule is

$$E_t = \sum_{i=1}^{\frac{n}{2}} E_i \quad = -\frac{h^5}{90}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) \quad = -\frac{(b-a)^5}{90n^5}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{90n^4}\,\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

Error in the Multiple Segment Simpson's 1/3rd Rule

The term $\dfrac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$ is an approximate average value of

$$f^{(4)}(x),\ a < x < b$$

Hence

$$E_t = -\frac{(b-a)^5}{90n^4}\bar{f}^{(4)}$$

where

$$\bar{f}^{(4)} = \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

THE END

Computational Physics