



Propagation of Errors

Computational Physics



Propagation of Errors

In numerical methods, the calculations are not made with exact numbers. How do these inaccuracies propagate through the calculations?



Example 1:

Find the bounds for the propagation in adding two numbers. For example if one is calculating $X + Y$ where

$$X = 1.5 \pm 0.05$$

$$Y = 3.4 \pm 0.04$$

Solution

Maximum possible value of $X = 1.55$ and $Y = 3.44$

Maximum possible value of $X + Y = 1.55 + 3.44 = 4.99$

Minimum possible value of $X = 1.45$ and $Y = 3.36$.

Minimum possible value of $X + Y = 1.45 + 3.36 = 4.81$

Hence

$$4.81 \leq X + Y \leq 4.99.$$



Propagation of Errors In Formulas

If f is a function of several variables $X_1, X_2, X_3, \dots, X_{n-1}, X_n$ then the maximum possible value of the error in f is

$$\Delta f \approx \left| \frac{\partial f}{\partial X_1} \Delta X_1 \right| + \left| \frac{\partial f}{\partial X_2} \Delta X_2 \right| + \dots + \left| \frac{\partial f}{\partial X_{n-1}} \Delta X_{n-1} \right| + \left| \frac{\partial f}{\partial X_n} \Delta X_n \right|$$



Example 2:

The strain in an axial member of a square cross-section is given by

$$\epsilon = \frac{F}{h^2 E}$$

Given

$$F = 72 \pm 0.9 \text{ N}$$

$$h = 4 \pm 0.1 \text{ mm}$$

$$E = 70 \pm 1.5 \text{ GPa}$$

Find the maximum possible error in the measured strain.



Example 2:

Solution

$$\begin{aligned}\epsilon &= \frac{72}{(4 \times 10^{-3})^2 (70 \times 10^9)} \\ &= 64.286 \times 10^{-6} \\ &= 64.286 \mu\end{aligned}$$

$$\Delta \epsilon = \left| \frac{\partial \epsilon}{\partial F} \Delta F \right| + \left| \frac{\partial \epsilon}{\partial h} \Delta h \right| + \left| \frac{\partial \epsilon}{\partial E} \Delta E \right|$$



Example 2:

$$\frac{\partial \epsilon}{\partial F} = \frac{1}{h^2 E} \quad \frac{\partial \epsilon}{\partial h} = -\frac{2F}{h^3 E} \quad \frac{\partial \epsilon}{\partial E} = -\frac{F}{h^2 E^2}$$

Thus

$$\begin{aligned} \Delta E &= \left| \frac{1}{h^2 E} \Delta F \right| + \left| \frac{2F}{h^3 E} \Delta h \right| + \left| \frac{F}{h^2 E^2} \Delta E \right| \\ &= \left| \frac{1}{(4 \times 10^{-3})^2 (70 \times 10^9)} \times 0.9 \right| + \left| \frac{2 \times 72}{(4 \times 10^{-3})^3 (70 \times 10^9)} \times 0.0001 \right| \\ &\quad + \left| \frac{72}{(4 \times 10^{-3})^2 (70 \times 10^9)^2} \times 1.5 \times 10^9 \right| \\ &= 5.3955 \mu \end{aligned}$$

Hence

$$\epsilon = (64.286 \mu \pm 5.3955 \mu)$$



Example 3:

Subtraction of numbers that are nearly equal can create unwanted inaccuracies. Using the formula for error propagation, show that this is true.

Solution

Let

$$z = x - y$$

Then

$$\begin{aligned} |\Delta z| &= \left| \frac{\partial z}{\partial x} \Delta x \right| + \left| \frac{\partial z}{\partial y} \Delta y \right| \\ &= |(1)\Delta x| + |(-1)\Delta y| \\ &= |\Delta x| + |\Delta y| \end{aligned}$$

So the relative change is

$$\left| \frac{\Delta z}{z} \right| = \frac{|\Delta x| + |\Delta y|}{|x - y|}$$



Example 3:

For example if

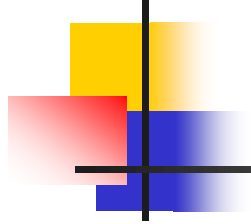
$$x = 2 \pm 0.001$$

$$y = 2.003 \pm 0.001$$

$$\left| \frac{\Delta z}{z} \right| = \frac{|0.001| + |0.001|}{|2 - 2.003|}$$

$$= 0.6667$$

$$= 66.67\%$$



THE END

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